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Preface

Intent

This text is designed to serve as a one-semester introduction to trigonometry and its applications for college students.

Assumptions

It is assumed that students have basic skills in solving linear and quadratic equations, working with radicals, and simple graphing as well as some acquaintance with, and access to, a scientific calculator.

There is no separate chapter of review material. Most students react negatively to such a chapter, and many teachers get bogged down in unnecessary details in such a chapter. In this book, material is reviewed as it is encountered during the exposition of the new material. Focusing on old skills when they become necessary provides a more interesting sequence for students. The best motivation for reviewing something is the fact that it is needed to learn something new. Examples go out of their way to review the algebraic skills being applied, as they first arise.



Calculators

It is assumed that students have constant access to a modern scientific/engineering calculator. In terms of approximate calculations, the text is completely calculator oriented. However, the usual exact radian/degree values, for $\frac{\pi}{6}(30^\circ)$, $\frac{\pi}{4}(45^\circ)$, and their multiples, are explicitly used as well.

Many students today have access to graphing calculators. The book is designed to be used *without* these calculators, but their use is encouraged. The text shows how to use these calculators for the many graphs that occur in trigonometry. The TEXAS INSTRUMENTS TI-81 is used to illustrate within the text. Appendix B, *Further uses for graphing calculators and computers*, shows further detail on using the TI-81 and also presents material on using the CASIO fx-7000 graphing calculator.

Content

Chapter 1 reviews the basic geometric foundations of right triangle trigonometry, followed by the trigonometry of the right triangle. The term trigonometric *ratio* is used to distinguish the definitions made in terms of the right triangle from those made in chapter 2 in terms of points in the plane. We begin with right triangle trigonometry for at least the following four reasons. First, a survey of mathematics teachers indicated a clear preference for this approach. Second, this allows students to start the course in territory that is probably familiar, degree measure and right triangles. Third, applications in this area can be drawn from everyday experience. Fourth, this approach is most helpful to those students who may be starting a physics or other technical course at the same time.

The secant, cosecant, and cotangent functions are defined as reciprocals of the cosine, sine and tangent functions. In this way students are exposed to identities immediately, and, after all, this is the way we view these functions in advanced applications.

The use of the calculator, including a basic use of the inverse trigonometric functions, is covered in this chapter.

The chapter ends with a section that introduces trigonometric equations, both conditional and identities. It is important for students to see equations involving trigonometric functions early and throughout the course. The topic of conditional equations is revisited in the context of radian measure in chapter 2. In this way, the inevitable chapter on trigonometric equations (chapter 5) becomes simply an expansion of previous material, instead of the cold shock it often is for students.

Chapter 2 begins with a review of *functions*, which are introduced in terms of sets of ordered pairs. This approach combines mathematical precision with pedagogical simplicity (as opposed to a long statement about rules associating something with something else). It also permits a much simpler, more natural development of the concepts of one-to-one and inverse functions and has no negative impact on acquiring

ability with $f(x)$ terminology. In addition, this approach helps avoid the documented problem that most students cannot separate the concept of a function from a defining expression (i.e., an equation).

This chapter then covers the trigonometry of angles in standard position. In this exposition of analytic trigonometry, we refer to trigonometric *functions* not ratios. The topic is first completely covered with familiar angle measure. Radians are then introduced, and the topic is revisited in these terms, including the solution of simple conditional equations in terms of radians.

Chapter 3 gives a more in-depth treatment of the properties of the trigonometric functions as functions from the real numbers to the real numbers. Domains, ranges, and graphs are stressed. An uncommon method of graphing periodic functions is introduced here. Period and phase shift are not treated separately but are integrated into one step. It is our experience that this independently discovered the method of graphing quite complicated functions is easily and quickly taught and learned. It also provides a basis for understanding what makes periodic functions intrinsically different from nonperiodic functions.

This chapter also supports the use of graphing calculators to obtain graphs, for those who have access to these calculators.

Chapter 4 discusses the inverse of any function, then fully treats the inverse trigonometric functions. Students will have some feeling for these functions from earlier chapters, where they were explicitly mentioned in the context of finding unknown angles and solving conditional equations. The simplification of expressions involving these functions, using reference triangles, is stressed. This skill is very important in a calculus course.

The definitions of the inverse cosecant and secant functions are made in terms of the inverse sine and cosine functions. Although less common than other possible definitions, this is in keeping with other sources and provides a great simplicity in learning and using these functions. Some calculus texts define the ranges of these functions differently than as presented here (for good reason, in that context), but the additional complexity is not pedagogically warranted for

an introduction. The definitions here, by the way, agree with those used in at least one popular symbolic algebra computer package.

Chapter 5 is a full treatment of trigonometric identities and conditional equations. The modern depth of coverage and the introduction in earlier chapters of most of the concepts should provide the basis for success for both teachers and students. Section 5-0 is explicitly designed to widen the scope of how students understand an expression to include trigonometric functions.

Chapter 6 presents topics that are applications of the previous material. The laws of sines and cosines are presented, and analytic vectors are introduced. The way in which the law of sines is presented should eliminate confusion for most students. The important fact that no triangle has more than one obtuse angle is stressed and used to solve triangles in a directed way.

Chapter 7 covers complex numbers, including De Moivre's theorem, and polar coordinates. The graphing of polar coordinates includes an independently discovered approach to the graphing of certain polar equations using the rectangular coordinate graph as a guide. This is a nontraditional approach, which the authors have used with success. It is easy to use, promotes a "feeling" for polar graphs, and reviews the graphing of trigonometric equations in rectangular coordinates.

Appendixes

Appendix A discusses using addition of ordinates for graphing. This topic is of much less practical value in the age of the electronic graphing device, but it can be a valuable pedagogical tool to promote the understanding of functions.

Appendix B presents more material on using graphing calculators. It can be used to enhance a course where every student has access to these wonderful devices.

Appendix C is the lengthy algebraic development of an identity, which may or may not be used in the course. This development has been set aside in an appendix because its inclusion within the text does not promote students' reading of the text or even a better understanding of the material.

Appendix D provides graphical material which the student may want to reproduce and have handy as an aid in studying.

Appendix E provides the answers to odd-numbered exercise problems, all chapter review problems, and all chapter test problems. Solutions are also provided for those trial exercise problems whose number is boxed in the exercise sets.

Exposition of the Material

We have attempted to write the exposition of the material in clear, understandable prose, in a logical order of development. Each section of a chapter is designed to provide accessible reading for students and clear examples of the skills that students are expected to master in that section. We have also tried to provide a cross section of *applications*, mainly in the exercises. These are designed to put the subject in a wider context of knowledge and therefore to pique students' interest. In particular, we have tried to show that trigonometry has become more important than ever in the age of the digital computer.

Each section of every chapter ends with a list of *mastery points*, which clearly states what students should know from that section. The exercise sets are designed to enable students to apply the material learned in the examples within the text, which attempt to provide a clear outline of the skills that the student must master. The exercise sets reinforce these skills explicitly, with many problems similar to the examples. Some problems require that students synthesize what has been learned, and a few require above-average efforts to solve. These are marked with the

symbol . The complete solutions to those problems whose numbers are contained in boxes, called *trial problems*, are given in Appendix E.

A set of core problems are indicated in each exercise set by having their numbers in color. These problems exemplify each of the mastery points. Thus, if students can do these problems without error or difficulty then they have mastered the skills presented in that section. This is provided for those students who do not have the time to work a larger subset of the

exercises. All students are still well advised to do the more difficult less skill oriented problems at the end of the exercise set when assigned.

Each chapter ends with a *chapter summary*, *chapter review*, and *chapter test*. The summary serves as a memory jogger, to be occasionally reviewed after the chapter is completed. The review consists of problems that enable students to practice the skills acquired throughout the chapter; these problems are keyed by section. The test is designed to allow students to practice a chapter test before seeing one in class. The material in the chapter test is removed from its exposition—this is the opportunity for students to see the material out of context, which is the situation when taking an in-class test.

Text Features

Some of the themes which we believe make this text special include the following.

- **Algorithmic** The text is explicit about procedures for accomplishing the skills required. These procedures are highlighted in the text. The examples explicitly follow these procedures.
- **Detailed** Skills and knowledge that are often assumed of students, but that in fact are not present, are explicitly covered. Rationalizing denominators is shown many times, as are many other algebraic manipulations, in the examples. Another example of this is the explicit coverage of the manipulation of the fractions involved in the radian measures, which are multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$, and achieving a feeling for the location on the unit circle of these measures. Still another is the algebraic manipulations often used in solving identities.
- **Gradual skill building** The level of algebraic competence required builds through the chapters. For example, solving equations in which a product equals zero is covered early, whereas factoring is postponed until the chapter on identities.

Identities and conditional equations are difficult, so they are introduced early, and revisited lightly in the chapters preceding the chapter on trigonometric equations.

Solving right triangles is so basic to trigonometry at all levels of abstraction that it must be second nature. For this reason, right triangles are stressed not only in the applications and theoretical development but also as a means of finding the values of trigonometric functions when given the value of one of them. This is most important in a calculus course.

- **Repetition of themes** There are many themes that are revisited over and over in the chapters. Examples include solving conditional equations and manipulating identities, solving right triangles, and the idea of reference angles. In addition, a great deal of material is explicitly repeated when radian measure is covered in the last third of chapter 2.

Vectors in two dimensions, complex numbers in polar form, and polar coordinates have a great deal of similarity in skills required, particularly in conversions between rectangular and polar forms. We have stressed these similarities.

- **Balance of concrete versus abstract** Most attention is paid to numeric manipulations, as in solving triangles and finding the values of trigonometric functions when given the value of one trigonometric function, but attention is paid to symbolic manipulation as well. Using these skills with symbolic manipulation is important in a calculus course, and they have not been short changed.

Some of the exercises, always toward the end of each exercise set, also stress more abstract thinking, as well as the investigation of related ideas.

Changes from the Previous Edition

All references to tables of values have been deleted in this edition. This simplifies the material in the earlier chapters, allowing some material to migrate toward the front of the text. Functions are now explicitly reviewed earlier, at the beginning of chapter 2. This better provides the setting for the introduction of the trigonometric functions.

The topic of radian measure occurs earlier, in chapter 2 also. This provides the setting for reviewing the trigonometric functions in a new context.

A section on graphing by addition of ordinates is moved to the appendixes. Introductory material on the inverse trigonometric functions and trigonometric equations is more thoroughly integrated into the earlier chapters.

The chapter on identities is refashioned to include a more explicit review of equation solving and algebraic manipulation. Several sections were combined, which allows a more coherent presentation of the material.

In the previous edition, vectors and complex numbers each got two sections. The material on vectors was simplified by lessening the stress on geometric vectors and proceeding more quickly to analytic vectors. Thus, vectors are now treated in one section. It also seemed pedagogically appropriate and feasible to combine the two sections on complex numbers into one, with only minor adjustments in coverage, at no loss of depth. A section on graphing polar equations has been added.

The appendix on graphing calculators is new, as is the discussion of these calculators within the text. It is only a matter of time and economics before these devices are universally used in mathematics courses, and they can certainly be used with this text.

An appendix was added containing material students may want to reproduce xerographically. In particular, rectangular and polar coordinate templates are provided.

Supplements

For the instructor

The *Instructor's Manual* includes an introduction to the text, a guide to the supplements that accompany *Trigonometry*, and reproducible chapter tests. Also included are a complete listing of all mastery points and suggested course schedules based on the mastery points. The final section of the *Instructor's Manual* contains answers to the reproducible materials.

The *Instructor's Solutions Manual* contains completely worked-out solutions to all of the exercises in the textbook.

Selected Overhead Transparencies are available to enhance classroom presentations.

WCB Computerized Testing Program provides you with an easy-to-use computerized testing and grade management program. No programming experience is required to generate tests randomly, by objective, by section, or by selecting specific test items. In addition, test items can be edited and new test items can be added. Also included with the *WCB Computerized Testing Program* is an on-line testing option which allows students to take tests on the computer. Tests can then be graded and the scores forwarded to the grade-keeping portion of the program.

The *Test Item File* is a printed version of the computerized testing program that allows you to examine all of the prepared test items and choose test items based on chapter, section, or objective. The objectives are taken directly from *Trigonometry*.

For the student

The *Student's Solutions Manual* introduces the student to the textbook and includes solutions to every-other odd-numbered section exercise and odd-numbered end-of-chapter exercise problems. It is available for student purchase.

Videotapes covering the major topics in each chapter are available. Each concept is introduced with a real-world problem and is followed by careful explanation and worked-out examples using computer-generated graphics. These videos can be used in the math lab for remediation or even the classroom to motivate or enhance the lecture. The videotapes are available free to qualified adopters.

The concepts and skills developed in *Trigonometry* are reinforced through the interactive *Software*.

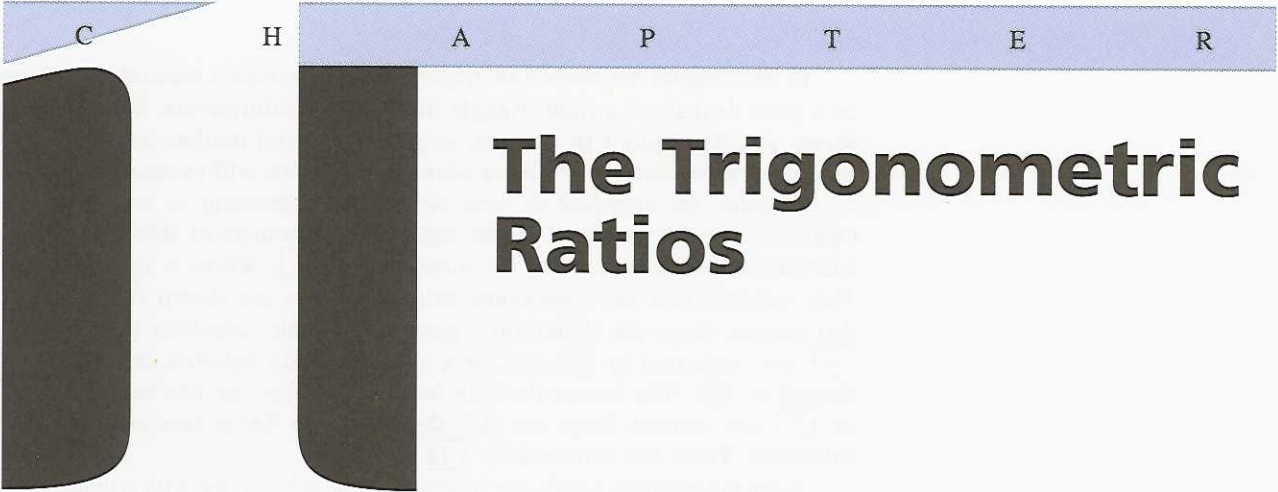
The Plotter is software for graphing and analyzing functions. This software simulates a graphing calculator on a PC. You may use it to do the technology

exercises even if you don't have a graphics calculator. A manual is included that describes operations and includes student exercises. The software is menu driven and has an easy-to-use window-type interface. The high-quality graphics can also be used for classroom presentation and demonstration. Students who go on to calculus classes will want to keep the software for future use.

Acknowledgments

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Throughout the development, writing, and production of this text, two WCB employees have been of such great value they deserve special recognition: Theresa Grutz and Eugenia M. Collins.

A decorative graphic at the top of the page. It features a horizontal bar with the letters C, H, A, P, T, E, R spaced out. Below the bar, there are two large, dark, rounded rectangular shapes. The first shape is on the left, and the second is on the right, partially overlapping the title text.

The Trigonometric Ratios

1-0 Introduction

Initially trigonometry was developed to express relationships between the sizes of the arcs in circles and the chords determining those arcs. (An arc is a portion of the circumference of a circle; a chord is a line segment going from one point on a circle to another.) These relationships were used in astronomy more than two thousand years ago to study what is called the celestial sphere. In fact, until the fifteenth century, trigonometry was mostly applied to spheres. This part of trigonometry is now called spherical trigonometry, and it is still used in navigation and astronomy.

After the fifteenth century, trigonometry was also used to relate the measure of angles in a triangle to the lengths of the sides of the triangle. The word “trigonometry,” which means “triangle measurement,” is credited to Bartholomaeus Pitiscus (1561–1613). Besides being used in surveying, trigonometry became important for the physics being developed by Sir Isaac Newton and others. Practically all the ideas of trigonometry had been developed by the eighteenth century.

Over the next two hundred years, trigonometry became more and more important as it was used to describe many physical phenomena, such as electricity, magnetism, and sound.

Today, in the age of the computer, trigonometry is being used even more. The computer can control manufacturing machines to great precision, but only if trigonometry is used to describe where this precision is to be applied. This whole area of engineering, called numerical control, relies heavily on the ideas we will study in this book. Computer graphics use aspects of projective geometry, which again uses the concepts of trigonometry. Voice recognition by computers uses a concept in mathematics called Fourier transforms, which again is built on these same ideas.

From medicine to manufacturing, from solar design to computer art, the ideas we study here are found over and over again.

In this chapter we learn how trigonometry provides a method for telling us a great deal about a right triangle from limited information. These simple ideas, widely applied in science, engineering, and mathematics, lay the groundwork for some of the more advanced ideas we will eventually need.

Students are expected to have access to engineering or scientific calculators to facilitate many of the calculations throughout this text. Many calculations are followed by the notation $\boxed{\text{CS } n}$, where n is an integer. This indicates that the appropriate calculator steps are shown at the end of that section. Steps are shown for a generic algebraic calculator (one with an $\boxed{=}$ key, indicated by $\textcircled{A}$) and for a generic postfix notation calculator (indicated by $\textcircled{P}$). The latter calculator has no $\boxed{=}$ key but has an $\boxed{\text{ENTER}}$ or $\boxed{\uparrow}$ key instead. Steps are also shown for the Texas Instruments TI-81 calculator. These are indicated by $\boxed{\text{TI-81}}$.

Some preliminary words are in order about calculating with a calculator. It is difficult to give a general rule for the number of digits that should appear in the final result of a calculation, and a discussion of this is outside the scope and intent of this book. The number of digits to which an approximate number should be rounded is specified throughout the text. The number of digits is chosen to represent a reasonable, if sometimes arbitrary, value.

Also, we will apply the most straightforward rounding rule to values. That is, if the first discarded digit is 5 or above we round up, otherwise we do not. Thus, 2.349 rounds to 2.35 to two decimal places, and to 2.3 to one decimal place.

1-1 Angle measurement, the right triangle, and the Pythagorean theorem

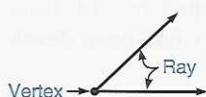


Figure 1-1

Angle measurement

An **angle** is composed of two rays, both beginning at what is called the **vertex** of the angle. Figure 1-1 shows a representation of an angle.

In modern geometry a method of angle measurement is assumed. We will assume that angles can be measured in **degrees**. The notation for degrees is $^\circ$. A common and useful interpretation of angle measure is “the amount of rotation” of one ray away from the other. In this context, 90° corresponds to a quarter-rotation, 180° to a half-rotation, 270° to a three-quarter rotation, and 360° to a full rotation. Naturally, the measurement of an angle does not necessarily imply any actual rotation. See figure 1-2.

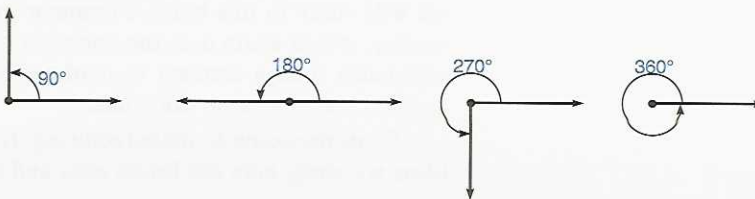


Figure 1-2

An angle with measure between 0° and 90° is said to be **acute**; an angle with measure 90° is **right**; an angle with measure between 90° and 180° is **obtuse**; and an angle with measure 180° is a **straight angle**.

One degree is divided into smaller units in two ways: using the degree, minute, second system and using the decimal degree system.

In the **degree, minute, second (DMS) system** a degree is divided into 60 equal parts called *minutes*, and each minute is divided into 60 equal parts called *seconds*. This is analogous to the way in which hours are divided into minutes and seconds on the clock. The notation for minutes is ' and the notation for seconds is ''. For example, $51^\circ 18' 22''$ means 51 degrees, 18 minutes, and 22 seconds.

In the **decimal degree system** the degree measure is written in decimal notation. For example, 2.53° is in decimal degrees and means 2 and 53 hundredths of a degree.

Calculators will generally not perform operations on values in the DMS system, so we must be able to convert from this system to decimal form. Many calculators are programmed to do this. They typically use keys marked $^\circ ' ''$ or $\rightarrow H$. This is illustrated in example 1-1 A for two typical calculators. Also note that we use the fact that there are 60 minutes in one degree, and $60^2 = 3,600$ seconds in one degree.

■ Example 1-1 A

Convert $46^\circ 42' 27''$ to decimal degrees to the nearest 0.001° .

Manually:

$$46^\circ + \left(\frac{42}{60}\right)^\circ + \left(\frac{27}{3600}\right)^\circ$$

Rewrite minutes and seconds as fractional parts of a degree

$$46.7075^\circ$$

$$46.708^\circ$$

$\boxed{CS 1}$

Round to the nearest 0.001°

Calculator A:

$$46 \boxed{^\circ ' ''} 42 \boxed{^\circ ' ''} 27 \boxed{^\circ ' ''}$$

Calculator B:

$$46.4227 \boxed{\rightarrow H}$$

"H" stands for hours

The right triangle

A closed figure composed of three straight sides will always include three angles, and these figures are therefore called **triangles**¹ (*tri* is from the Latin for three). (Actually the angles are formed by extending the sides, which are line segments, to form rays.)

A useful property of triangles is that *the measures of the three angles of a triangle always add up to 180°* . This fact was known thousands of years ago and its first known demonstration appears in Euclid's *Elements*, an ancient Greek book on geometry. This allows us to find the measure of the third angle in a triangle if we know the measures of the other two: if we add the measures

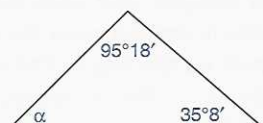
¹Note that, unless otherwise specified, all references to lines, triangles, etc., are to these concepts as they occur in plane, or Euclidean, geometry. Fortunately, this is the geometry with which the reader is most likely to be familiar, but it is worth mentioning that mathematics recognizes other "types" of geometry, called non-Euclidean geometries. In fact, several of these geometries find applications in modern physics.

of the two known angles and subtract this from the known total, 180° , the result must be the measure of the third angle.

Note We often denote angles using Greek letters. The letters most often used are α (alpha), β (beta), γ (gamma), and θ (theta).

■ Example 1-1 B

Find the measure of angle α in the triangle in the figure.



$$\begin{aligned}\alpha &= 180^\circ - (95^\circ18' + 35^\circ8') \\ &= 180^\circ - 130^\circ26' = 179^\circ60' - 130^\circ26' = 49^\circ34'\end{aligned}$$

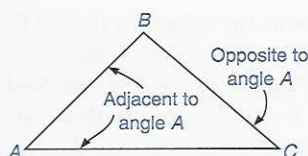


Figure 1-3

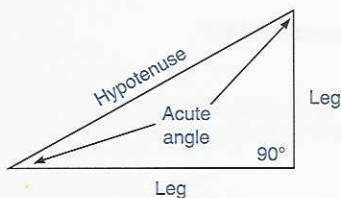


Figure 1-4

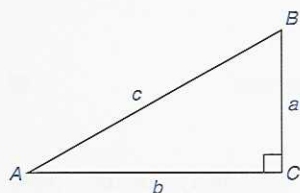


Figure 1-5

When we say that a side of a triangle is **opposite** to an angle, we are referring to the side that is not used to form the angle. When we say that a side of a triangle is **adjacent** to an angle, we mean that the side actually forms one side of the angle. For example, in triangle ABC in figure 1-3, we would say that side BC is opposite to angle A , while sides AB and AC are each adjacent to angle A .

A **right triangle** is a triangle in which one of the angles is a right angle (90°). In such a triangle, two of the angles must be acute (less than 90°), since their measures must add up to 90° . The side of a right triangle that is opposite the right angle is called the **hypotenuse**, and the sides that are adjacent to the right angle are sometimes called **legs**. See figure 1-4.

One particular way to label right triangles is widely used. Unless we are told otherwise, in a right triangle we always label the right angle C and the two acute angles A and B . The lengths of the legs are always labeled a and b , with a opposite angle A and b opposite angle B . The hypotenuse is always labeled c . This is illustrated in figure 1-5.

Note The symbol $\square$ denotes a right angle.

The Pythagorean theorem

We often use the following theorem.² It is one of the most important facts in mathematics.

The Pythagorean theorem

In a right triangle with legs having lengths a and b and hypotenuse having length c ,

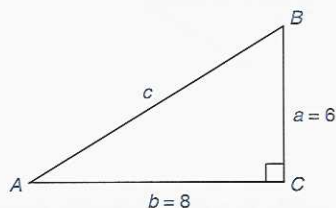
$$a^2 + b^2 = c^2$$

We use the Pythagorean theorem to find the length of the hypotenuse of a right triangle if we know the lengths of the two legs.

²The word *theorem* means a statement that has been proved to be true, and the proof of this theorem is credited to the Greek mathematician Pythagoras (sixth century B.C.), who is said to have sacrificed an ox as an offering of thanks. In the last two thousand years literally hundreds of proofs of this theorem have been given.

■ Example 1-1 C

If the two legs of a right triangle have lengths 6 and 8, what is the length of the hypotenuse?



To use the Pythagorean theorem, we label one leg a and the other b . See the figure. Let $a = 6$ and $b = 8$ and use the theorem as follows:

$$\begin{aligned} a^2 + b^2 &= c^2 \text{ so } 6^2 + 8^2 = c^2 \\ 100 &= c^2 \end{aligned}$$

To find c we take the principal square root of 100:

$$\begin{aligned} \sqrt{100} &= c \\ 10 &= c \end{aligned}$$

The Pythagorean theorem also provides a way to find the length of one leg of a right triangle if the hypotenuse and the other leg are known.

■ Example 1-1 D

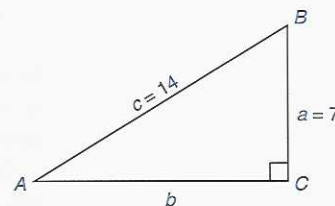
If one leg of a right triangle has length 7, and the hypotenuse has length 14, find the length of the other leg. Find the answer both exactly and to the nearest tenth.

Let $a = 7$ and b be the unknown side, as in the figure. Then,

$$a^2 + b^2 = c^2$$

so

$$\begin{aligned} 7^2 + b^2 &= 14^2 \\ 49 + b^2 &= 196 \\ b^2 &= 147 \\ b &= \sqrt{147} \\ b &= \sqrt{(49)(3)} \\ b &= \sqrt{49}\sqrt{3} \\ b &= 7\sqrt{3} \\ b &\approx 12.1 \end{aligned}$$



Exact solution

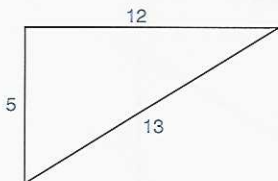
Approximate solution³ CS 2

It can also be shown that if $a^2 + b^2 = c^2$, then that triangle is a right triangle, and the right angle is opposite side c .

³When we write something like $7\sqrt{3} \approx 12.1$, we mean that $7\sqrt{3}$ is approximately 12.1. This use of the symbol $\approx$ is adhered to in this text to signify approximate values.

■ **Example 1-1 E**

1. Is the triangle in the figure a right triangle?



If the triangle is a right triangle, then the hypotenuse would be the side having length 13 (*the hypotenuse of a right triangle is always the longest side*). Therefore the legs would have lengths 5 and 12. We now add the squares of the lengths of the legs and see if this sum equals the square of the hypotenuse, 13.

$$5^2 + 12^2 = 25 + 144 = 169, \text{ and} \\ 13^2 = 169$$

Thus, $a^2 + b^2 = c^2$ and, therefore, the triangle is a right triangle.

Note In part 1 of example 1-1 E, we stated that the hypotenuse of a right triangle is always the longest side. This is because *in any triangle, whether or not it is a right triangle, the longest side is always opposite the largest angle*. Similarly, the shortest side is always opposite the smallest angle.

2. The sides of a triangle have lengths 32, 53, and 62. Is the triangle a right triangle?

Using the Pythagorean theorem,

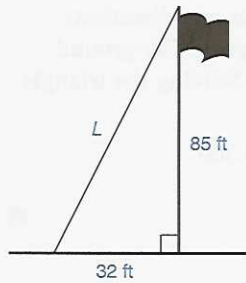
$$32^2 + 53^2 = 1,024 + 2,809 = 3,833 \text{ and } 62^2 = 3,844$$

Since $a^2 + b^2 \neq c^2$, we see that the triangle is not a right triangle. ■

As we said earlier, the Pythagorean theorem is one of the most important facts in mathematics because it has so many applications to practical situations and technical problems. Many situations in science and technology can be described, or modeled, using right triangles. If we know that a physical situation can be described in terms of right triangles, then all of the mathematics that applies to right triangles can be used to learn more about the situation or to solve a given problem.

In practice we must often make simplifying assumptions about the situation. For example, we may assume that a telephone pole makes a right angle with the ground, when in fact it is unlikely that the angle is exactly 90° , or we may assume that the earth is flat, that “level” roads are perfectly level, etc.

■ Example 1-1 F



1. A guy wire is to be attached to the top of a flagpole and anchored in the ground at a point 32 feet from the base of the flagpole. If the pole is 85 feet high, how long will the guy wire have to be, to the nearest foot?

We first draw a figure to describe the problem, as shown. We assume that a flagpole is constructed to form as close to a 90° angle with the ground as possible. Also, we ignore the fact that the wire will actually sag somewhat and therefore is not a true straight line.

We can see from the figure that the unknown length of the guy wire L is the hypotenuse of a right triangle in which the legs have lengths 85 and 32. Thus, the Pythagorean theorem provides the answer.

$$85^2 + 32^2 = L^2$$

$$8,249 = L^2$$

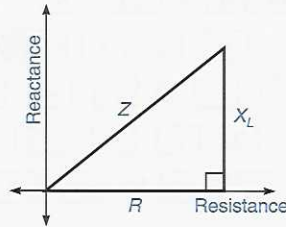
$$\sqrt{8,249} = L$$

$$90.8 \approx L$$

CS 3

To the nearest foot, the guy wire must be 91 feet long.

2. In the theory of alternating current⁴ in electronics, the total **circuit impedance** Z in an inductive circuit can be found if we know the **inductive reactance** X_L and the **resistance** R by using the impedance diagram shown in the figure.



Suppose $R = 320$ ohms and $X_L = 160$ ohms. Find Z to the nearest ten ohms.

Using the Pythagorean theorem,

$$Z^2 = R^2 + X_L^2$$

$$Z^2 = 320^2 + 160^2$$

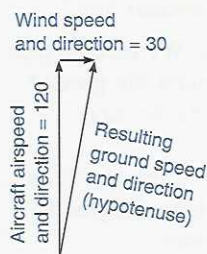
$$Z^2 = 128,000$$

$$Z = \sqrt{128,000}$$

$$Z \approx 357.8$$

$$Z \approx 360 \text{ ohms, to the nearest ten ohms}$$

⁴As in many other examples and problems in this text, the reader may not be familiar with the terminology and/or the facts involved in the application being illustrated. Please note, however, that where the necessary background is not provided, the problem is phrased so that it is clear to the reader which mathematical operation is required to achieve the desired result.



3. An aircraft is flying with an airspeed of 120 knots⁵ and a heading of due north. A 30-knot wind is blowing from the west. What is the aircraft's ground speed, to the nearest knot?

It is a fact of physics that we can represent the speeds and directions given above in a right triangle as illustrated in the figure. The ground speed of the aircraft is the length of the hypotenuse. Solving the triangle shows that the ground speed is 124 knots:

$$\begin{aligned}(\text{ground speed})^2 &= 120^2 + 30^2 = 15,300 \\ \text{ground speed} &= \sqrt{15,300} \\ &\approx 124\end{aligned}$$

Calculator steps

1. $\textcircled{A}$ 46 $\boxed{+}$ 42 $\boxed{\div}$ 60 $\boxed{+}$ 27 $\boxed{\div}$ 3600 $\boxed{=}$ Display $\boxed{46.7075}$
 $\textcircled{P}$ 46 $\boxed{\text{ENTER}}$ 42 $\boxed{\text{ENTER}}$ 60 $\boxed{\div}$ $\boxed{+}$ 27 $\boxed{\text{ENTER}}$
 3600 $\boxed{\div}$ $\boxed{+}$
 $\boxed{\text{TI-81}}$ 46 $\boxed{+}$ 42 $\boxed{\div}$ 60 $\boxed{+}$ 27 $\boxed{\div}$ 3600 $\boxed{\text{ENTER}}$
2. $\textcircled{A}$ 7 $\boxed{\times}$ 3 $\boxed{\sqrt{x}}$ $\boxed{=}$ Display $\boxed{12.12435566}$
 $\textcircled{P}$ 7 $\boxed{\text{ENTER}}$ 3 $\boxed{\sqrt{x}}$ $\boxed{\times}$
 $\boxed{\text{TI-81}}$ 7 $\boxed{\times}$ $\boxed{\sqrt{}}$ 3 $\boxed{\text{ENTER}}$
3. $\textcircled{A}$ 85 $\boxed{x^2}$ $\boxed{+}$ 32 $\boxed{x^2}$ $\boxed{=}$ $\boxed{\sqrt{x}}$ Display $\boxed{90.82400564}$
 $\textcircled{P}$ 85 $\boxed{x^2}$ 32 $\boxed{x^2}$ $\boxed{+}$ $\boxed{\sqrt{x}}$
 $\boxed{\text{TI-81}}$ $\boxed{\sqrt{}}$ $\boxed{(}$ 85 $\boxed{x^2}$ $\boxed{+}$ 32 $\boxed{x^2}$ $\boxed{)}$ $\boxed{\text{ENTER}}$

Mastery points

Can you

- State whether a given angle is acute, right, straight, or obtuse?
- Convert from the degree, minute, second system to the decimal degree system?
- Find the third angle of any triangle when you know the other two angles?
- Give the definition of right triangle and draw and label one using the conventional notation?
- State the Pythagorean theorem?
- Use the Pythagorean theorem to find one side of a right triangle when you know the other two sides?

⁵One knot means one nautical mile per hour. A nautical mile is $\frac{1}{60}$ of a degree on the earth's circumference. It is approximately 6,080.27 feet. Nautical miles are often used in the navigation of ships and aircraft.

Exercise 1-1

Convert each angle to its measure in decimal degrees. Round the answer to the nearest 0.001° where necessary. Also, state whether each angle is acute, obtuse, straight, or right.

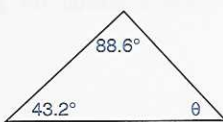
- | | | | |
|------------------------|------------------------|---------------------|-------------------|
| 1. $13^\circ 25'$ | 2. $111^\circ 56'$ | 3. $0^\circ 12'$ | 4. $42^\circ 37'$ |
| 5. $25^\circ 33' 19''$ | 6. $87^\circ 2' 13''$ | 7. $165^\circ 47'$ | 8. $19^\circ 15'$ |
| 9. $33^\circ 5' 55''$ | 10. $0^\circ 19' 12''$ | 11. $159^\circ 59'$ | 12. $20^\circ 1'$ |

13. Draw a representation of a right triangle and label it using the conventional labeling (using A, B, C, a, b, c).

14. State the Pythagorean theorem from memory.

In the following problems find the measure of the angle θ .

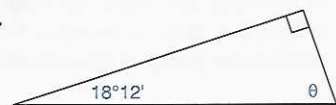
15.



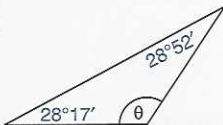
16. 82.65°



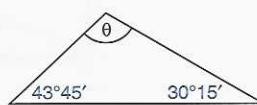
17.



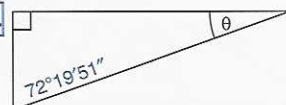
18.



19.

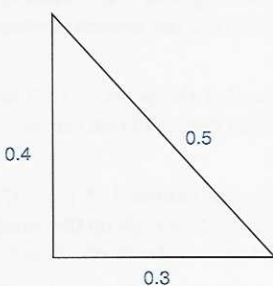


20.

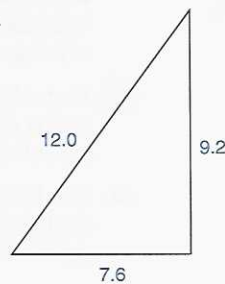


In the following problems state whether each triangle is a right triangle or not; if the triangle is a right triangle, state the length of the hypotenuse.

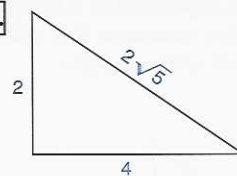
21.



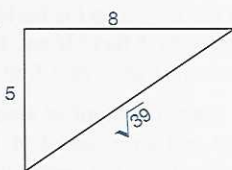
22.



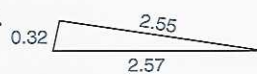
23.



24.



25.



26.

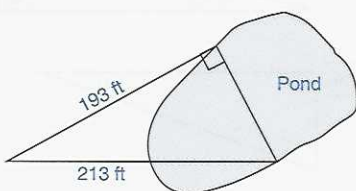


In the following problems two of the three sides of a right triangle are given. Use the Pythagorean theorem to calculate the length of the missing side; leave your answer in exact form and also approximate your answer to the same number of decimal places as the data (if necessary).

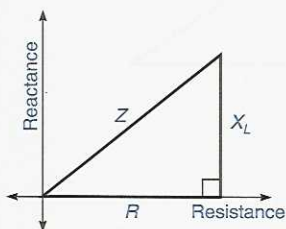
	a	b	c		a	b	c		a	b	c
27.	9	12	?	28.	10	?	26	29.	?	8	10
30.	5	10	?	31.	12	?	18	32.	?	6.8	9.2
33.	$\sqrt{5}$	3	?	34.	13.2	19.6	?	35.	100	150	?
36.	0.66	1.42	?	37.	$\sqrt{7}$	3	?	38.	4	?	$\sqrt{23}$
39.	6.3	?	15.0	40.	2	?	3	41.	$3\sqrt{2}$	$4\sqrt{5}$	?
42.	$3\sqrt{2}$	?	$4\sqrt{5}$	43.	?	19	28	44.	1,002	3,512	?
45.	1	1	?	46.	30	?	50				

Solve the following problems.

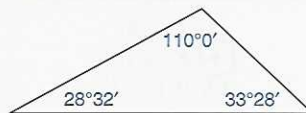
47. A flagpole is 55 feet tall and is supported by a wire attached at the top of the pole and to the ground 26 feet from the base of the pole. How long is the wire, to the nearest foot?
48. A flagpole is 93 feet tall and is supported by a guy wire that is 157 feet long, attached at the top of the pole and to the ground some distance from the base of the pole. Find the distance of the wire's ground attachment point from the base of the pole, to the nearest foot.
49. A surveyor has made the measurements shown in the diagram in order to compute the width of a pond. How wide is the pond, to the nearest 0.1 foot?



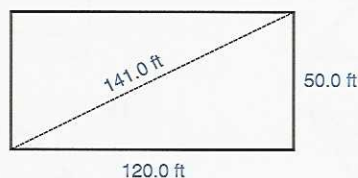
This diagram is called an impedance diagram. It is used to compute total impedance in a certain electronic circuit. X_L means inductive reactance, R is resistance, and Z is impedance. All units are ohms.



50. If X_L is 40.0 ohms and R is 56.6 ohms, calculate the total impedance Z to the nearest 0.1 ohm.
51. Suppose that X_L is 5.68 ohms and R is 19.25 ohms. Find Z to the nearest 0.01 ohm.
52. If $Z = 213$ ohms and $R = 183$ ohms, find X_L to the nearest ohm.
53. If $X_L = 2,150$ ohms and $Z = 4,340$ ohms, find R to the nearest 10 ohms.
54. A triangular piece of land has been surveyed and the results are shown in the diagram. What can you say about the accuracy of the survey? Give a reason for your answer.



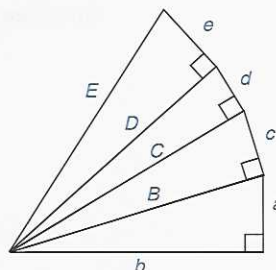
55. A rectangular piece of land has been surveyed and the results are shown in the diagram. What can you say about the accuracy of the survey? Give a reason for your answer.



56. A machinist has to cut a rectangular piece of steel along its diagonal. The saw that will be used can cut this type and thickness of steel at the rate of 0.75 inch per minute. If the piece is 13.8 inches long and 9.6 inches wide, calculate how many minutes (to the nearest minute) it will take to cut the piece.
57. Do problem 56 but assume that the piece is 15.0 centimeters (cm) long and 10.5 cm wide and that the saw will cut at 0.8 cm per minute.
58. The ladder on a fire truck can extend 125 feet. If the truck is 25 feet from a building, how high up the building can the ladder reach, to the nearest tenth of a foot?
59. If the fire truck of problem 58 moves 5 more feet from the building (to 30 feet), does the height up the building that the ladder can reach decrease by 5 feet? If not, how much does it decrease, to the nearest tenth of a foot?
60. Find the ground speed, to the nearest knot, of an aircraft flying with a heading due east and an airspeed of 132 knots if there is a wind blowing from the north at 23 knots.
61. Find the ground speed, to the nearest knot, of an aircraft flying with a heading due west and an airspeed of 105 knots if there is a wind blowing from the north at 18 knots.
62. Find the ground speed, to the nearest knot, of an aircraft flying with a heading due south and an airspeed of 178 knots if there is a wind blowing from the east at 25 knots.

63. Find the speed relative to the ground, to the nearest knot, of a boat heading directly across a river at 16 knots if the current is moving at 4.3 knots.
64. To the nearest $\frac{1}{4}$ inch, find the length of the diagonal of an $8\frac{1}{2}$ -inch by 11-inch piece of paper.
65. In the "Mathematics of Warfare" by F. W. Lanchester (from *The World of Mathematics* by James R. Newman), Mr. Lanchester presents the idea that, all other things being equal, the strengths of fighting forces add in a manner proportional to the squares of their numbers. Referring to the diagram, this means that a force of size B is equal to two forces of sizes a and b ; that C is equal to the combined strengths of a , b , and c ; etc. Assuming

that forces a , b , c , d , and e are of size 20, 5, 12, 8, and 10, respectively, find the size of force E that is equivalent to the combined strengths of these forces. Find this force to the nearest unit.



1-2 The trigonometric ratios

Trigonometry has existed in one form or another for more than two thousand years. The creator of trigonometry is said to have been the Greek Hipparchus of the second century B.C. The Hindus and, primarily, the Arabs continued developing the subject. In the fifteenth and sixteenth centuries, the Germans developed trigonometry into the form presented here.

The primary trigonometric ratios

We first define the **sine**, **cosine**, and **tangent** ratios, abbreviated sin, cos, and tan, respectively. We refer to these as the primary trigonometric ratios.

Primary trigonometric ratios

If θ is either of the two acute angles in a right triangle (refer to figure 1-6), then:

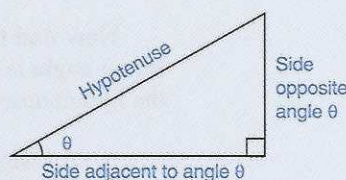


Figure 1-6

Ratio	Definition
$\sin \theta =$	$\frac{\text{length of side opposite } \theta}{\text{length of hypotenuse}}$
$\cos \theta =$	$\frac{\text{length of side adjacent to } \theta}{\text{length of hypotenuse}}$
$\tan \theta =$	$\frac{\text{length of side opposite } \theta}{\text{length of side adjacent to } \theta}$

Note "sin θ " means "the sine of angle θ " and is read "sine theta."
 "cos θ " means "the cosine of angle θ " and is read "cosine theta."
 "tan θ " means "the tangent of angle θ " and is read "tangent theta."

An abbreviated version of these definitions is

$$\sin \theta = \frac{\text{opp}}{\text{hyp}}$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}}$$

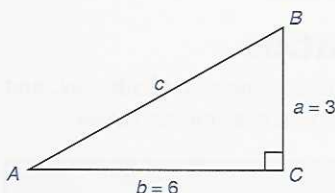
$$\tan \theta = \frac{\text{opp}}{\text{adj}}$$

These ratios are used in astronomy, surveying, engineering, science, and mathematics. In fact, there is virtually no area of science and technology that does not use them.

■ Example 1-2 A

- Find the sine, cosine, and tangent ratios for angles A and B in right triangle ABC , where $a = 3$ and $b = 6$.

Recall that the right angle is always labeled C , side a is opposite angle A , and side b is opposite angle B . We show this and the given data in the figure.



First, we find c by the Pythagorean theorem.

$$a^2 + b^2 = c^2$$

$$3^2 + 6^2 = c^2$$

$$45 = c^2$$

$$\sqrt{45} = c$$

$$\sqrt{(9)(5)} = c$$

$$\sqrt{9}\sqrt{5} = c$$

$$3\sqrt{5} = c$$

Now find the sine ratio for each angle A and B . Since the sine ratio for an angle is the length of the side opposite the angle over the length of the hypotenuse, we have

$$\begin{aligned}\sin A &= \frac{\text{side opposite angle } A}{\text{hypotenuse}} = \frac{a}{c} = \frac{3}{3\sqrt{5}} = \frac{1}{\sqrt{5}} \\ &= \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}\end{aligned}$$

$$\begin{aligned}\sin B &= \frac{\text{side opposite angle } B}{\text{hypotenuse}} = \frac{b}{c} = \frac{6}{3\sqrt{5}} = \frac{2}{\sqrt{5}} \\ &= \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}\end{aligned}$$

To find the cosine ratio we form the ratios of the sides adjacent to each angle over the length of the hypotenuse.

$$\cos A = \frac{\text{side adjacent to angle } A}{\text{hypotenuse}} = \frac{b}{c} = \frac{6}{3\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

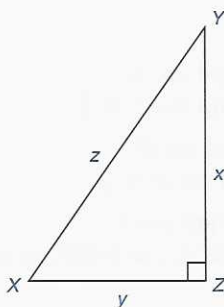
$$\cos B = \frac{\text{side adjacent to angle } B}{\text{hypotenuse}} = \frac{a}{c} = \frac{3}{3\sqrt{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

The tangent of an angle is the length of the side opposite the angle over the length of the side adjacent to the angle. Thus,

$$\tan A = \frac{\text{side opposite angle } A}{\text{side adjacent to angle } A} = \frac{a}{b} = \frac{3}{6} = \frac{1}{2}$$

$$\tan B = \frac{\text{side opposite angle } B}{\text{side adjacent to angle } B} = \frac{b}{a} = \frac{6}{3} = 2$$

2. Find the sine, cosine, and tangent ratios, in terms of the values x and y only, for the angle labeled X in the right triangle in the figure.



Using the Pythagorean theorem, we find the hypotenuse:

$$z^2 = x^2 + y^2$$

$$z = \sqrt{x^2 + y^2}$$

Since the sine of an acute angle in a right triangle is the length of the side opposite the angle divided by the length of the hypotenuse, then

$$\sin X = \frac{x}{\sqrt{x^2 + y^2}}; \text{ similarly, } \cos X = \frac{y}{\sqrt{x^2 + y^2}} \text{ and } \tan X = \frac{x}{y}. \quad \blacksquare$$

The reciprocal trigonometric ratios

The final three trigonometric ratios are called the **cosecant** (csc), **secant** (sec), and **cotangent** (cot) of an acute angle of a right triangle. These ratios are the reciprocals of the three ratios above.

Reciprocal trigonometric ratios

If θ represents either acute angle of a right triangle, then

$$\csc \theta = \frac{1}{\sin \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad \cot \theta = \frac{1}{\tan \theta}$$

Note The definitions of $\csc \theta$, $\sec \theta$, and $\cot \theta$ given here are equivalent to the following definitions:

$$\csc \theta = \frac{\text{hyp}}{\text{opp}}, \quad \sec \theta = \frac{\text{hyp}}{\text{adj}}, \quad \cot \theta = \frac{\text{adj}}{\text{opp}}$$

We could use these as the definitions for these three ratios.

It is worth stressing that the following pairs of ratios are reciprocals:

cosine and secant

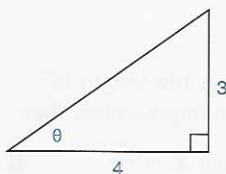
sine and cosecant

tangent and cotangent

This means that if we know one ratio, we can invert it to find the other ratio in the pair.

■ **Example 1-2 B**

1. $\sin A = \frac{2}{3}$. Find $\csc A$.
Invert $\frac{2}{3}$, giving $\csc A = \frac{3}{2}$
2. $\cot B = 5$. Find $\tan B$.
Invert $\frac{5}{1}$ to get $\tan B = \frac{1}{5}$
3. $\sec A = 1.6$. Find $\cos A$.
Invert 1.6 to get $\frac{1}{1.6}$, or 0.625, so $\cos A = 0.625$. ■

■ **Example 1-2 C**

1. Given the triangle in the figure, find the six trigonometric ratios of θ .

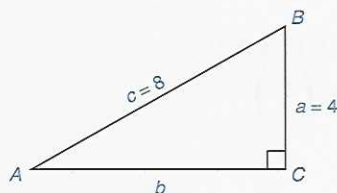
Since we are not told the length of the hypotenuse, we calculate it, using the Pythagorean theorem:

$$3^2 + 4^2 = 9 + 16 = 25, \text{ and } \sqrt{25} = 5$$

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{3}{5}, \quad \cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{4}{5}, \quad \tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{3}{4},$$

$$\csc \theta = \frac{1}{\sin \theta} = \frac{1}{\frac{3}{5}} = \frac{5}{3}, \quad \sec \theta = \frac{1}{\cos \theta} = \frac{1}{\frac{4}{5}} = \frac{5}{4},$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$



2. Find $\sin B$ and $\cos B$ for the triangle in the figure.

First we find the length of side b , using the Pythagorean theorem:

$$a^2 + b^2 = c^2 \text{ so } 16 + b^2 = 64, \text{ or } b^2 = 48, \text{ so } b = \sqrt{48} = \sqrt{(16)(3)} = 4\sqrt{3}.$$

$$\sin B = \frac{b}{c} = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$$

$$\cos B = \frac{a}{c} = \frac{4}{8} = \frac{1}{2}$$

The fundamental identity of trigonometry

Recall that an identity is an equation that is true for any valid replacement of the variable. The following trigonometric identity is so important that it is often called the **fundamental identity of trigonometry**.

The fundamental identity of trigonometry

If θ is either acute angle in a right triangle, then

$$\sin^2\theta + \cos^2\theta = 1$$

Concept

If we square both the sine and the cosine ratios for a given angle and add these quantities, we always get 1.

- Note**
1. In chapter 2 we see that θ does not have to be acute.
 2. The notation $\sin^2\theta$ means $(\sin \theta)^2$, and $\cos^2\theta$ means $(\cos \theta)^2$.

■ Example 1-2 D

1. Show that the fundamental identity of trigonometry applies to the results of part 2 of example 1-2 C.

Recall that in part 2 of example 1-2 C the angle is called B , not θ . Also,

$$\sin B = \frac{\sqrt{3}}{2} \text{ and } \cos B = \frac{1}{2}.$$

$$\begin{aligned} \sin^2 B + \cos^2 B &= (\sin B)^2 + (\cos B)^2 \\ &= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \\ &= \frac{3}{4} + \frac{1}{4} \\ &= 1 \end{aligned}$$

2. Prove that the fundamental identity of trigonometry applies to angle A in any right triangle ABC .

$$\begin{aligned}\sin^2 A + \cos^2 A &= (\sin A)^2 + (\cos A)^2 \\ &= \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 \\ &= \frac{a^2}{c^2} + \frac{b^2}{c^2} \\ &= \frac{a^2 + b^2}{c^2}\end{aligned}$$

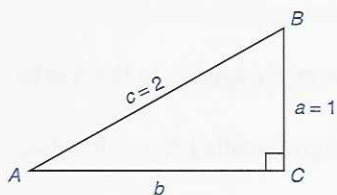
and, since $a^2 + b^2 = c^2$,

$$\begin{aligned}&= \frac{c^2}{c^2} \\ &= 1\end{aligned}$$

Finding other trigonometric ratios from a known ratio

If we know one of the trigonometric ratios of an angle, we can construct a right triangle with an angle for which that ratio is true. We can use this triangle to compute the other five trigonometric ratios.

■ Example 1-2 E



In right triangle ABC , $\sin A = \frac{1}{2}$. Draw a triangle in which $\sin A$ is $\frac{1}{2}$, and use this to compute the other five trigonometric ratios for angle A .

Since $\sin A$ is $\frac{a}{c}$, we see that $\frac{a}{c} = \frac{1}{2}$. Thus, a right triangle in which $a = 1$ and $c = 2$ would work. This is shown in the figure. We then compute b by the Pythagorean theorem,

$$\begin{aligned}a^2 + b^2 &= c^2 \\ 1^2 + b^2 &= 2^2 \\ b^2 &= 3 \\ b &= \sqrt{3}\end{aligned}$$

and then compute the other five ratios for angle A :

$$\begin{aligned}\cos A &= \frac{\sqrt{3}}{2}, \tan A = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3}, \sec A = \frac{2}{\sqrt{3}} \text{ or } \frac{2\sqrt{3}}{3}, \\ \csc A &= 2, \cot A = \sqrt{3}\end{aligned}$$

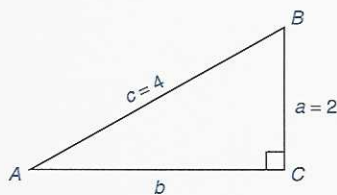


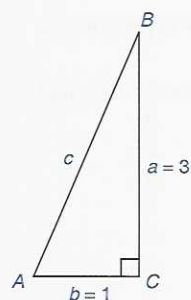
Figure 1-7

A logical question is whether any other triangles would work in example 1-2 E; the answer is yes. In fact, since $\frac{2}{4}$ reduces to $\frac{1}{2}$, we could use the triangle in figure 1-7. However, our values for the six ratios would not change.

For example, we find that $b = \sqrt{12}$, or $2\sqrt{3}$; thus, $\cos A = \frac{2\sqrt{3}}{4}$, which reduces to the same value, $\frac{\sqrt{3}}{2}$. The other ratios will also reduce to the same values we already obtained.

Similarly, we could start with any fraction that is equivalent to $\frac{1}{2}$. This means that we could use an unlimited number of triangles in such problems. This is true because of the properties of similar triangles, discussed in section 1-3.

■ Example 1-2 F



1. $\cot B = 3$. Draw a right triangle for which this is true and compute the other five trigonometric ratios for B .

We know that if $\cot B = 3$, then $\tan B = \frac{1}{3}$ (see example 1-2 B) and $\tan B = \frac{\text{opp}}{\text{adj}} = \frac{b}{a}$. Thus, a triangle in which $b = 1$ and $a = 3$ would work.

This is shown in the figure.

Using the Pythagorean theorem, we find c :

$$c^2 = a^2 + b^2 = 3^2 + 1^2 = 10$$

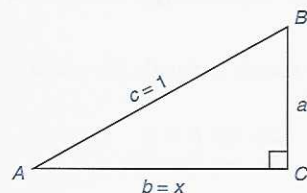
$$c = \sqrt{10}$$

With this we can now compute the four remaining ratios:

$$\sin B = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}, \quad \cos B = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10},$$

$$\sec B = \frac{\sqrt{10}}{3}, \quad \csc B = \sqrt{10}$$

2. In right triangle ABC , $\cos A = x$. Draw a right triangle for which this is true and find $\tan B$ in terms of x .



If we think of x as $\frac{x}{1}$ we see that $\cos A$ in the triangle shown in the figure is x .

From the Pythagorean theorem we find a :

$$c^2 = a^2 + b^2$$

$$1^2 = a^2 + x^2$$

$$1 - x^2 = a^2$$

$$\sqrt{1 - x^2} = a$$

Now we can find $\tan B$:

$$\tan B = \frac{b}{a} = \frac{x}{\sqrt{1 - x^2}}$$

Mastery points

Can you

- State the definitions of the six trigonometric ratios?
- Find the six trigonometric ratios of an angle in a right triangle when you know the lengths of the sides?
- State and use the fundamental identity of trigonometry?
- Find the remaining five ratios for an angle if given one of the six ratios for an angle in a right triangle?

Exercise 1-2

1. Draw a right triangle, label it in the standard way (using A, B, C, a, b, c), and use it to define the six trigonometric ratios for both acute angles.

In the following problems you are given parts of right triangle ABC ; use this information to compute the six trigonometric ratios for the angle specified.

a	b	c	Find ratios for this angle	a	b	c	Find ratios for this angle
2. 3	4		A	3. 3	4		B
4. 5		13	A	5. 1	3		B
6. 4	$\sqrt{10}$		A	7. 5	$\sqrt{7}$		B
8. 2		$\sqrt{13}$	A	9. 2		$\sqrt{5}$	B
10. 4		7	A	11. 12	13		B
12. 5	12		A	13. 6		10	A
14. 10		15	A	15. $\sqrt{3}$	4		A
16. x	y		A	17. x		z	B
18. 1	1		A	19. 1		2	B
20. 5		8	A	21. 9	5		B

22. You will learn in section 1-3 that if the sine ratio for an acute angle is more than 0.5, then the angle is larger than 30° . In a right triangle, $a = 3$ and $c = 5$; is angle B more or less than 30° ?

In the following problems you are given one of the trigonometric ratios for an angle. Use this to sketch a triangle for which the ratio is true, and then use this triangle to find the other five trigonometric ratios for that angle.

- | | | | |
|----------------------------|----------------------------|-----------------------------|--------------------|
| 23. $\sin A = \frac{4}{5}$ | 24. $\cos B = \frac{1}{4}$ | 25. $\cos A = 0.5$ | 26. $\tan B = 4$ |
| 27. $\sec A = 3$ | 28. $\cot B = 0.2$ | 29. $\sin A = \frac{5}{13}$ | 30. $\csc B = 1.6$ |
| 31. $\cos A = 0.9$ | | | |

Solve the following problems.

32. Show that the fundamental identity of trigonometry applies to the results of problems 27 and 29.
33.  Show that the fundamental identity of trigonometry applies to angle B in any right triangle ABC .
34. In right triangle ABC , $\sin A = x$. Sketch a triangle for which this is true and use it to find $\sin B$ in terms of the variable x . (Hint: $x = \frac{x}{1}$.)
35. In right triangle ABC , $\tan A = x$. Sketch a triangle for which this is true and use it to find $\sin B$ in terms of the variable x . (See the hint for problem 34.)
36. In right triangle ABC , $\sec A = x$. Sketch a triangle for which this is true and use it to find $\sec B$ in terms of the variable x .
37. In right triangle ABC , $\tan B = x$. Sketch a triangle for which this is true and use it to find $\tan A$ in terms of the variable x .

1-3 Angle measure and the values of the trigonometric ratios

Trigonometric ratios for equal angles are equal

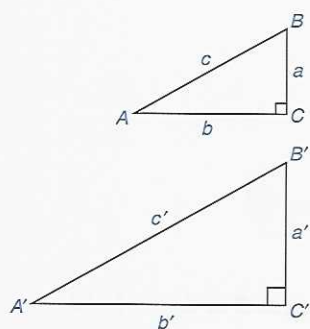


Figure 1-8

It is possible to have angles of the same measure in different right triangles. Angles A and A' in figure 1-8 are examples. The trigonometric ratios will have the same value for these angles. This is because these triangles are *similar*—that is, they have the same shape, but perhaps different sizes. It is a theorem of geometry that corresponding ratios in similar figures are equal. Thus, in figure 1-8, $\frac{a}{c} = \frac{a'}{c'}$, so $\sin A = \sin A'$. For the same reasons, the other five trigonometric ratios are also equal.

Note Read A' as "A-prime," B' as "B-prime," etc.

These facts mean that the values of the trigonometric ratios for an acute angle do not depend upon the particular right triangle in which it appears. *For an acute angle with a given measure, the values of the trigonometric ratios will always be the same.*

Values for angles of measure 30° , 45° , and 60°

We now find the trigonometric ratios for some special angles— 30° , 45° , and 60° . Figure 1-9 shows an equilateral triangle, which is a triangle in which all sides have equal length. We label this length c .

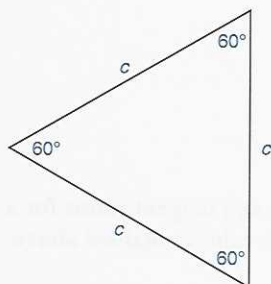


Figure 1-9

We construct the line AC , as shown in figure 1-10, which forms a right triangle with acute angles $A = 30^\circ$ and $B = 60^\circ$. The length a is half of c , so $a = \frac{c}{2}$. We find b next.

$$b^2 = c^2 - a^2 = c^2 - \left(\frac{c}{2}\right)^2 = \frac{4c^2}{4} - \frac{c^2}{4}$$

$$b^2 = \frac{3c^2}{4}$$

$$b = \frac{\sqrt{3}}{2}c$$

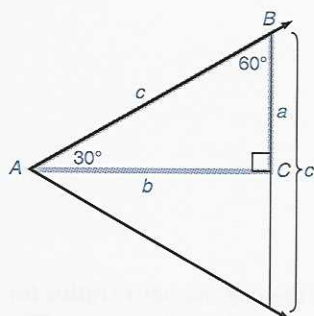


Figure 1-10

Angle A is a 30° angle, so we will write $\sin 30^\circ$ to mean the value of the sine ratio associated with an angle of measure 30° . Thus,

$$\begin{aligned}\sin 30^\circ &= \frac{a}{c} = \frac{\frac{c}{2}}{c} = \frac{1}{2} \\ \cos 30^\circ &= \frac{b}{c} = \frac{\frac{\sqrt{3}}{2}c}{c} = \frac{\sqrt{3}}{2} \\ \tan 30^\circ &= \frac{a}{b} = \frac{\frac{c}{2}}{\frac{\sqrt{3}}{2}c} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}\end{aligned}$$

The values for a 60° angle can be obtained from angle B in figure 1-10.

$$\begin{aligned}\sin 60^\circ &= \frac{b}{c} = \frac{\frac{\sqrt{3}}{2}c}{c} = \frac{\sqrt{3}}{2} \\ \cos 60^\circ &= \frac{a}{c} = \frac{\frac{c}{2}}{c} = \frac{1}{2} \\ \tan 60^\circ &= \frac{b}{a} = \frac{\frac{\sqrt{3}}{2}c}{\frac{c}{2}} = \sqrt{3}\end{aligned}$$

In the exercises we will compute the sine, cosine and tangent ratios for a 45° angle; these are shown in table 1-1, along with the values obtained above.

	Sine	Cosine	Tangent
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Table 1-1

General values

It is actually impossible to find the exact values of the trigonometric ratios for most angles. Tables of approximate values were calculated long ago. The earliest known table of trigonometric values, for the equivalent of the sine ratio, was created by Hipparchus of Nicaea about 150 B.C. In the second century A.D. Ptolemy constructed a table of values of the sine ratio for acute angles

in increments of one-quarter degree. Today we use calculators to approximate these values. When using a calculator it is important that the calculator be in **degree mode**. This means that the calculator is expecting the measure of the angle in decimal degrees. *Check your calculator's manual to make sure it is in degree mode.* This is usually done with a key marked **DRG** or simply **DEG**. “DRG” means degrees, radians, grads. We discuss radian measure in a later section. Grads, or grades, is the metric measure for an angle. There are 100 grads in a right angle. We will not use this measure in this text.

To select degree mode on the Texas Instruments TI-81 it is necessary to select “Deg” under the **MODE** feature. To do this, select **MODE**, darken in the “Deg” mode indicator (use the four cursor-moving “arrow” keys) and select **ENTER**.

■ Example 1-3 A

Find each value rounded to four decimal places.

1. $\sin 34.51^\circ$ 34.51 **sin** Display **0.5665500655**
 $\sin 34.51^\circ \approx 0.5666$ **TI-81** **SIN** 34.51 **ENTER**

2. $\tan 84.6^\circ$ 84.6 **tan** Display **10.57889499**
 $\tan 84.6^\circ \approx 10.5789$ **TI-81** **TAN** 84.6 **ENTER**

3. $\sec 33.5^\circ$
 Since there is no secant key on a calculator we use the fact that $\sec 33.5^\circ = \frac{1}{\cos 33.5^\circ}$. Compute $\cos 33.5^\circ$ and divide it into one; the **1/x** key is designed for this type of situation.

33.5 **cos** **1/x** Display **1.199204943**
TI-81 **(** **COS** 33.5 **)** **x⁻¹**
ENTER

Note that on the TI-81 the **1/x** key is the **x⁻¹** key.
 $\sec 33.5^\circ \approx 1.1992$

4. $\cot 87.23^\circ$ $\cot 87.23^\circ = \frac{1}{\tan 87.23^\circ}$, so use
 87.23 **tan** **1/x**
 Display **0.04838332158**
TI-81 **(** **TAN** 87.23 **)** **x⁻¹**
ENTER

$\cot 87.23^\circ \approx 0.0484$

5. $\cos 13^\circ 43'$

Recall that angles in the DMS system must be converted to decimal degrees. We show the calculation with and without special calculator keys. (See also example 1-1 A.)

No special keys: 13 $\boxed{+}$ 43 $\boxed{\div}$ 60 $\boxed{=}$ $\boxed{\cos}$

Display $\boxed{0.9714801855}$

Calculator A: 13 $\boxed{\circ ' ''}$ 43 $\boxed{\circ ' ''}$ $\boxed{\cos}$

Calculator B: 13.43 $\boxed{\rightarrow H}$ $\boxed{\cos}$

$\boxed{\text{TI-81}}$ $\boxed{\text{COS}}$ $\boxed{[}$ 13 $\boxed{+}$ 43 $\boxed{\div}$ 60 $\boxed{]}$
 $\boxed{\text{ENTER}}$

$\cos 13^\circ 43' \approx 0.9715$

Finding an angle from a known trigonometric ratio

It is important to be able to reverse the operations discussed above. For example, if θ is an acute angle and $\sin \theta = \frac{1}{2}$, what is θ ? We can see from table 1-1 that θ must be 30° . The calculator is programmed to solve this problem. This is done with the **inverse trigonometric ratios** called the inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$) and inverse tangent ($\tan^{-1}$) ratios. The superscript -1 does *not* indicate a reciprocal value in the way that, say, $2^{-1} = \frac{1}{2}$. We will study these ratios in more detail later. For now we illustrate how to find the acute angle whose sine, cosine, or tangent value is known.

For this, most calculators use the appropriate key ($\sin$, $\cos$, $\tan$), prefixed by another key such as $\boxed{\text{SHIFT}}$, $\boxed{2\text{nd}}$, $\boxed{\text{INV}}$ or $\boxed{\text{ARC}}$. The appropriate function is generally shown above the key itself. The *result is always an angle in decimal degrees* (when the calculator is in degree mode). We will show the necessary two keystrokes as one.

$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$
$\boxed{\sin}$	$\boxed{\cos}$	$\boxed{\tan}$

Example 1-3 B

Find θ in the following problems using the calculator. Assume θ is an acute angle. Round the answer to the nearest 0.01° .

1. $\cos \theta = 0.4602$

We need to calculate $\cos^{-1} 0.4602$.

0.4602 $\boxed{\cos^{-1}}$ Display $\boxed{62.59998611}$

$\boxed{\text{TI-81}}$ $\boxed{\text{COS}^{-1}}$.4602 $\boxed{\text{ENTER}}$

$\theta \approx 62.60^\circ$

2. $\tan \theta = 1.2231$

Calculate $\tan^{-1} 1.2231$.

1.2231 $\boxed{\tan^{-1}}$ Display $\boxed{50.73075144}$

$\boxed{\text{TI-81}}$ $\boxed{\text{TAN}^{-1}}$ 1.2231 $\boxed{\text{ENTER}}$

$\theta \approx 50.73^\circ$

3. $\csc \theta = 1.0551$

We use the fact that if $\csc \theta = 1.0551$, then $\sin \theta = \frac{1}{1.0551}$. Thus we

compute $\sin^{-1}\left(\frac{1}{1.0551}\right)$. Use the $\boxed{1/x}$ key. (On the TI-81 the $\boxed{1/x}$ key is the $\boxed{x^{-1}}$ key.)

1.0551 $\boxed{1/x}$ $\boxed{\sin^{-1}}$ Display $\boxed{71.40162609}$

$\boxed{\text{TI-81}}$ $\boxed{\text{SIN}^{-1}}$ 1.0551 $\boxed{x^{-1}}$ $\boxed{\text{ENTER}}$

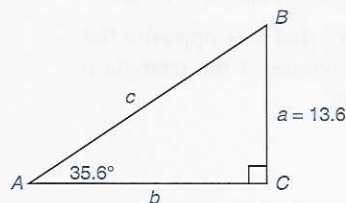
$\theta \approx 71.40^\circ$

Solving right triangles

One application of trigonometry that occurs in many situations is *solving right triangles*—this means *discovering the lengths of all sides and the measures of all angles* of the triangle. We will round the values we compute to the same number of decimal places as the given data.

These types of situations fall into two categories, ones in which we know one side and one acute angle and others in which we know two sides and no angles. Each category is illustrated in example 1-3 C.

■ Example 1-3 C



Solve the following right triangles.

1. $A = 35.6^\circ$, $a = 13.6$ (one side, one angle)

To solve this triangle we need to find the lengths of sides b and c and the measure of angle B . Since angle C is always 90° , angles A and B total 90° . Thus, angle B is $90^\circ - 35.6^\circ = 54.4^\circ$. We now note that

$$\sin A = \frac{a}{c}, \text{ so that}$$

$$\sin 35.6^\circ = \frac{13.6}{c}$$

$$c \sin 35.6^\circ = 13.6 \quad \text{Multiply each member by } c$$

$$c = \frac{13.6}{\sin 35.6^\circ} \quad \text{Divide each member by } \sin 35.6^\circ$$

$$c \approx 23.4 \quad \boxed{\text{CS 1}}$$

Now we find b by noting that $\tan A = \frac{a}{b}$.

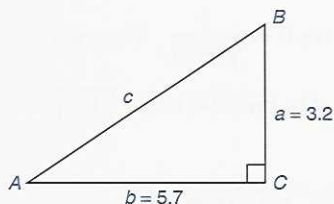
$$\tan 35.6^\circ = \frac{13.6}{b}$$

$$b \tan 35.6^\circ = 13.6 \quad \text{Multiply each member by } b$$

$$b = \frac{13.6}{\tan 35.6^\circ} \quad \text{Divide each member by } \tan 35.6^\circ$$

$$b \approx 19.0 \quad \boxed{\text{CS 2}}$$

Since we know the lengths of all sides and all angles we have solved the triangle. To summarize, $a = 13.6$, $b \approx 19.0$, $c \approx 23.4$, $A = 35.6^\circ$, $B = 54.4^\circ$, $C = 90^\circ$.



2. $a = 3.2$, $b = 5.7$ (two sides)

We can find the length of side c by the Pythagorean theorem.

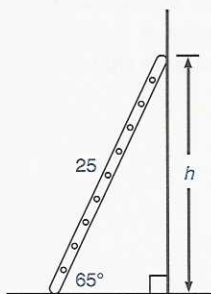
$$\begin{aligned} c^2 &= a^2 + b^2 \\ c^2 &= 3.2^2 + 5.7^2 \\ c^2 &= 42.73 \\ c &= \sqrt{42.73} \\ c &\approx 6.5 \end{aligned}$$

We can find angle A by noting that $\tan A = \frac{a}{b}$.

$$\begin{aligned} \tan A &= \frac{3.2}{5.7} \\ A &= \tan^{-1}\left(\frac{3.2}{5.7}\right) \quad \boxed{\text{CS 3}} \\ A &\approx 29.3^\circ \end{aligned}$$

We know that the sum of the measures of A and B is 90° , so $B = 90^\circ - 29.3^\circ = 60.7^\circ$. This completes the process of finding the lengths of all sides and measures of all angles. Thus, $a = 3.2$, $b = 5.7$, $c \approx 6.5$, $A \approx 29.3^\circ$, $B \approx 60.7^\circ$, $C = 90^\circ$.

3. A tag on a 25-foot ladder states that, for safety reasons, the angle that the ladder makes with the ground should not exceed 65° . How high can the ladder reach without exceeding this angle, to the nearest 0.1 feet?



We need to find h in the figure. If we observe that h is opposite the known angle and that the length of the hypotenuse of the triangle is known, we see that we can use the sine ratio.

$$\begin{aligned} \sin 65^\circ &= \frac{h}{25} \\ 25 \sin 65^\circ &= h && \text{Multiply each member by 25} \\ 22.7 &\approx h && \boxed{\text{CS 4}} \end{aligned}$$

The ladder can reach a height of approximately 22.7 feet without exceeding a 65° angle with the ground. ■

Calculator steps

1. (A) 13.6 $\div$ 35.6 $\sin$ $=$ Display 23.36276130
 (P) 13.6 ENTER 35.6 $\sin$ $\div$
TI-81 13.6 $\div$ SIN 35.6 ENTER
2. (A) 13.6 $\div$ 35.6 $\tan$ $=$ Display 18.99627899
 (P) 13.6 ENTER 35.6 $\tan$ $\div$
TI-81 13.6 $\div$ TAN 35.6 ENTER

3. (A) $3.2 \div 5.7 = \tan^{-1}$ Display 29.31000707

(P) $3.2 \text{ ENTER } 5.7 \div \tan^{-1}$

TI-81 $\text{TAN}^{-1} (3.2 \div 5.7) \text{ ENTER}$

4. (A) $25 \times 65 \sin =$ Display 22.65769468

(P) $25 \text{ ENTER } 65 \sin \times$

TI-81 $25 \times \text{SIN } 65 \text{ ENTER}$

Mastery points

Can you

- Compute the value of the trigonometric ratios for a given acute angle, using a calculator?
- Compute the value of an acute angle, given the value of a trigonometric ratio, using a calculator?
- Solve a right triangle when given one side and one acute angle?
- Solve a right triangle when given two sides?

Exercise 1-3

Use a calculator to find four-decimal-place approximations for the following.

- | | | | |
|-------------------------|-------------------------|-------------------------|-------------------------|
| 1. $\sin 31.28^\circ$ | 2. $\cos 85.23^\circ$ | 3. $\tan 11.95^\circ$ | 4. $\sec 40.08^\circ$ |
| 5. $\cot 28.87^\circ$ | 6. $\csc 5.15^\circ$ | 7. $\sin 40.28^\circ$ | 8. $\tan 76.23^\circ$ |
| 9. $\sec 66.47^\circ$ | 10. $\sin 35.56^\circ$ | 11. $\sin 78.33^\circ$ | 12. $\cos 17.45^\circ$ |
| 13. $\sin 78^\circ 33'$ | 14. $\cos 17^\circ 45'$ | 15. $\cos 85^\circ 28'$ | 16. $\tan 40^\circ 41'$ |
| 17. $\tan 35^\circ 8'$ | 18. $\cos 23^\circ 24'$ | 19. $\cos 56^\circ 24'$ | 20. $\cot 13^\circ 3'$ |
| 21. $\sin 48^\circ 8'$ | 22. $\tan 33^\circ 38'$ | 23. $\sec 86^\circ 22'$ | |

24. A surveyor needs to compute R in the following formula as part of finding the area of the segment of a circle:

$$R = \frac{LC}{2 \sin I}. \text{ Find } R \text{ to three decimal places if } LC = 425.0 \text{ feet and } I = 13.2^\circ.$$

25. Compute R using the formula of the previous problem if $LC = 611.1$ meters and $I = 18^\circ 20'$. Round the answer to two decimal places.

26. In the mathematical modeling of an aerodynamics problem the following equation arises:

$$y = x \cos A \cos B - x^2 \cos A \sin B - x^3 \sin A$$

Compute y to two decimal places if $x = 2.5$, $A = 31^\circ$, and $B = 17^\circ$.

27. Compute y to two decimal places using the formula of problem 26 if $x = 1.2$, $A = 10^\circ$, and $B = 15^\circ$.
28. The average power in an AC circuit is given by the formula $P = VI \cos \theta$. Compute P (in watts) if $V = 120$ volts, $I = 2.3$ amperes, and $\theta = 45^\circ$, to the nearest 0.1 watt.

29. Compute P using the formula of problem 28 if $V = 42$ volts, $I = 25$ amperes, and $\theta = 45^\circ$, to the nearest 0.1 watt.
30. A formula that relates the distance across the flats of a piece of hexagonal stock in relation to the distance across the corners is $f = 2r \cos \theta$. A machinist needs to compute f for a piece of stock in which $r = 28$ millimeters (mm) and $\theta = 30^\circ$. Compute f to the nearest 0.1 mm.
31. Using the formula of problem 30 find r if $f = 21.4$ inches and $\theta = 25^\circ$.
32. Using the formula of problem 30 find θ to the nearest 0.1 if $f = 36.8$ millimeters and $r = 24.0$.

Find the unknown acute angle θ to the nearest 0.01° .

- | | | |
|----------------------------|---------------------------------------|---------------------------------------|
| 34. $\sin \theta = 0.3746$ | 35. $\sin \theta = 0.8007$ | 36. $\cos \theta = 0.1028$ |
| 37. $\tan \theta = 1.8807$ | 38. $\sin \theta = 0.9484$ | 39. $\cos \theta = 0.8515$ |
| 40. $\tan \theta = 1.0014$ | 41. $\sin \theta = \frac{35.9}{68.3}$ | 42. $\cos \theta = \frac{8.25}{12.5}$ |
| 43. $\tan \theta = 2$ | 44. $\csc \theta = 1.1243$ | 45. $\sec \theta = 4.8097$ |
| 46. $\cot \theta = 2.5$ | 47. $\sec \theta = \frac{6.45}{2.35}$ | 48. $\csc \theta = \sqrt{10.8}$ |

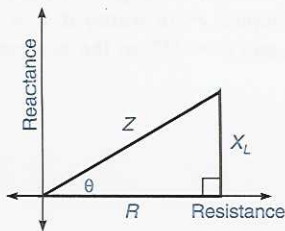
In the following problems you are given one side and one angle of a right triangle. Solve the triangle. Round all answers to the same number of decimal places as the data.

- | | | |
|--------------------------------|---------------------------------|--------------------------------|
| 49. $a = 15.2, B = 38.3^\circ$ | 50. $a = 12.6, B = 17.9^\circ$ | 51. $a = 11.1, A = 13.7^\circ$ |
| 52. $a = 5.25, A = 70.3^\circ$ | 53. $b = 0.672, A = 29.4^\circ$ | 54. $b = 15.2, A = 81.3^\circ$ |
| 55. $b = 21.8, B = 78.0^\circ$ | 56. $b = 2.14, B = 50.4^\circ$ | 57. $c = 10.0, A = 15.0^\circ$ |
| 58. $c = 3.45, A = 46.2^\circ$ | 59. $c = 122, B = 65.5^\circ$ | 60. $c = 31.5, B = 62.0^\circ$ |

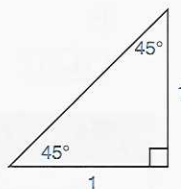
In the following problems you are given two sides of a right triangle. Solve the triangle. Round all lengths to the same number of decimal places as the data and all angles to the nearest 0.1° .

- | | | |
|---------------------------|--------------------------|--------------------------|
| 61. $a = 13.1, b = 15.6$ | 62. $a = 5.67, b = 8.91$ | 63. $a = 0.22, b = 1.34$ |
| 64. $a = 2.82, b = 1.09$ | 65. $a = 17.8, c = 25.2$ | 66. $a = 311, c = 561$ |
| 67. $b = 51.3, c = 111.0$ | 68. $b = 4.55, c = 5.66$ | 69. $a = 12.0, c = 13.0$ |
| 70. $a = 33.1, c = 41.0$ | 71. $b = 84.0, c = 90.1$ | |

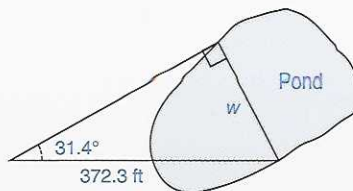
72. The figure illustrates an impedance diagram used in electronics theory. If Z (impedance) = 10.35 ohms and X_L (inductive reactance) = 4.24 ohms, find θ (phase angle) to the nearest degree and R (resistance) to the nearest 0.01 ohm.



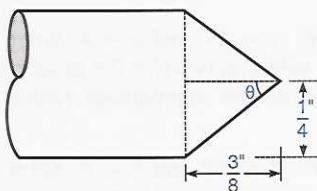
33. Find the exact values for the sine, cosine, and tangent ratios for an angle of measure 45° by proceeding in the following manner. Draw an *isosceles* right triangle—a right triangle in which the two legs have the same length. Label this length one. Observe that the two acute angles must be 45° . Now find the length of the hypotenuse and use the definitions of the trigonometric ratios to find the desired values.



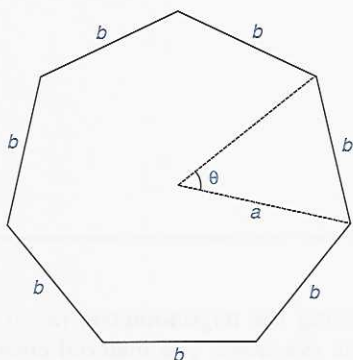
73. Use the impedance diagram of problem 72 to find Z if $\theta = 24.2^\circ$ and $X_L = 22.6$ ohms.
74. The diagram illustrates the measurements a surveyor made to find the width w of a pond; compute the width to the nearest foot.



75. The diagram illustrates the tip of a threading tool; find angle θ to the nearest degree.



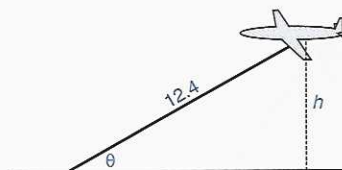
76. The diagram illustrates a piece of wood that is being mass produced to form the bottom of a planter. Find dimension a in the figure to the nearest 0.01 inch if $b = 8\frac{1}{4}$ inches. Note: angle θ is $\frac{360^\circ}{7}$.



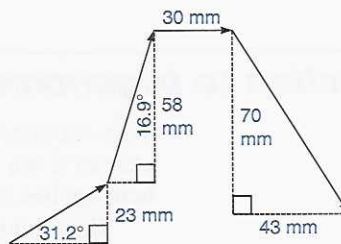
77. A formula found in electronics is $E = \frac{P}{I \cos \theta}$, where E is voltage, P is power, I is current, and θ is phase angle. Find E (in volts) if $P = 45.0$ watts, $I = 2.5$ amperes, and $\theta = 15^\circ$. Round the answer to the nearest 0.1 volt.
78. The diagram illustrates the wind triangle problem in air navigation. A plane has an airspeed of 155 mph and heading of due north. It is flying in a wind from the west with a speed of 30 mph. Find the ground speed S and the ground direction θ , each to the nearest unit.



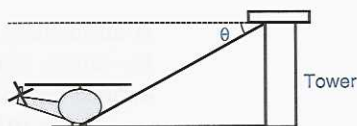
79. An **angle of elevation** is an angle formed by one horizontal ray and another ray that is above the horizontal. Angle θ in the diagram is the angle of elevation to an aircraft that radar shows has a slant distance of 12.4 miles from the radar site. If θ is 30.1° , find the elevation h of the aircraft, to the nearest 100 feet. Remember that 1 mile = 5,280 feet.



80. If it is known that an aircraft is flying at 28,500 feet and the angle of elevation of a radar beam tracking the aircraft is 8.2° , what is the slant distance d from the radar to the aircraft, to the nearest 100 feet?
81. The diagram illustrates the path of a laser beam on an optics table. Compute the total distance traveled by the beam to the nearest millimeter.

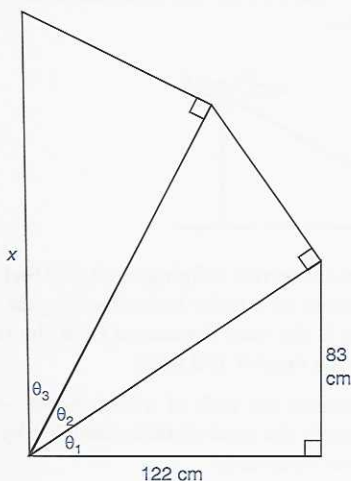


82. An **angle of depression** is an angle formed by one horizontal ray and another ray that is below the horizontal. Angle θ in the figure is the angle of depression formed by the line of sight of an observer in an airport control tower looking at a helicopter on the ground. If θ is 17.2° and the tower is 257 feet high, how far is the aircraft from the base of the tower, to the nearest foot.



83. If an aircraft is 1.23 miles from the foot of the tower in problem 82, what is θ , to the nearest 0.1° ? (Remember, 1 mile = 5,280 ft.)

84. The diagram is a top view of a portion of a spiral staircase that an architect has designed. If $\theta_1 = \theta_2 = \theta_3$, find the length x to one decimal place. (Caution: Carry out your calculations to as many digits as practical to avoid an accumulation of errors.)



85. If the architect of problem 84 revises the plans so that $\theta_2 = \theta_1 + 5^\circ$ and $\theta_3 = \theta_2 + 5^\circ$, find x to one decimal place.
86. In right triangle ABC , $A = 45^\circ$ and $b = 4$. Solve this triangle using exact values only. (Use the exact values for $\sin 45^\circ$ etc., and do not approximate radicals as decimals.)
87. In right triangle ABC , $B = 60^\circ$ and $b = 8$. Solve this triangle using exact values only. (Use the exact values for $\sin 60^\circ$ etc., and do not approximate radicals as decimals.)

1-4 Introduction to trigonometric equations

This section introduces equations involving the trigonometric ratios. In chapter 2 we learn about the trigonometric *functions*. The material covered here applies to these functions as well.

Equations can be categorized as identities and conditional equations. An **identity** is an equation that is true for every allowed value of its variable (or variables). For example,

$$2(x + 3) = 2x + 6$$

is an identity, since the left member and right member of the equation represent the same value, regardless of the value of x . Similarly

$$\frac{3x^2}{3x} = x$$

is an identity; the left member equals the right member for every value of x for which both members are defined. Observe that the left member is not defined for the value 0, so the identity is true for all real values *except* zero.

A **conditional equation** is an equation that is true only for some, but not all, values that may replace the variable. For example,

$$6x = 12$$

is true if and only if x is replaced by 2, and

$$x^2 = 9$$

is true if and only if x is replaced by 3 or -3 .

Identities

We have seen the reciprocal ratio identities

$$\csc \theta = \frac{1}{\sin \theta}, \sec \theta = \frac{1}{\cos \theta}, \cot \theta = \frac{1}{\tan \theta}$$

Similarly,

$$\sin \theta = \frac{1}{\csc \theta}, \cos \theta = \frac{1}{\sec \theta}, \tan \theta = \frac{1}{\cot \theta}$$

are identities.

Knowing these identities permits us to simplify certain trigonometric expressions.

■ Example 1-4 A

Simplify each trigonometric expression.

1. $\csc \theta \sin \theta$

$$\begin{aligned} \csc \theta \sin \theta &= \frac{1}{\sin \theta} \cdot \sin \theta \\ &= \frac{1}{1} \end{aligned} \quad \csc \theta = \frac{1}{\sin \theta}$$

Thus, $\csc \theta \sin \theta = 1$.

2. $\frac{1 - \csc \theta}{\csc \theta}$

$$\begin{aligned} \frac{1 - \csc \theta}{\csc \theta} &= \frac{1}{\csc \theta} - \frac{\csc \theta}{\csc \theta} \\ &= \sin \theta - 1 \end{aligned} \quad \begin{aligned} \frac{a - b}{c} &= \frac{a}{c} - \frac{b}{c} \\ \sin \theta &= \frac{1}{\csc \theta} \end{aligned}$$

Thus, $\frac{1 - \csc \theta}{\csc \theta} = \sin \theta - 1$. The right member is considered simpler because it is not a rational expression. ■

Two more useful identities are

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \cot \theta = \frac{\cos \theta}{\sin \theta}$$

To see why the first is true consider angle A in figure 1-11. Note that

$$\tan A = \frac{a}{b}, \text{ and } \frac{\sin A}{\cos A} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{c} \cdot \frac{c}{b} = \frac{a}{b} \text{ also. It is left as an exercise to}$$

show that the same is true for angle B and that the second identity is true.

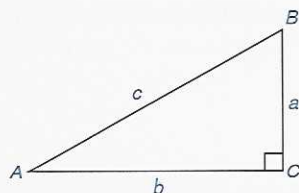


Figure 1-11

■ **Example 1-4 B**

Simplify each expression.

1. $\cot \alpha (\sin \alpha - \tan \alpha)$

$$\cot \alpha (\sin \alpha - \tan \alpha)$$

$$\cot \alpha \sin \alpha - \cot \alpha \tan \alpha$$

$$\frac{\cos \alpha}{\sin \alpha} \sin \alpha - \frac{1}{\tan \alpha} \tan \alpha$$

$$\cos \alpha - 1$$

$$a(b - c) = ab - ac$$

$$\cot \alpha = \frac{\cos \alpha}{\sin \alpha}; \cot \alpha = \frac{1}{\tan \alpha}$$

Note We replaced $\cot \alpha$ by $\frac{\cos \alpha}{\sin \alpha}$ in one term and by $\frac{1}{\tan \alpha}$ in another term. We use whichever identity better suits the rest of the term.

$$\text{Thus, } \cot \alpha (\sin \alpha - \tan \alpha) = \cos \alpha - 1.$$

2. $\sec \theta (\cos \theta - \cot \theta)$

$$\sec \theta (\cos \theta - \cot \theta)$$

$$\sec \theta \cos \theta - \sec \theta \cot \theta$$

$$\frac{1}{\cos \theta} \cos \theta - \frac{1}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta}$$

$$1 - \frac{1}{\sin \theta}$$

$$1 - \csc \theta$$

$$\text{Thus } \sec \theta (\cos \theta - \cot \theta) = 1 - \csc \theta.$$

We also use the fundamental identity of trigonometry (section 1-2):

$$\sin^2 \theta + \cos^2 \theta = 1$$

■ **Example 1-4 C**

1. Simplify the expression $\frac{1}{\sec^2 \theta} + \frac{1}{\csc^2 \theta}$.

$$\frac{1}{\sec^2 \theta} + \frac{1}{\csc^2 \theta}$$

$$\frac{1}{(\sec \theta)^2} + \frac{1}{(\csc \theta)^2}$$

$$\left(\frac{1}{\sec \theta}\right)^2 + \left(\frac{1}{\csc \theta}\right)^2$$

$$(\cos \theta)^2 + (\sin \theta)^2$$

$$1$$

 $\sec^2 \theta$ means $(\sec \theta)^2$, $\csc^2 \theta$ means $(\csc \theta)^2$

$$\frac{1}{x^2} = \left(\frac{1}{x}\right)^2$$

$$\frac{1}{\sec \theta} = \cos \theta, \frac{1}{\csc \theta} = \sin \theta$$

The fundamental identity of trigonometry

$$\text{Thus, } \frac{1}{\sec^2 \theta} + \frac{1}{\csc^2 \theta} = 1.$$

2. Simplify the expression $1 - \sin^2 \beta$.

$$1 - \sin^2 \beta$$

$$(\sin^2 \beta + \cos^2 \beta) - \sin^2 \beta$$

$$\cos^2 \beta$$

$$\text{Thus, } 1 - \sin^2 \beta = \cos^2 \beta.$$

$$1 = \sin^2 \beta + \cos^2 \beta$$

3. Verify the fundamental identity of trigonometry for $\theta = 60^\circ$.

$$\begin{aligned}\sin^2 60^\circ + \cos^2 60^\circ &= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \sin 60^\circ = \frac{\sqrt{3}}{2}, \cos 60^\circ = \frac{1}{2} \\ &= \frac{3}{4} + \frac{1}{4} \\ &= 1\end{aligned}$$

Conditional trigonometric equations

Section 1-3 showed that if we have an equation like $\tan \theta = 4$, then one value of θ is $\tan^{-1} 4 \approx 76^\circ$. This is an example of a simple conditional trigonometric equation. Solutions to such equations rely on the inverse sine, cosine, and tangent functions as illustrated in section 1-3.

■ Example 1-4 D

Solve the following conditional equations to the nearest 0.1° .

1. $2 \sin x = 1$

$$\begin{aligned}\sin x &= \frac{1}{2} \\ x &= 30^\circ\end{aligned}$$

Divide both members by 2
Table 1-1, section 1-3

2. $5 \sin x = 3$

$$\begin{aligned}\sin x &= \frac{3}{5} \\ x &= \sin^{-1} \frac{3}{5} \\ x &\approx 36.9^\circ\end{aligned}$$

Divide both members by 5

CS 1

3. $\sin 5x = 0.8$

$$\begin{aligned}5x &= \sin^{-1} 0.8 \\ x &= \frac{\sin^{-1} 0.8}{5} \text{ or } \frac{1}{5} \sin^{-1} 0.8 \\ x &\approx 10.6^\circ\end{aligned}$$

CS 2

4. $4 \cos 3\alpha = 3$

$$\begin{aligned}\cos 3\alpha &= \frac{3}{4} \\ 3\alpha &= \cos^{-1} \frac{3}{4} \\ \alpha &= \frac{\cos^{-1} \frac{3}{4}}{3} \text{ or } \frac{1}{3} \cos^{-1} \frac{3}{4} \\ \alpha &\approx 13.8^\circ\end{aligned}$$

Divide both members by 4

Divide both members by 3

CS 3

Observe in example 1-4 D when it is proper to divide, and when it is not. An expression like

$$10 \sin x$$

indicates the product of 10 and $\sin x$. Thus, for example, the expression

$$\frac{10 \sin x}{5} = \frac{10}{5} \sin x = 2 \sin x$$

However,

$$\frac{\sin 10x}{5}$$

would not reduce. This is because the 5 is not dividing a product. The expression “ $\sin 10x$ ” does not represent multiplication.

An expression like $\sin \frac{10x}{5}$ can be simplified to $\sin 2x$, since the 5 is dividing the product $10x$.

Calculator steps

- (A) $3 \div 5 = \sin^{-1}$

(P) $3 \text{ ENTER } 5 \div \sin^{-1}$

$\text{TI-81} \quad \text{SIN}^{-1} \quad (\quad 3 \div 5 \quad) \quad \text{ENTER}$
- (A) $.8 \sin^{-1} \div 5 =$

(P) $.8 \sin^{-1} 5 \div$

$\text{TI-81} \quad \text{SIN}^{-1} \quad .8 \div 5 \quad \text{ENTER}$
- (A) $3 \div 4 = \cos^{-1} \div 3 =$

(P) $3 \text{ ENTER } 4 \div \cos^{-1} 3 \div$

$\text{TI-81} \quad \text{COS}^{-1} \quad (\quad 3 \div 4 \quad) \quad \div 3 \quad \text{ENTER}$

Mastery points

Can you

- Simplify simple trigonometric expressions?
- Solve simple equations involving the trigonometric ratios?

Exercise 1-4

Simplify the following trigonometric expressions.

- $\tan \theta \cot \theta$
- $\sec \theta \cos \theta$
- $\cos \theta (1 - \sec \theta)$
- $\cot \alpha (\tan \alpha + \sin \alpha)$
- $\sec \theta (\cot \theta + \cos \theta - 1)$
- $\frac{\cos \theta - 1}{\sin \theta}$
- $\frac{\cos \alpha - \sin \alpha}{\cos \alpha}$
- $\frac{\sin \theta + \cos \theta - 2}{\cos \theta}$
- $1 - \cos^2 \theta$
- $\cos \theta \cos \theta + \sin^2 \theta$
- $\cos \beta (\sec \beta - \cos \beta)$
- $-\sin \theta (\sin \theta - \csc \theta)$
- $(\cos \theta + \sin \theta)(\cos \theta - \sin \theta) + 2 \sin^2 \theta$
- Verify by computation that the fundamental identity is true when $\theta = 30^\circ$.
- Using approximate values check the fundamental identity when
a. $\theta = 16^\circ 50'$ b. $\theta = 50^\circ$
- Use the two identities $\cot \theta = \frac{1}{\tan \theta}$ and $\tan \theta = \frac{\sin \theta}{\cos \theta}$ to show that $\cot \theta = \frac{\cos \theta}{\sin \theta}$.
- Use approximate values to show that $\tan \theta = \frac{\sin \theta}{\cos \theta}$ when $\theta = 32^\circ 40'$.
- Show that $\tan \theta = \frac{\sin \theta}{\cos \theta}$ when $\theta = 60^\circ$ (use values from table 1-1).

Use identities to show that the left side of each equation can be simplified to become the right side.

19. $\tan x(\cot x + \csc x) = 1 + \sec x$

21. $\sin \beta(\cot \beta - \csc \beta + \sin \beta) = \cos \beta - \cos^2 \beta$

23. $\cos \alpha(\csc \alpha + \sec \alpha) = \cot \alpha + 1$

20. $\csc \alpha(\cos \alpha - \sin \alpha) = \cot \alpha - 1$

22. $\frac{\sin x - \cos x}{\sin x} = 1 - \cot x$

24. $\tan \beta(\cot \beta - \cos \beta) = 1 - \sin \beta$

Solve the following conditional equations to the nearest 0.1° .

25. $2 \cos x = 1$

26. $\sqrt{3} \tan x = 1$

27. $2 \sin x = \sqrt{3}$

28. $\sqrt{2} \cos x = 1$

29. $5 \sin x = 1$

30. $3 \sin x = 2$

31. $2 \tan x = 9$

32. $4 \cot x = 3$

33. $\frac{\sin x}{3} = \frac{2}{11}$

34. $\frac{\csc x}{3} = 2$

35. $\sin 3x = \frac{1}{2}$

36. $\cos 2x = \frac{1}{2}$

37. $\tan 2x = \sqrt{3}$

38. $\sin 2x = 0.8$

39. $\csc 3x = 3$

40. $4 \sin 2x = 3$

41. $2 \cos 4x = 1$

42. $3 \sin 2x = 0.75$

43. $2 \tan 3x = 8$

44. $\frac{1}{2} \sin 3x = \frac{1}{4}$

45. Show that in any right triangle ABC $\tan \theta = \frac{\sin \theta}{\cos \theta}$ is true when angle θ is angle B .

Chapter 1 summary

- A degree can be divided into smaller parts in one of two ways—into minutes and seconds or into decimal parts.
- The sum of the measures of the three angles of any triangle is 180° .
- A right triangle is a triangle in which one of the angles is a right (90°) angle. The side of a right triangle that is opposite the right angle is called the hypotenuse, and the sides that form the right angle are called the legs.
- The standard method of labeling right triangles is to label the right angle C and the two acute angles A and B . The sides are always labeled a and b , with a opposite angle A and b opposite angle B . The hypotenuse is always labeled c .
- The Pythagorean theorem** Given a right triangle with legs a and b and hypotenuse c , then $a^2 + b^2 = c^2$.
- If $a^2 + b^2 = c^2$ in a triangle, then that triangle is a right triangle, and the right angle is opposite side c .
- If θ is one of the two acute angles in a right triangle, then

$$\sin \theta = \frac{\text{length of leg opposite } \theta}{\text{length of hypotenuse}}$$

$$\cos \theta = \frac{\text{length of leg adjacent to } \theta}{\text{length of hypotenuse}}$$

$$\tan \theta = \frac{\text{length of leg opposite } \theta}{\text{length of leg adjacent to } \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}, \csc \theta = \frac{1}{\sin \theta}, \cot \theta = \frac{1}{\tan \theta}$$

- Fundamental identity of trigonometry** If θ is either acute angle in a right triangle, then

$$\sin^2 \theta + \cos^2 \theta = 1$$

- The following pairs of ratios are reciprocals:
 - cosine and secant
 - sine and cosecant
 - tangent and cotangent
- If we know one of the trigonometric ratios of an angle, we can find a triangle for which that ratio is true. We can use this triangle to compute the other five trigonometric ratios for that angle.
- The value of a trigonometric ratio depends only on the measure of the angle and not on the right triangle in which it appears.
- A right triangle is said to be solved once the lengths of its three sides and the measures of its two acute angles are known. To solve a right triangle, we must already know at least one side and either another side or one acute angle.
- Several important trigonometric identities** are

$$\sin \theta = \frac{1}{\csc \theta}, \cos \theta = \frac{1}{\sec \theta}, \tan \theta = \frac{1}{\cot \theta},$$

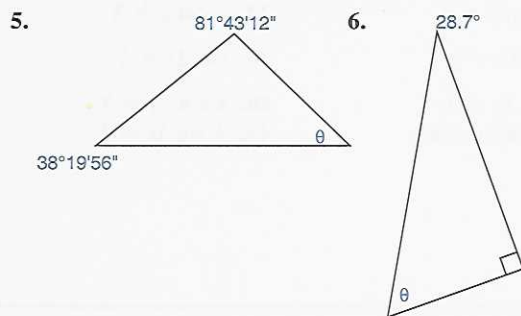
$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \cot \theta = \frac{\cos \theta}{\sin \theta}$$

Chapter 1 review

[1–1] Convert each angle to its measure in decimal degrees. Round your answer to the nearest 0.01° . Also, state whether each angle is acute, obtuse, or right.

1. $17^\circ 34' 17''$ 2. $84^\circ 9'$ 3. $125^\circ 37'$ 4. $39^\circ 45' 43''$

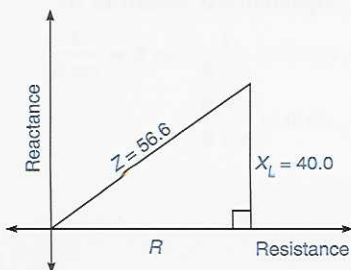
Find the measure of the missing angle, θ .



In the following problems two of the three sides of a right triangle are given. Use the Pythagorean theorem to calculate the length of the missing side; give your answer in exact form and in approximate form, rounded to one decimal place if necessary.

a	b	c
7. 8	12	?
8. 9	?	26
9. ?	8	13

10. A flagpole is 46 feet tall and is supported by a wire attached at the top of the pole and to the ground 25 feet from the base of the pole. How long is the wire (to the nearest foot)?
11. The diagram is called an impedance diagram and is used in computing total impedance in a certain electronics circuit. It shows that inductive reactance X_L is 40.0 ohms and that impedance Z is 56.6 ohms. Calculate the resistance R to the nearest tenth of an ohm.



[1–2] In the following problems you are given parts of right triangle ABC . Use this information to compute the six trigonometric ratios for the angle specified.

a	b	c	Find ratios for this angle
12. 3	7		A
13. 5		15	B
14. 2	$\sqrt{10}$		A

15. In right triangle ABC , $\sin A = \frac{4}{5}$. Draw a triangle for which this is true and use it to find $\sin B$.
16. In right triangle ABC , $\tan A = 5$. Draw a triangle for which this is true and use it to find $\cos B$.
17. In right triangle ABC , $\sec A = 3$. Draw a triangle for which this is true and use it to find $\tan B$.
18. In right triangle ABC , $\tan B = 0.8$. Draw a triangle for which this is true and use it to find $\sec A$.

[1–3] Use an electronic calculator to find four-decimal-place approximations for the following.

19. $\sin 15.5^\circ$ 20. $\sec 68.2^\circ$ 21. $\tan 17.9^\circ$
 22. $\sin 40.8^\circ$ 23. $\tan 16.3^\circ$ 24. $\sec 25.7^\circ$
 25. $\cot 31^\circ 20'$ 26. $\sec 85^\circ 40'$ 27. $\tan 31^\circ 30'$
 28. $\csc 43^\circ 10'$ 29. $\tan 63^\circ 30'$ 30. $\sec 86^\circ 40'$

31. Find four-digit approximations for all six trigonometric ratios for
 a. 53.20° b. $53^\circ 20'$

32. A surveyor needs to compute R in the following formula as part of finding the area of the segment of a circle.

$$R = \frac{LC}{2 \sin I}. \text{ Find } R \text{ to three decimal places if } LC = 315.2 \text{ meters and } I = 22.6^\circ.$$

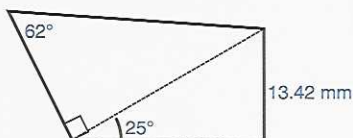
Find the unknown acute angle θ to the nearest 0.1° .

33. $\sin \theta = 0.6314$ 34. $\cos \theta = 0.1382$
 35. $\tan \theta = \frac{35.9}{50.0}$ 36. $\sec \theta = 2.351$
 37. $\csc \theta = 1.425$ 38. $\cot \theta = 6.310$

In the following problems two of the five values that must be known to solve right triangle ABC are given. Find the other three values. Round all answers to the same accuracy as the data.

a	b	c	A	B
39. 23.3			56.1°	
40.	11.4	23.0		
41. 4.00	8.55			
42.		66.0		63.1°

43. Ground radar shows that an aircraft is 22.6 kilometers from the radar site, at an angle of elevation of 11.2° . Find the aircraft's altitude to the nearest 0.01 kilometer.
44. The diagram is a top view of a portion of a machine part. Find the length x to the nearest tenth of a millimeter.



[1–4]

45. Show that the fundamental identity of trigonometry is true for an angle of measure 45° . (Use exact values.)

Simplify the following trigonometric expressions:

46. $\cot \theta \sec \theta$ 47. $\csc \theta (\sin \theta - \tan \theta)$
 48. $\frac{\sin \theta - 1}{\sin \theta}$ 49. $\frac{\cos \alpha + 2 - \cot \alpha}{\cos \alpha}$
 50. $(1 + \sin \theta)(1 - \sin \theta)$
 51. Solve $3 \sin 2x = 2$ for x to the nearest 0.1° .

Chapter 1 test

1. Convert $26^\circ 27' 43''$ to its measure in decimal degrees. Round your answer to the nearest 0.001° .
2. Find the measure of the missing angle θ .



3. In right triangle ABC , $a = 6$ and $c = 12$. Find b . Leave your answer in exact form.
4. A flagpole is 46 feet tall and is supported by an 87-foot guy wire attached at the top of the pole and to the ground. How far from the base of the pole is the ground attachment point of the wire (to the nearest 0.1 foot)?
5. In right triangle ABC , $\sec A = \frac{5}{3}$. Draw a triangle for which this is true and use it to find $\tan B$.

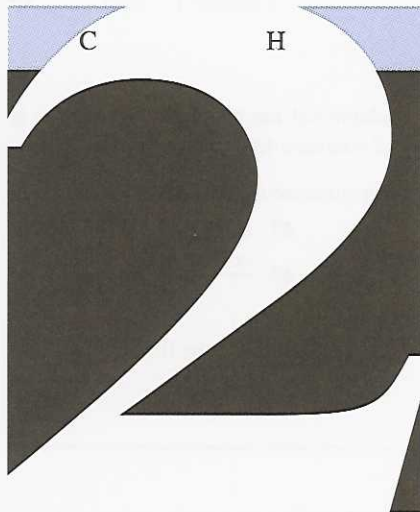
Find approximations for the following, to four decimal places.

6. $\sin 25.5^\circ$ 7. $\sec 78.3^\circ$ 8. $\cot 31^\circ 50'$

9. The average power, in watts, in an AC circuit is given by the formula $P = VI \cos \theta$. Compute P (in watts) if $V = 220$ volts, $I = 4.1$ amperes, and $\theta = 48^\circ$ (to the nearest 0.1 watt).

Find the unknown acute angle θ to the nearest 0.1° .

10. $\sin \theta = 0.1314$ 11. $\sec \theta = 4.121$
12. In right triangle ABC , $a = 3.8$ and $A = 21.4^\circ$. Solve the triangle, rounding answers to the nearest tenth.
13. In right triangle ABC , $a = 5.2$ and $b = 7.9$. Solve the triangle, rounding answers to the nearest tenth.
14. If the angle of depression from an aircraft to a ground point 13.6 miles away is 12.5° , how high is the aircraft flying, to the nearest ten feet?
15. Simplify the expression $\cos \theta (\sec \theta - \cos \theta)$.
16. Solve $\sec 5x = 5$ for x to the nearest 0.1° .



The Trigonometric Functions

In this chapter we widen the application of trigonometric concepts to include angles of any degree measure. We then learn about another method of angle measurement, called radian measure. This is the system of measurement most often used in higher mathematics, engineering, and the sciences. We begin by reviewing the idea of function.

2-1 Functions

The concept of a function is basic to higher mathematics. It provides a way to describe, or model, many real-world situations. For example, the temperature of a metal bar, heated at one end, varies with the distance from the heated end. We say that the temperature along the bar is a function of the distance along the bar. The number of pounds of tomatoes sold in a given geographic area may vary with the retail price per pound. We say that the number of pounds sold is a function of retail price. In short, any time a change in one measurable quantity can be linked to a change in another measurable quantity, the idea of function can be used to give precise meaning to the idea.

A useful definition of function is the following:

Function

A function is a set of ordered pairs having the property that no first element of the ordered pairs repeats.

For example, $f = \{(1,3), (4,9), (-2,6)\}$ is a function since, first, it is a set of ordered pairs and second, no first element of these ordered pairs repeats—that is, they are all different. However, $g = \{(1,3), (4,9), (4,6)\}$ is not a function since, although it is a set of ordered pairs, one of the first elements (4) is repeated.

If the weight of some type of steel bar is 1.5 pounds per foot and the bar comes in 1-, 2-, 5-, 8-, and 10-foot lengths, then a function that describes the weight of a bar would be $\{(1,1.5), (2,3), (5,7.5), (8,12), (10,15)\}$, where, of course, the first element of each pair is the length of the bar and the second element is the weight.

The set of all first elements of the ordered pairs in a function is called the **domain** of the function, and the set of all second elements is called the **range** of the function. The domain of f (above) is $\{-2, 1, 4\}$, and the range of f is $\{3, 6, 9\}$.

One to one

A function is said to be one to one if no second element of the ordered pairs repeats.

For example, the function f mentioned above is one to one, whereas the function $h = \{(1,5), (2,9), (3,9)\}$ is not one to one since there is a second element, 9, of the ordered pairs that repeats.

■ Example 2-1 A

For each set of ordered pairs listed,

- a. State whether the set is a function or not.
 - b. If a function, state the domain and range.
 - c. If a function, state whether it is one to one or not.
1. $\{(-2,8), (5,9), (100,19)\}$
 - a. The set is a function since no first element repeats.
 - b. The domain is $\{-2, 5, 100\}$, and the range is $\{8, 9, 19\}$.
 - c. It is one to one since no second element repeats.
 2. $\{(-20,-10), (-3,0), (-3,1), (22,15)\}$
 - a. The set is not a function since the first element -3 repeats.
 3. $\{(-5,3), (-1,9), (12,9), (256,256)\}$
 - a. The set is a function since no first element repeats.
 - b. The domain is $\{-5, -1, 12, 256\}$, and the range is $\{3, 9, 256\}$.
 - c. The function is not one to one because the second element 9 is repeated.

If we reverse the first and second elements in each ordered pair of a one-to-one function f we get a new function. For example, consider the function f .

$$f = \{(2,3), (5,9), (7,16)\}$$

If we reverse the ordered pairs, we get

$$\{(3,2), (9,5), (16,7)\}$$

which is also a function. If we reverse the ordered pairs of the function

$$h = \{(1,5), (2,9), (3,9)\}$$

we get

$$\{(5,1), (9,2), (9,3)\}$$

which is not a function (because one of the first elements, 9, is repeated). This first element repeated because a second element repeated in function h . Note that h was not a one-to-one function.

Now consider the situation in a general way. If we reverse the elements of each ordered pair in a one-to-one function, then no first element in the resulting set will repeat since no second element in the original function was repeated. Thus, reversing the ordered pairs of a one-to-one function produces a function.

If we reverse the elements of each ordered pair in a function that is not one to one, the resulting set cannot be a function, since if the function was not one to one there was a second element that repeated, which becomes a repeated first element in the resulting set. Taken together these statements prove the following theorem:

Theorem

Reversing the elements of the ordered pairs of a function produces a function if and only if the function is one to one.

We call the function produced by reversing the ordered pairs of a one-to-one function f the **inverse function** f^{-1} .

Note A symbol for function with a superscript “ -1 ” does not mean the same thing as an expression with an exponent “ -1 .” Although 3^{-1} means $\frac{1}{3^1}$ or $\frac{1}{3}$, and x^{-1} means $\frac{1}{x}$, the symbol f^{-1} does *not* mean $\frac{1}{f}$ if f represents a function.

Example 2-1 B

In each of the given sets of ordered pairs,

- a. Determine if the set is a function.
 - b. If a function, state the domain and range.
 - c. If a one-to-one function, state its inverse function.
1. $f = \{(1,3), (1,5), (2,5), (4,9)\}$
 - a. f is not a function since a first element, 1, repeats.
 2. $g = \{(-2,-8), (0,2), (2,3), (8,8)\}$
 - a. g is a function since no first element repeats.
 - b. The domain of g is $\{-2, 0, 2, 8\}$, and the range is $\{-8, 2, 3, 8\}$.
 - c. g is one to one since no second element repeats. Therefore, it has an inverse.

$$g^{-1} = \{(-8,-2), (2,0), (3,2), (8,8)\}$$
 3. $h = \{(-1,0), (0,0), (1,2)\}$
 - a. h is a function since no first element repeats.
 - b. The domain of h is $\{-1, 0, 1\}$, and the range is $\{0, 2\}$.
 - c. h is not one to one because there is a second element, 0, which repeats. Therefore, h does not have an inverse function.

Function notation

We often describe an ordered pair of a function by using “ f of x ,” or “ $f(x)$ ” notation. For example, in the function f is defined as $f = \{(-2,6), (1,3), (4,9)\}$ we would say

$$\begin{array}{ll} f(-2) = 6 & \text{“}f \text{ of } -2 \text{ is } 6\text{”} \\ f(1) = 3 & \text{“}f \text{ of } 1 \text{ is } 3\text{”} \\ f(4) = 9 & \text{“}f \text{ of } 4 \text{ is } 9\text{”} \end{array}$$

Thus, $f(x)$ notation is a way of describing what range element is associated with a given domain element.

Note $f(-2)$ is read “ f of -2 ” or “ f at -2 .” Also, letters other than “ f ” can be used. We could write “ $g(x)$ ” or “ $h(x)$,” for example.

■ Example 2-1 C

In the function $f = \{(-100,10), (-50,20), (0,30)\}$ state

a. $f(-50)$ b. $f(10)$

a. $f(-50)$ is 20, since the range element associated with -50 is 20.

b. $f(10)$ does not exist. This is because 10 is not in the domain, so we have no way to relate it to some element in the range. ■

Since most functions contain an infinite number of ordered pairs, we cannot describe them with a list. In these cases we use a rule. The rule is usually combined with $f(x)$ notation. For example, we might describe a function f with the rule

$$f(x) = 5x - 3$$

This rule tells us that to form an ordered pair that belongs to the function, where the domain element is x , compute $5x - 3$. This is the range element. If $x = 2$, then $5x - 3$ becomes $5(2) - 3 = 7$, so the ordered pair $(2,7)$ is in the function f . Usually we write

$$\begin{array}{l} f(x) = 5x - 3 \\ f(2) = 5(2) - 3 \\ f(2) = 7 \end{array}$$

The statement $f(2) = 7$, verbalized “ f of 2 is 7,” means that for the domain element 2 the range element is 7.

■ Example 2-1 D

1. If a function f is described by the rule

$$f(x) = -3x + 1$$

form the ordered pairs in f for the domain elements (a) -2 , (b) 3 , and (c) 5 .

a. $f(-2) = -3(-2) + 1$

$f(-2) = 7$, so $(-2,7)$ is an ordered pair in f .

b. $f(3) = -3(3) + 1$

$f(3) = -8$, so $(3,-8)$ is an ordered pair in f .

c. $f(5) = -14$, so $(5,-14)$ is an ordered pair in f .

2. If a function g is described by the rule

$$g(x) = x^2 - 2x + 3$$

form the ordered pairs in g for the domain elements (a) -5 , (b) $\sqrt{2}$, and (c) 10 .

- a. $g(-5) = (-5)^2 - 2(-5) + 3$
 $g(-5) = 38$, so $(-5, 38)$ is an ordered pair in g .
- b. $g(\sqrt{2}) = (\sqrt{2})^2 - 2(\sqrt{2}) + 3$
 $g(\sqrt{2}) = 2 - 2\sqrt{2} + 3 = 5 - 2\sqrt{2}$,
 so $(\sqrt{2}, 5 - 2\sqrt{2})$ is an ordered pair in g .
- c. $g(10) = 10^2 - 2(10) + 3$
 $g(10) = 83$, so $(10, 83)$ is an ordered pair in g . ■

In these examples we never stated exactly what the domain of each function was. If we are not told what the domain is, we always use all real numbers for which the rule makes sense. This is called the **implied domain**.

For example, if the rule for a function f were $f(x) = \frac{3}{x-2}$, then the domain would be every real number except 2, since $f(2)$ would be $\frac{3}{2-2} = \frac{3}{0}$, which is undefined. If $f(x) = \frac{x}{x^2-1}$ were the rule that described a function, the domain would be all the real numbers except ± 1 , since either value would make the denominator of the expression 0.

The trigonometric ratios are functions in the sense of our definition. They can be viewed as sets of ordered pairs in which no first element repeats. For example the sine ratio can be described as the ordered pairs (degree measure of angle, sine of angle). It would include, for example, the ordered pairs $\left(30^\circ, \frac{1}{2}\right)$, $\left(45^\circ, \frac{\sqrt{2}}{2}\right)$, $\left(60^\circ, \frac{\sqrt{3}}{2}\right)$, etc.

Mastery points

Can you

- State the definition of a function?
- Determine if a set is a function?
- State whether a function is one to one?
- State the inverse of a one-to-one function?
- Use $f(x)$ notation to form ordered pairs in a function?

Exercise 2-1

In each of the following sets of ordered pairs:

- a. Determine if the set is a function.
- b. If a function, state the domain and range.
- c. If a one-to-one function, state the inverse.

1. $f = \{(3,5), (4,5), (6,9), (7,10)\}$

3. $h = \{(-2,-2), (3,4), (4,3)\}$

2. $g = \{(-4,-3), (-1,1), (1,3), (2,5)\}$

4. $f = \{(0.5,2), (1.5,3), (2,4), (2.5,5)\}$

5. $g = \{(1,4), (1,5), (5,9), (6,10)\}$

7. $f = \{(1,1), (2,2), (3,3), (4,4)\}$

6. $h = \{(-10,-5), (10,5), (12,20), (20,30)\}$

8. $g = \{(-3,5), (5,8), (8,13), (13,21)\}$

Assume $f = \{(-3,6), (-2,9), (-1,0), (0,19), (4\frac{1}{2},7), (2\pi,11), (200,220), (300,7)\}$ in problems 9 and 10.

9. Find

a. $f(-2)$ b. $f(0)$ c. $f(2\pi)$ d. $f(7)$ e. $f(250)$

10. Find

a. $f(-1)$ b. $f(4\frac{1}{2})$ c. $f(\pi)$ d. $f(300)$

In each of the following problems a rule that describes a function is given. Form the ordered pairs that this function contains for the following domain elements:

a. -2 b. 0 c. $\sqrt{3}$ d. $\frac{1}{2}$ e. 5

11. $f(x) = 5x - 3$

12. $g(x) = 2 - 3x$

13. $h(x) = (x - 3)(x + 2)$

14. $f(x) = x^2 - 2x + 3$

15. $g(x) = x^2 + x - 1$

16. $h(x) = x^2$

17. $f(x) = 3x^4 - x^2 + 2$

18. $g(x) = 1 - x^2$

19. $h(x) = \frac{x}{x+3}$

20. $f(x) = \frac{4}{x^2 - 1}$

21. The statement $t(x) = 500 - 2x$ states the functional relationship between the temperature t of an iron rod at a point x centimeters from the heated end. Find the temperature (in degrees centigrade) for points that are (a) 3 cm, (b) 12 cm, and (c) 104 cm from the heated end.

2-2 The trigonometric functions—definitions

There are many situations where we have to think of angles as being nonacute. This is often a situation in which we wish to describe an amount of rotation. For example, a ship may turn through an angle of 215° , a computed tomography (CT) scanner used in medical diagnosis may move through an angle of 360° , or a surveyor may find the measure of the angle at one corner of a piece of land to be $165^\circ 20'$. For these situations, we often place the angle in a rectangular (x - y) coordinate system.

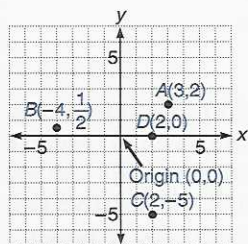


Figure 2-1

The x - y (rectangular) coordinate system

We graph using the x - y rectangular coordinate system. Recall that an **ordered pair** is a pair of numbers listed in parentheses, separated by a comma. In the ordered pair (x,y) x is called the **first component** and y is called the **second component**; $(5,-3)$, $(9,3)$, and $(4,\frac{2}{3})$ are examples of ordered pairs. The graphing system we use is formed by sets of vertical and horizontal lines; one vertical line is called the y -axis, and one horizontal line is called the x -axis. The geometric plane (flat surface) that contains this system of lines is called the **coordinate plane**. See figure 2-1.

The **graph** of an ordered pair is the geometric point in the coordinate plane located by moving left or right, as appropriate, according to the first component of the ordered pair, and vertically a number of units corresponding to the second component of the ordered pair. The graphs of the points $A(3,2)$, $B(-4,\frac{1}{2})$, $C(2,-5)$, and $D(2,0)$ are shown in the figure. The first and second elements of the ordered pair associated with a geometric point in the coordinate plane are called its **coordinates**.

Angles in standard position

Angle in standard position

An angle in standard position is formed by two rays, one of which always lies on the nonnegative portion of the x -axis. This ray is called the **initial side**. The second ray is called the **terminal side**. It may be in any quadrant or along any axis.

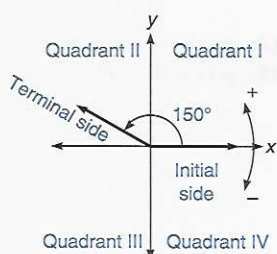


Figure 2-2

Figure 2-2 shows an angle in standard position, with measure 150° . Observe the labeling of the quadrants formed by the x -axis and the y -axis. They are called quadrants I through IV as shown. We generally use the word “angle” instead of the phrase “angle in standard position.”

If the measure of the angle is *positive*, we picture the terminal side as having moved away from the initial side in a *counterclockwise* direction; if the measure of the angle is *negative* we picture the terminal side as having moved away from the initial side in a *clockwise* direction. If an angle’s measure is greater than 360° or less than -360° we consider the angle to have “gone around” more than once. Several examples of angles in standard position are shown in figure 2-3. In part c we show the angle as a 360° revolution, followed by an additional 200° turn.

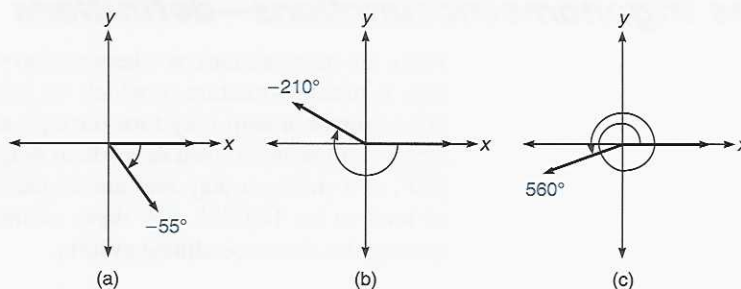


Figure 2-3

Angles which have the same terminal side are said to be **coterminal**. (All angles in standard position have the same initial side.) The 150° angle in figure 2-2 and the -210° angle in figure 2-3 (b) are coterminal. We can see this when we realize that in each case the angle formed by the negative side of the x -axis and the terminal side of each angle is 30° . Since $\pm 360^\circ$ represents one complete revolution, coterminal angles are angles whose degree measures differ by an integer multiple of 360° . This forms the basis for our definition.

Coterminal angles

Two angles¹ α and β are said to be coterminal if

$$\alpha = \beta + k(360^\circ), \quad k \text{ an integer.}$$

Concept

Two angles are coterminal if the difference of their degree measures is evenly divisible by 360° .

¹Remember, α is the Greek letter alpha, and β is the Greek letter beta.

Example 2-2 A

In each case find a coterminal angle with measure x such that $0^\circ \leq x < 360^\circ$.

1. 875°

$$875^\circ - 360^\circ = 515^\circ$$

Subtract 360° until x is found

$$515^\circ - 360^\circ = 155^\circ$$

The required angle is 155°

We could have done this more elegantly by computing $875^\circ - 2(360^\circ)$.

2. $-1,000^\circ$

$$-1,000^\circ + 360^\circ = -640^\circ$$

Add 360° until x is found

$$-640^\circ + 360^\circ = -280^\circ$$

$$-280^\circ + 360^\circ = 80^\circ$$

Or solve by computing $-1,000^\circ + 3(360^\circ) = -1,000^\circ + 1,080^\circ = 80^\circ$. ■

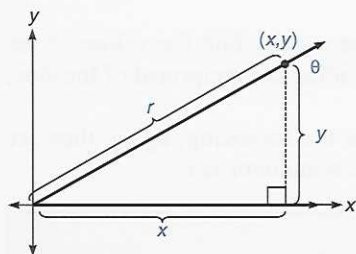


Figure 2-4

The trigonometric functions

We now define the six trigonometric functions. They have the same names as the six trigonometric ratios, and the same abbreviations. The trigonometric ratios are functions with domain the set of *acute* angles. The trigonometric functions have the set of *all* angles as their domain. We used the word ratio to distinguish the two sets of functions. For acute angles the trigonometric functions are essentially the same as the trigonometric ratios. The following definition refers to figure 2-4.

The trigonometric functions

Let θ be an angle in standard position, and let (x, y) be any point on the terminal side of the angle, except $(0, 0)$. Let $r = \sqrt{x^2 + y^2}$ be the distance from the origin to the point. Then,

$$\sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r}, \quad \tan \theta = \frac{y}{x}$$

$$\csc \theta = \frac{r}{y}, \quad \sec \theta = \frac{r}{x}, \quad \cot \theta = \frac{x}{y}$$

- Note**
1. We define r so that $r > 0$.
 2. If x or y in the point (x, y) is zero, then those ratios with x or y in the denominator are not defined.
 3. Unlike the trigonometric ratios, the trigonometric functions can take on negative values.
 4. We sometimes call the cosecant, secant, and cotangent functions the *reciprocal trigonometric functions*.

It can be proven that for a given angle, *it does not matter what point on the terminal side is chosen; the values of the trigonometric functions will be the same*. This is illustrated in example 2-2 D.

It can also be seen that *coterminal angles have the same values for the trigonometric functions*. This is because two coterminal angles have the same terminal side, and the definitions depend solely on a point on the terminal side.

The definitions of the trigonometric functions imply the following identities for all values of θ for which any denominator is nonzero. These identities look identical to those for the trigonometric ratios; however, those were shown to be true only for acute angles in right triangles.

Reciprocal function identities

$$\begin{aligned}\csc \theta &= \frac{1}{\sin \theta}, & \sin \theta &= \frac{1}{\csc \theta} \\ \sec \theta &= \frac{1}{\cos \theta}, & \cos \theta &= \frac{1}{\sec \theta} \\ \cot \theta &= \frac{1}{\tan \theta}, & \tan \theta &= \frac{1}{\cot \theta}\end{aligned}$$

To see that the first reciprocal function identity is true observe that $\csc \theta = \frac{r}{y} = \frac{1}{\frac{y}{r}} = \frac{1}{\sin \theta}$. It is left as an exercise to show that the rest of these are true.

Using the reciprocal function identities we can usually find the values of the cosecant, secant, and cotangent functions by finding the reciprocal of the sine, cosine, and tangent functions.

Two other identities that can be useful are the following; again, they are true only for those values of θ for which no denominator is 0.

Tangent/cotangent identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

Example 2-2 B

Show that $\tan \theta = \frac{\sin \theta}{\cos \theta}$ is an identity for the trigonometric functions.

We show that each member of the equation is equivalent to the same thing.

$\tan \theta$	$\frac{\sin \theta}{\cos \theta}$	
	$\frac{y}{\frac{r}{x}}$	
$\frac{y}{x}$	$\frac{r}{x}$	Apply the definitions
	$\frac{y}{r} \cdot \frac{r}{x}$	
	$\frac{y}{x}$	Algebra of division
	$\frac{y}{x}$	Reduce $\frac{ry}{rx}$

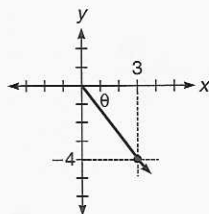
Thus, $\tan \theta = \frac{y}{x}$ and $\frac{\sin \theta}{\cos \theta} = \frac{y}{x}$, so $\tan \theta = \frac{\sin \theta}{\cos \theta}$. ■

Example 2-2 C illustrates finding values of the trigonometric functions for an angle in standard position, given a point on the terminal side of the angle.

■ **Example 2-2 C**

In each problem a point on the terminal side of an angle θ is given. Use it to find the trigonometric functions for that angle. Also, make a sketch of the angle.

1. $(3, -4)$



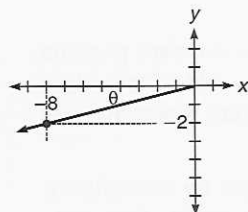
$$\begin{aligned} r &= \sqrt{x^2 + y^2} && \text{Definition} \\ &= \sqrt{3^2 + (-4)^2} && \text{Replace } x, y \\ &= 5 \end{aligned}$$

$$\sin \theta = \frac{y}{r} = -\frac{4}{5}, \csc \theta = \frac{1}{\sin \theta} = -\frac{5}{4}$$

$$\cos \theta = \frac{x}{r} = \frac{3}{5}, \sec \theta = \frac{1}{\cos \theta} = \frac{5}{3}$$

$$\tan \theta = \frac{y}{x} = -\frac{4}{3}, \cot \theta = \frac{1}{\tan \theta} = -\frac{3}{4}$$

2. $(-8, -2)$



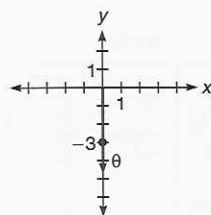
$$\begin{aligned} r &= \sqrt{x^2 + y^2} && \text{Definition} \\ &= \sqrt{(-8)^2 + (-2)^2} && \text{Replace } x, y \\ &= \sqrt{68} = 2\sqrt{17} \end{aligned}$$

$$\sin \theta = \frac{y}{r} = \frac{-2}{2\sqrt{17}} = -\frac{1}{\sqrt{17}} = -\frac{\sqrt{17}}{17}, \csc \theta = -\sqrt{17}$$

$$\cos \theta = \frac{x}{r} = \frac{-8}{2\sqrt{17}} = -\frac{4}{\sqrt{17}} = -\frac{4\sqrt{17}}{17}, \sec \theta = -\frac{\sqrt{17}}{4}$$

$$\tan \theta = \frac{y}{x} = \frac{-2}{-8} = \frac{1}{4}, \cot \theta = 4$$

3. $(0, -3)$



$$r = \sqrt{0^2 + (-3)^2} = 3$$

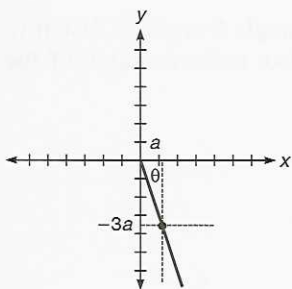
$$\sin \theta = \frac{y}{r} = \frac{-3}{3} = -1, \csc \theta = \frac{1}{\sin \theta} = -1$$

$$\cos \theta = \frac{x}{r} = \frac{0}{3} = 0, \sec \theta = \frac{1}{\cos \theta} = \frac{1}{0}, \text{undefined}$$

$$\tan \theta = \frac{y}{x} = \frac{-3}{0}, \text{undefined; } \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{0}{-1} = 0$$

As illustrated here, when the tangent function is undefined the cotangent function is defined. In this one case it is useful to use the appropriate

tangent/cotangent identity or equivalently $\cot \theta = \frac{x}{y}$.

4. $(a, -3a)$, $a > 0$

Since the x -coordinate is positive the angle is in quadrants I or IV; since the y -coordinate is negative, we choose quadrant IV for our sketch.

$$r = \sqrt{a^2 + (-3a)^2} = \sqrt{10a^2} = a\sqrt{10}$$

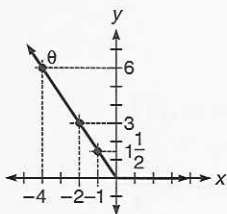
$$\sin \theta = -\frac{3a}{a\sqrt{10}} = -\frac{3}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}, \csc \theta = -\frac{\sqrt{10}}{3}$$

$$\cos \theta = \frac{a}{a\sqrt{10}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}, \sec \theta = \sqrt{10}$$

$$\tan \theta = -\frac{3a}{a} = -3, \cot \theta = -\frac{1}{3}$$

Example 2-2 D illustrates that the value of the trigonometric functions for a given angle depend only on the measure of the angle and not on the point that is chosen on its terminal side.

Example 2-2 D



The point $(-2, 3)$ is on the terminal side of an angle θ in standard position. Find two other points that would also be on the terminal side of this angle and then compute the value of the sine, cosine, and tangent functions with all three points.

We can find other points on the terminal side of this angle by multiplying both values, -2 and 3 , by the same amount. Thus, if we double them we obtain the point $(-4, 6)$. If we take half of each we obtain $(-1, 1\frac{1}{2})$. All three points are shown in the figure.

The computations for the three points are shown in the table. The same results are obtained regardless of which point is used to perform the calculation.

Point	x	y	r	$\sin \theta$	$\cos \theta$	$\tan \theta$
$(-1, 1\frac{1}{2})$	-1	$\frac{3}{2}$	$\frac{\sqrt{13}}{2}$	$\frac{\frac{3}{2}}{\frac{\sqrt{13}}{2}} = \frac{3\sqrt{13}}{13}$	$\frac{-1}{\frac{\sqrt{13}}{2}} = -\frac{2\sqrt{13}}{13}$	$\frac{\frac{3}{2}}{-1} = -\frac{3}{2}$
$(-2, 3)$	-2	3	$\sqrt{13}$	$\frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$	$\frac{-2}{\sqrt{13}} = -\frac{2\sqrt{13}}{13}$	$\frac{3}{-2} = -\frac{3}{2}$
$(-4, 6)$	-4	6	$2\sqrt{13}$	$\frac{6}{2\sqrt{13}} = \frac{3\sqrt{13}}{13}$	$\frac{-4}{2\sqrt{13}} = -\frac{2\sqrt{13}}{13}$	$\frac{6}{-4} = -\frac{3}{2}$

Mastery points

Can you

- When given an angle θ , find a positive coterminal angle with measure x such that $0^\circ \leq x < 360^\circ$?
- Sketch an angle and find the values of the six trigonometric functions when given a point on the terminal side of the angle?

Exercise 2-2

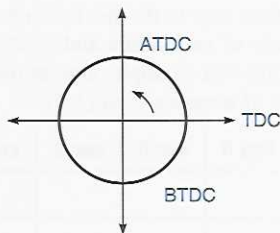
In problems 1–17,

a. Draw the initial and terminal side of the given angle.

b. State the measure of the smallest nonnegative angle that is coterminal with the given angle.

- | | | | | |
|-------------------|------------------|----------------------|------------------|--------------------|
| 1. 420° | 2. -40° | 3. 230° | 4. $1,000^\circ$ | 5. $1,800.6^\circ$ |
| 6. $1,260^\circ$ | 7. 547.9° | 8. $2,000^\circ$ | 9. -870° | 10. 625° |
| 11. 525° | 12. -610° | 13. $-1,530.3^\circ$ | 14. 390° | 15. -720° |
| 16. -11.9° | 17. -313° | | | |

18. An automobile engine is timed to fire the spark plug for cylinder 1 at 8° BTDC (before top dead center), which, for our purposes, is -8° . Assuming this engine rotates in a counterclockwise direction, what is the equivalent amount ATDC (after TDC) (i.e., the least nonnegative angle coterminal with it)?



19. If an automobile engine is timed to fire at 13° BTDC, what is the equivalent amount ATDC?

20. If an automobile engine is timed to fire at 8.6° BTDC, what is the equivalent amount ATDC?

21. If an automobile engine is timed to fire at 6.1° BTDC, what is the equivalent amount ATDC?

22. In an electronic circuit with an inductive component to the impedance, the current follows the voltage. For example, the current may follow the voltage by 15° , in which case we could say the phase angle of the current is -15° , relative to the voltage. We could just as easily say that the phase angle of the voltage is 345° , relative to the current. Find the phase angle of the voltage relative to the current if the phase angle of the current relative to the voltage is (a) -88° , (b) -24.33° , (c) $-35^\circ 56'$, (d) -16.56° (e) $-0^\circ 14'$, (f) -0.14° . (Find the least nonnegative coterminal angle in each case.)

In the following problems you are given a point that lies on the terminal side of an angle in standard position. In each case, compute the value of all six trigonometric functions for the angle.

- | | | | | |
|----------------------|----------------------|------------------------------|----------------------|------------------------------|
| 23. $(3, 6)$ | 24. $(-2, 5)$ | 25. $(-5, 8)$ | 26. $(-7, -8)$ | 27. $(2, -2)$ |
| 28. $(3, 0)$ | 29. $(-1, 4)$ | 30. $(0, -4)$ | 31. $(-10, -15)$ | 32. $(3, \sqrt{5})$ |
| 33. $(-\sqrt{2}, 6)$ | 34. $(3, -\sqrt{6})$ | 35. $(-\sqrt{3}, -\sqrt{2})$ | 36. $(1, -\sqrt{3})$ | 37. $(\sqrt{6}, -\sqrt{10})$ |

In the following problems you are given a point that lies on the terminal side of an angle in standard position. In each case, compute the value of all six trigonometric functions for the angle. Assume $a > 0$, $b > 0$.

- | | | | | |
|------------------------|-----------------------------------|----------------|----------------------|-----------------------|
| 38. $(b, -2b)$ | 39. $(2a, -a)$ | 40. $(-a, -a)$ | 41. $(\sqrt{2}b, b)$ | 42. $(3a, \sqrt{3}a)$ |
| 43. $(\frac{b}{2}, b)$ | 44. $(-\frac{a}{3}, \frac{a}{2})$ | | | |

45. Show that the identity $\sec \theta = \frac{1}{\cos \theta}$ is true, except where $\cos \theta = 0$.

46. Show that the identity $\cot \theta = \frac{1}{\tan \theta}$ is true, except where $\tan \theta = 0$.

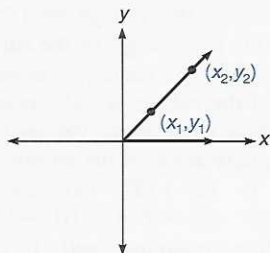
47. Show that the identity $\cot \theta = \frac{\cos \theta}{\sin \theta}$ is true, except where $\sin \theta = 0$.

48. Show that the identity $\cos \theta = \frac{1}{\sec \theta}$ is true.

49. Show that the identity $\sin \theta = \frac{1}{\csc \theta}$ is true.

 To solve the following two problems, we must recall that the equation of a nonvertical straight line can be put in the form $y = mx + b$, where m is the slope and b is the y -intercept. If a straight line passes through the origin then $b = 0$ and the equation becomes $y = mx$.

- 50.** Show that if two different points lie on the terminal side of an angle in standard position, then using either point gives the same value for the sine function. For the sake of simplicity assume the terminal side is not vertical or horizontal. Represent the points as (x_1, y_1) and (x_2, y_2) . Note that these points lie on the same line. The equation of any line that passes through the origin is of the form $y = mx$, so we know that for the same value of m , $y_1 = mx_1$ and $y_2 = mx_2$. This means that the points (x_1, y_1) and (x_2, y_2) can be rewritten as (x_1, mx_1) and (x_2, mx_2) . Use these versions of the points to compute the length r for each point. Then show that the value of the sine function is the same when computed using either point.



- 51.** Show that if the trigonometric function values are the same for two points, then these points lie on the terminal side of the same angle. Assume for simplicity that these points are not located on either the x -axis or the y -axis. To show that the points lie on the terminal side of the same angle we must show that both points lie on the same line and are in the same quadrant. Let (x_1, y_1) and (x_2, y_2) represent the two points, and consider the value of the tangent function as given by each point. This can be used to show that $y_1 = mx_1$ and $y_2 = mx_2$ (for the same value of m). This means that the two points lie on the same line. Now explain why they must be in the same quadrant.

- 52.** Fill in the table below. One way to do this is to choose points on the terminal side of each angle and apply the definitions of each function. For example, a point on the terminal side of an angle of measure 0° is $(1, 0)$.

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\cot \theta$
0°						
90°						
180°						
270°						

2-3 Values for any angle—the reference angle/ASTC procedure

The values of the trigonometric functions for an angle of any measure are related to the values for the acute angles of the first quadrant. These values (for the first quadrant) are the same as those for the trigonometric ratios for acute angles. The values of the trigonometric functions for any angle have a sign and a “size” (absolute value). We first discuss the sign of the basic trigonometric functions, then the size.

The ASTC rule—the signs of the trigonometric functions by quadrant

The **sign** of the value of a trigonometric function for an angle *depends on the quadrant in which the angle terminates*. Figure 2-5 shows the quadrants in which the sine, cosine, and tangent functions are positive. (They are negative in the other quadrants.)

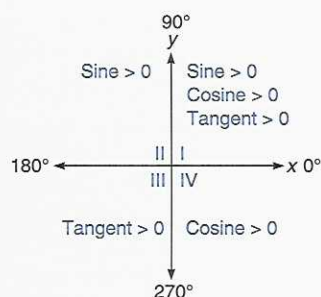


Figure 2-5

The figure shows that the sine function is positive in quadrants I and II, and therefore negative in quadrants III and IV. This is because the sine function is defined by the ratio $\frac{y}{r}$; since r is always positive this ratio is positive where y is positive, in quadrants I and II. Since the cosine function is $\frac{x}{r}$ and $r > 0$, the cosine is positive where x is positive: quadrants I and IV. The tangent function is defined by $\frac{y}{x}$, so it is positive where x and y are both positive (quadrant I) or both negative (quadrant III).

Figure 2-5 should be memorized; it represents the ASTC rule.

The ASTC rule

In quadrant I,	All the trigonometric functions are positive.
In quadrant II, the	Sine function is positive.
In quadrant III, the	Tangent function is positive.
In quadrant IV, the	Cosine function is positive.

One memory aid is the sentence “All Students Take Calculus.”

Since the sign of the reciprocal of a value is the same as the value, the sign of the cosecant function is the same as the sign of the sine function, that of the secant function is the same as that of the cosine function, and the sign of the cotangent function is the same as that of the tangent function.

The ASTC rule can be used to determine in which quadrant a given angle terminates.

■ Example 2-3 A

Determine in which quadrant the given angle θ terminates.

1. $\sin \theta < 0$, $\tan \theta > 0$

If $\sin \theta < 0$ then θ terminates in quadrants III or IV.

If $\tan \theta > 0$ then θ terminates in quadrants I or III.

Thus, for both conditions to be true, θ must terminate in quadrant III.

2. $\cos \theta < 0$, $\sin \theta > 0$

$\cos \theta < 0$ means θ terminates in quadrant II or III.

$\sin \theta > 0$ means θ terminates in quadrant I or II.

Thus, θ terminates in quadrant II. ■

Reference angles

Angles whose degree measures are integer multiples of 90° , such as 0° , $\pm 90^\circ$, $\pm 180^\circ$, $\pm 270^\circ$, etc. are called **quadrantal angles** because their terminal sides fall between two quadrants. All other angles are nonquadrantal angles. A **reference angle** for a nonquadrantal angle is the acute angle formed by the terminal side of the angle and the x -axis. A reference angle is not defined for

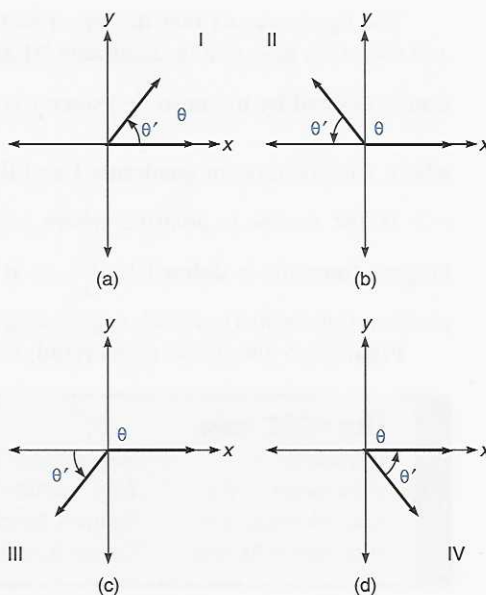


Figure 2-6

quadrantal angles. Figure 2-6 shows a reference angle, θ' (theta-prime) for an angle θ terminating in each quadrant.

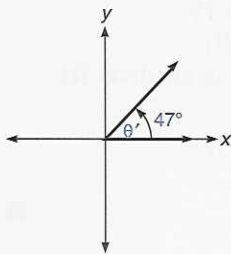
A reference angle is always acute (between 0° and 90°) and is always formed by the terminal side of the angle and the x -axis (never the y -axis). As will be illustrated in example 2-3 B, a good way to find a reference angle is to sketch the angle itself. This should make clear what computation to perform.

■ Example 2-3 B

Compute and sketch the reference angle for each angle.

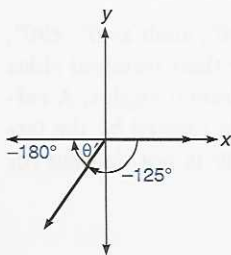
1. 47°

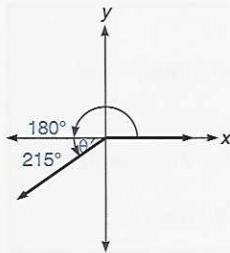
This angle terminates in quadrant I. The reference angle is the same as the angle itself, 47° .



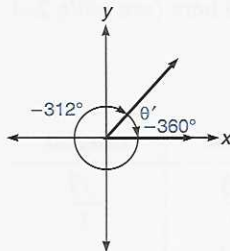
2. -125°

This angle terminates in quadrant III. The positive difference between -125° and -180° is $180^\circ - 125^\circ = 55^\circ$, which is the value of the reference angle.



3. 215°

This angle terminates in quadrant III also. Here the reference angle is $215^\circ - 180^\circ = 35^\circ$.

4. -312°

This angle terminates in quadrant I. The value of the reference angle is the positive difference between -312° and $-360^\circ = 360^\circ - 312^\circ = 48^\circ$.

It can be seen that if $0^\circ < \theta < 360^\circ$ then the reference angle θ' can be found according to the following formulas.

$$\begin{aligned} \theta \text{ in quadrant I: } & \theta' = \theta \\ \theta \text{ in quadrant II: } & \theta' = 180^\circ - \theta \\ \theta \text{ in quadrant III: } & \theta' = \theta - 180^\circ \\ \theta \text{ in quadrant IV: } & \theta' = 360^\circ - \theta \end{aligned}$$

The absolute value of the trigonometric functions for any angle

The **absolute value** of a trigonometric function for any angle is the same as the trigonometric ratio for the corresponding reference angle. Figure 2-7 illustrates this idea for the angle 150° . If an angle of measure 150° is in standard position, then we find the values of the trigonometric functions by taking a point on its terminal side (the point $B(x,y)$ in the figure), and using the definitions of these functions in terms of x , y , and r .

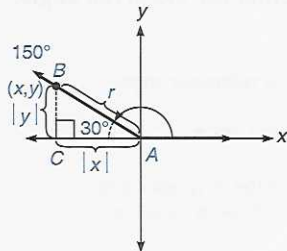


Figure 2-7

As seen in the figure, the absolute value of $\sin 150^\circ$ is $\frac{|y|}{r}$. This is also the value of the trigonometric ratio for the reference angle, with measure 30° : $\sin 30^\circ = \frac{\text{length of side opposite } 30^\circ}{\text{length of hypotenuse}}$. We know from section 1-3 that $\sin 30^\circ = \frac{1}{2}$. Thus, in absolute value, $|\sin 150^\circ| = \sin 30^\circ = \frac{1}{2}$. Since we know that the sine function is positive in quadrant II, $\sin 150^\circ = \frac{1}{2}$.

The reference angle/ASTC procedure

The facts discussed in the previous paragraphs provide a method for finding the *exact* values of the basic trigonometric functions for any nonquadrantal angle whose reference angle is 30° , 45° , or 60° . We call this the reference angle/ASTC procedure.

Reference angle/ASTC procedure

To find the exact value of a trigonometric function for a nonquadrantal angle whose reference angle is 30° , 45° , or 60° :

1. Find the value of the reference angle.
2. Find the value of the appropriate trigonometric ratio for the reference angle from table 1–1, section 1–3.
3. Determine the sign of this value using the ASTC rule (figure 2–5).

For convenience, the needed table and figure are repeated here (see table 2–1 and figure 2–8).

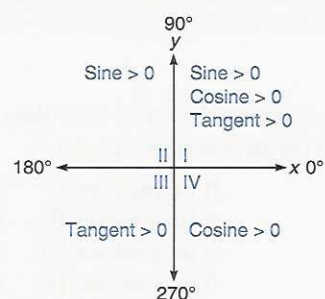


Figure 2–8

	Sine	Cosine	Tangent
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Table 2–1

Example 2–3 C

Find the exact value of the given trigonometric function for the given angle.

1. $\cos 210^\circ$

$$\theta' = 210^\circ - 180^\circ = 30^\circ$$

Find the value of the reference angle

$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$

Table 2–1 (memorized value)

$$\cos 210^\circ = -\frac{\sqrt{3}}{2}$$

A 210° angle terminates in quadrant III, where the cosine function is negative

2. $\tan (-45^\circ)$

$$\theta' = 45^\circ$$

Find the value of the reference angle

$$\tan 45^\circ = 1$$

Table 2–1 (memorized value)

$$\tan (-45^\circ) = -1$$

A -45° angle terminates in quadrant IV, where the tangent function is negative

3. $\tan 840^\circ$

$$840^\circ - 2(360^\circ) = 120^\circ$$

120° and 840° are coterminal

$$\theta' = 180^\circ - 120^\circ = 60^\circ$$

Reference angle for 840°

$$\tan 60^\circ = \sqrt{3}$$

Memorized value

$$\tan 840^\circ = -\sqrt{3}$$

840° terminates in quadrant II, where the tangent function is negative

The values of the trigonometric functions for quadrantal angles can be found by selecting any point on the terminal side of the angle and using the definitions.

■ Example 2-3 D

Find the values of the six trigonometric functions for the angle with measure 900° .

$$900^\circ - 2(360^\circ) = 180^\circ, \text{ so } 900^\circ \text{ and } 180^\circ \text{ are coterminal angles.}$$

The point $(-1, 0)$ is on the terminal side of a 180° angle, and is therefore on the terminal side of a 900° angle. Use this point to find the values for 900° .

$$r = \sqrt{(-1)^2 + 0^2} = 1, x = -1, y = 0.$$

$$\sin 900^\circ = \frac{y}{r} = \frac{0}{1} = 0, \csc 900^\circ = \frac{1}{\sin 900^\circ} = \frac{1}{0}; \text{ undefined}$$

$$\cos 900^\circ = \frac{x}{r} = \frac{-1}{1} = -1, \sec 900^\circ = \frac{1}{\cos 900^\circ} = \frac{1}{-1} = -1$$

$$\tan 900^\circ = \frac{y}{x} = \frac{0}{-1} = 0, \cot 900^\circ = \frac{1}{\tan 900^\circ} = \frac{1}{0}; \text{ undefined}$$

Approximate values of the trigonometric functions—calculators

Approximate values of the trigonometric functions are calculated using the same calculator keys as for the trigonometric ratios; for acute angles the ratios and functions have the same values. Recall from section 1-3 that the calculator must be in degree mode when entering angle measure in degrees.

■ Example 2-3 E

Find four decimal place approximations to the following function values.

Make sure the calculator is in degree mode.

1. $\sin 133^\circ$ 133 Display
 $\sin 133^\circ \approx 0.7314$ 133

2. $\tan (-18^\circ)$ 18 Display
 $\tan (-18^\circ) \approx -0.3249$ 18

3. $\sec (-335.6^\circ)$ $\sec (-335.6^\circ) = \frac{1}{\cos (-335.6^\circ)}$
 335.6
 335.6

 $\sec (-335.6^\circ) \approx 1.0981$ Display

Solutions to trigonometric equations

Recall from chapter 1 that we use the inverse trigonometric functions to solve trigonometric equations of the form $\sin \theta = k$, $\cos \theta = k$, $\tan \theta = k$, where k is a known constant. In particular, to find one value of θ in each equation, we use the following facts:

if $\sin \theta = k$, then one solution for θ is $\theta = \sin^{-1} k$

if $\cos \theta = k$, then one solution for θ is $\theta = \cos^{-1} k$

if $\tan \theta = k$, then one solution for θ is $\theta = \tan^{-1} k$

In chapters 4 and 5 we will examine this situation in more depth, but for now we will simply rely on these facts, and on the fact that these inverse trigonometric functions are programmed into calculators as seen in section 1–3.

■ Example 2–3 F

Find one solution to each trigonometric equation, to the nearest 0.1° .

1. $\sin \theta = -0.8500$
 $\theta = \sin^{-1}(-0.8500) \approx -58.2^\circ$
2. $\cos \theta = -0.8500$
 $\theta = \cos^{-1}(-0.8500) \approx 148.2^\circ$

Mastery points

Can you

- Determine in which quadrant an angle terminates when given the signs of two of the trigonometric function values for that angle?
- Compute and sketch the reference angle for a given nonquadrantal angle θ with given degree measure?
- Find the exact value of any trigonometric function for an angle whose reference angle is 30° , 45° , or 60° , using the reference angle/ASTC procedure?
- Find the exact value of any trigonometric function for a quadrantal angle?
- Find the approximate value of any trigonometric function using a calculator?
- Find the approximate value of one solution to an equation of the form $\sin \theta = k$, $\cos \theta = k$, $\tan \theta = k$?

Exercise 2–3

In the following problems you are given the sign of two of the trigonometric functions of an angle in standard position. State in which quadrant the angle terminates.

- | | | | |
|--|---|---|--|
| 1. $\sin \theta > 0$, $\cos \theta < 0$ | 2. $\sec \theta < 0$, $\tan \theta > 0$ | 3. $\cos \theta > 0$, $\tan \theta > 0$ | 4. $\cot \theta < 0$, $\csc \theta > 0$ |
| 5. $\tan \theta < 0$, $\csc \theta < 0$ | 6. $\sec \theta > 0$, $\csc \theta < 0$ | 7. $\csc \theta > 0$, $\cos \theta < 0$ | 8. $\tan \theta > 0$, $\sin \theta < 0$ |
| 9. $\sec \theta > 0$, $\sin \theta < 0$ | 10. $\cot \theta > 0$, $\sin \theta > 0$ | 11. $\sin \theta < 0$, $\sec \theta < 0$ | |

For each of the following angles, find the measure of the reference angle θ' .

- | | | | | |
|--------------------|--------------------|-------------------|--------------------|--------------------|
| 12. 39.3° | 13. 164.2° | 14. 213.2° | 15. 427.1° | 16. -16.8° |
| 17. -255.3° | 18. -100.4° | 19. 130.7° | 20. -671.3° | 21. -181.0° |
| 22. 512.8° | 23. -279.5° | 24. 292.3° | 25. -252° | 26. 312° |

Find the exact trigonometric function value for each angle.

- | | | | | |
|----------------------|------------------------|------------------------|------------------------|------------------------|
| 27. $\sin 135^\circ$ | 28. $\cos 120^\circ$ | 29. $\sin 210^\circ$ | 30. $\cos 330^\circ$ | 31. $\tan 300^\circ$ |
| 32. $\sin 240^\circ$ | 33. $\sin(-120^\circ)$ | 34. $\cos(-315^\circ)$ | 35. $\cos 660^\circ$ | 36. $\csc(-315^\circ)$ |
| 37. $\cot 300^\circ$ | 38. $\sin 450^\circ$ | 39. $\cos(-450^\circ)$ | 40. $\tan(-540^\circ)$ | 41. $\csc 90^\circ$ |
| 42. $\sin 840^\circ$ | 43. $\sin(-690^\circ)$ | 44. $\cot 215^\circ$ | 45. $\sec 150^\circ$ | 46. $\tan 330^\circ$ |

Find the trigonometric function value for each angle to four decimal places.

- | | | | | |
|------------------------|-----------------------|--------------------------|----------------------------|--------------------------|
| 47. $\sin 113.4^\circ$ | 48. $\cos 88.2^\circ$ | 49. $\tan 214.6^\circ$ | 50. $\csc 345^\circ 10'$ | 51. $\cot 412^\circ$ |
| 52. $\tan 527.2^\circ$ | 53. $\sec(-13^\circ)$ | 54. $\sin(-88^\circ)$ | 55. $\cos(-355^\circ 20')$ | 56. $\tan(-248.6^\circ)$ |
| 57. $\csc 285.3^\circ$ | 58. $\sec 211^\circ$ | 59. $\cos(-133.2^\circ)$ | 60. $\sin(-293^\circ 50')$ | |

Find one approximate solution to each equation, to the nearest 0.1° .

- | | | | | |
|--------------------------|---------------------------------|---------------------------|---------------------------|----------------------------------|
| 61. $\sin \theta = 0.25$ | 62. $\sin \theta = \frac{1}{3}$ | 63. $\cos \theta = -0.5$ | 64. $\cos \theta = 0.813$ | 65. $\tan \theta = -\frac{8}{5}$ |
| 66. $\tan \theta = 3$ | 67. $\sin \theta = -0.59$ | 68. $\cos \theta = -0.18$ | | |

69. In a certain electrical circuit the instantaneous voltage E (in volts) is found by the formula $E = 156 \sin(\theta + 45^\circ)$. Compute E to the nearest 0.01 volt for the following values of θ :
a. 0° b. 45° c. 100° d. -200° e. 13.3° f. -45°
70. In a certain electrical circuit the instantaneous current I (in amperes) is found by the formula $I = 1.6 \cos(800t)^\circ$. Find I to the nearest 0.01 ampere for the following values of t :
a. 0 b. 0.25 c. 0.85 d. 1 e. -1 f. -2.5 g. -0.02
71. If a force of 200 pounds is applied to a rope to drag an object, the actual force tending to move the object horizontally is $f(\theta) = 200 \cos \theta$, where θ is the angle the rope makes with the horizontal. Compute the force tending to move the object horizontally if the angle of the rope is
a. 0° b. 25° c. 50°
72. If a rocket is moving through the air at a speed of 1,200 mph, at an angle of θ° with the horizontal, then the rate at which it is rising is $v(\theta) = 1200 \sin \theta$. Find the rate at which a rocket moving at 1200 mph is rising if the angle it makes with the horizontal is
a. 50° b. 60° c. 70° d. 80°
73. Use the values 30° and 60° to see if the statement $\sin(2\theta) = 2 \sin \theta$ is true. (Let θ be 30° .)
74. Use the values 30° and 60° to see if the statement $\sin \frac{\theta}{2} = \frac{\sin \theta}{2}$ is true. (Let θ be 60° .)
75. Use the values 30° , 60° , 90° to see if the statement $\sin(\alpha + \beta) = \sin \alpha + \sin \beta$ is true.

2-4 Finding values from other values—reference triangles

Finding a general angle from a value and quadrant

In sections 1-3 and 1-4 we learned how to find the degree measure of an acute angle if we know the value of one of the trigonometric ratios for that angle. We used the inverse sine, cosine, or tangent function as appropriate. We are now dealing with angles of any measure, but the same procedure can be used to find the value of a reference angle. From this we can find the least nonnegative measure for an angle.

As is illustrated in example 2-4 A, we *always find a reference angle θ' by finding the inverse sine, cosine, or tangent function value for a positive value*. We use a positive value to obtain an acute angle (all reference angles are acute). We could summarize the procedure as follows.

Finding the least nonnegative measure of an angle from a trigonometric function value and information about a quadrant.

1. If necessary use the ASTC rule² to determine the quadrant for the terminal side of the angle.
2. Use $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ to find θ' . Use the absolute value of the given trigonometric function value.
3. Apply θ' to the correct quadrant to determine the value of θ .

Note We find the “least nonnegative value.” There are actually an unlimited number of values, since the trigonometric values are the same for all coterminal angles.

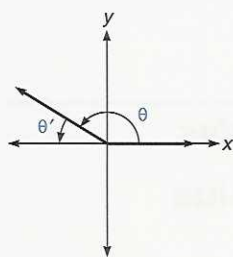
In section 2-3 we saw formulas that find θ' if $0^\circ < \theta < 360^\circ$. These formulas can be solved for θ if necessary and thus provide a formula for finding θ given θ' .

Relationship between θ and θ' if $0^\circ < \theta < 360^\circ$

θ in Quadrant I:	$\theta' = \theta$	$\theta = \theta'$
θ in Quadrant II:	$\theta' = 180^\circ - \theta$	$\theta = 180^\circ - \theta'$
θ in Quadrant III:	$\theta' = \theta - 180^\circ$	$\theta = \theta' + 180^\circ$
θ in Quadrant IV:	$\theta' = 360^\circ - \theta$	$\theta = 360^\circ - \theta'$

It is interesting to observe that the formulas are the “same” when solved for θ and θ' for every quadrant except quadrant III.

Example 2-4 A



Find the least nonnegative measure of θ to the nearest 0.1° .

1. $\sin \theta = 0.5150$ and $\cos \theta < 0$

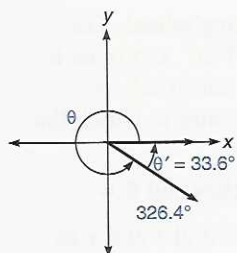
Since $\sin \theta > 0$ and $\cos \theta < 0$, θ terminates in quadrant II (see the figure). We find the acute reference angle θ' just as we did in section 1-2.

$$\theta' = \sin^{-1} 0.5150 \approx 31.0^\circ$$

$$\text{Thus, } \theta \approx 180^\circ - 31.0^\circ = 149.0^\circ.$$

Note The calculator can be used to verify our result by checking that $\sin 149^\circ \approx 0.5150$ and that $\cos 149^\circ < 0$.

²Section 2-3.



2. $\tan \theta = -0.6644$ and $\sin \theta < 0$

Since $\tan \theta < 0$ and $\sin \theta < 0$, θ terminates in quadrant IV.

$$\theta' = \tan^{-1} 0.6644 \approx 33.6^\circ$$

Note we use the positive value 0.6644

$$\theta = 360^\circ - \theta' \approx 326.4^\circ$$

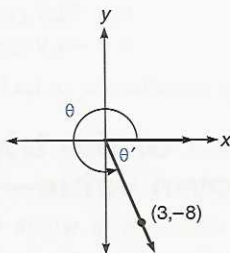
Example 2-4 B applies several of the things we have been studying.

■ Example 2-4 B

The point $(3, -8)$ is on the terminal side of θ .

- Draw a representation of θ .
- Find the exact value of the trigonometric functions for θ .
- Find the least nonnegative measure of θ , to the nearest 0.1° .

a. The representation is shown in the figure.



b. Applying the definitions of section 2-2:

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + (-8)^2} = \sqrt{73}$$

$$\sin \theta = \frac{y}{r} = \frac{-8}{\sqrt{73}} = -\frac{8\sqrt{73}}{73}, \quad \csc \theta = \frac{1}{\sin \theta} = -\frac{\sqrt{73}}{8}$$

$$\cos \theta = \frac{x}{r} = \frac{3}{\sqrt{73}} = \frac{3\sqrt{73}}{73}, \quad \sec \theta = \frac{1}{\cos \theta} = \frac{\sqrt{73}}{3}$$

$$\tan \theta = \frac{y}{x} = -\frac{8}{3}, \quad \cot \theta = \frac{1}{\tan \theta} = -\frac{3}{8}$$

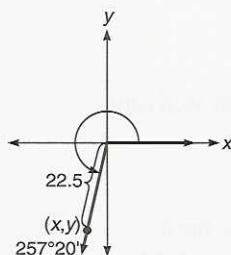
c. We can find θ' by using the fact that $\tan \theta' = |\tan \theta| = \frac{8}{3}$, so that

$$\theta' = \tan^{-1} \frac{8}{3} \approx 69.4^\circ, \text{ so}$$

$$\theta \approx 360^\circ - 69.4^\circ = 290.6^\circ$$

There are many places in science and technology where we find applications for trigonometric functions. With the advent of numerically controlled, or computer-controlled, machines these applications are becoming more common.

Example 2-4 C



A technician is setting up a numerically controlled grinding wheel. The starting position for the wheel must be at an angle of $257^\circ 20'$ and must be 22.5 inches from the origin (assuming the machine uses our usual x - y coordinate system). Find the x - and y -coordinates of the point at which the grinding wheel must start, to the nearest tenth of an inch.

The figure illustrates the situation. We have $r = 22.5$ inches and $\theta = 257^\circ 20'$. By definition, $\sin \theta = \frac{y}{r}$ and $\cos \theta = \frac{x}{r}$, so we find x and y as follows:

$$\sin 257^\circ 20' = \frac{y}{22.5}$$

$$y = 22.5 \sin 257^\circ 20'$$

$$y \approx -22.0 \text{ inches}$$

CS 1

$$\cos 257^\circ 20' = \frac{x}{22.5}$$

$$x = 22.5 \cos 257^\circ 20'$$

$$x \approx -4.9 \text{ inches}$$

Thus, the starting coordinates, in inches, for the grinder are $(-4.9, -22.0)$. ■

Exact values of the trigonometric functions from a known value—reference triangles

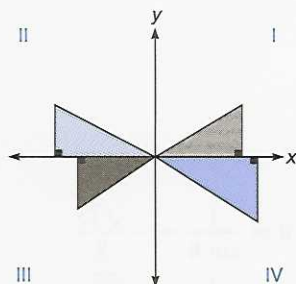
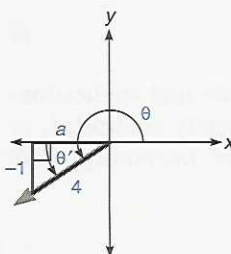


Figure 2-9

There are many situations in which we know the exact value of one of the trigonometric functions for a given angle and need to find the exact value of one or more of the remaining five trigonometric functions for the same angle. We can do this by using a reference triangle, which is a convenient way of combining the idea of reference angle and right triangle. A **reference triangle** is a right triangle with one leg on the x -axis and one leg parallel to the y -axis. The acute angle on the x -axis is the reference angle for the angle in question. *The lengths of the legs of a reference triangle are treated as directed distances (i.e., positive or negative); the hypotenuse is always positive.* This is illustrated in example 2-4 D. Figure 2-9 shows a reference triangle for each quadrant.

Example 2-4 D



In each case draw a representation of angle θ and use a reference triangle to help find the values of the other five trigonometric functions. Also, find the least positive value of θ to the nearest 0.1° .

1. $\sin \theta = -\frac{1}{4}$ and $\tan \theta > 0$

We know θ terminates in quadrant III since $\sin \theta < 0$ and $\tan \theta > 0$.

We construct a right triangle in quadrant III in which one acute angle is a reference angle. This is shown in the figure. We label the hypotenuse 4 and the directed side opposite θ' as -1 . Thus,

$$\sin \theta' = \frac{\text{length of side opposite } \theta'}{\text{length of hypotenuse}} = -\frac{1}{4}.$$

$$a^2 + (-1)^2 = 4^2 \quad \text{Find the value of } |a| \text{ using the Pythagorean theorem; since we are squaring values this theorem works for directed distances}$$

$$a^2 = 15$$

$$a = \pm\sqrt{15}$$

We choose $a = -\sqrt{15}$ since it is negative as a directed distance.

We can now use the definitions of the trigonometric ratios for θ' along with the directed distances to find the remaining trigonometric function values for θ .

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = -\frac{\sqrt{15}}{4}, \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{-1}{-\sqrt{15}} = \frac{\sqrt{15}}{15}$$

$$\csc \theta = \frac{1}{\sin \theta} = -4, \quad \sec \theta = \frac{1}{\cos \theta} = -\frac{4}{\sqrt{15}} = -\frac{4\sqrt{15}}{15}$$

$$\cot \theta = \frac{1}{\tan \theta} = \sqrt{15}$$

We now find an approximation to θ .

$$\sin \theta' = \frac{1}{4}, \text{ so } \theta' = \sin^{-1} \frac{1}{4} \approx 14.5^\circ, \text{ so } \theta \approx 180^\circ + 14.5^\circ = 194.5^\circ.$$

Note The reference triangle works because it is equivalent to finding a point on the terminal side of θ and applying the definitions of the trigonometric functions (section 2-2). The reference triangle above was equivalent to finding the point $(-\sqrt{15}, -1)$ to be on the terminal side of angle θ . (Figure 2-10)

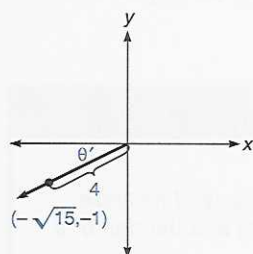
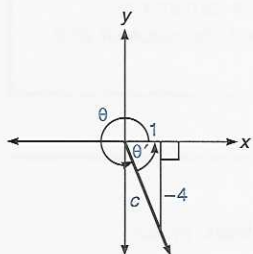


Figure 2-10



2. $\cot \theta = -\frac{1}{4}$ and $270^\circ < \theta < 360^\circ$

If $\cot \theta = -\frac{1}{4}$ then $\tan \theta = -4$. The figure shows a reference triangle for an angle in quadrant IV with tangent -4 .

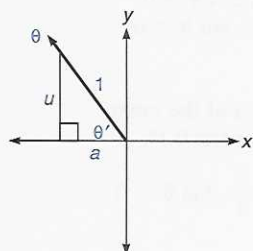
$$c^2 = 1^2 + (-4)^2$$

$$c = \sqrt{17}$$

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{-4}{\sqrt{17}} = -\frac{4\sqrt{17}}{17}, \quad \cos \theta = \frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17},$$

$$\csc \theta = \frac{1}{\sin \theta} = -\frac{\sqrt{17}}{4}, \quad \sec \theta = \frac{1}{\cos \theta} = \sqrt{17},$$

$$\tan \theta' = 4, \text{ so } \theta' = \tan^{-1} 4 \approx 76.0^\circ, \text{ and } \theta = 360^\circ - \theta' \approx 284^\circ.$$



3. $\sin \theta = u$ and θ terminates in quadrant II

The figure shows a reference triangle in quadrant II, where $\sin \theta' = u$.

$$u^2 + a^2 = 1^2 \quad \text{Find the value of side } a$$

$$a = \pm \sqrt{1 - u^2}$$

$$a = -\sqrt{1 - u^2} \quad \text{Choose } a < 0 \text{ as a directed distance}$$

$$\cos \theta = \frac{a}{1} = a = -\sqrt{1 - u^2},$$

$$\tan \theta = \frac{u}{a} = \frac{u}{-\sqrt{1 - u^2}} = -\frac{u}{\sqrt{1 - u^2}},$$

$$\sec \theta = \frac{1}{\cos \theta} = -\frac{1}{\sqrt{1 - u^2}}, \quad \csc \theta = \frac{1}{\sin \theta} = \frac{1}{u},$$

$$\cot \theta = \frac{1}{\tan \theta} = -\frac{\sqrt{1 - u^2}}{u}$$

Since we do not know the actual value of u we cannot make a determination of an approximate value for angle θ . ■

Calculator steps

1. (A) $22.5 \times (257 + 20 \div 60) \sin =$
 Display -21.9524013
- (P) $22.5 \text{ ENTER } 257 \text{ ENTER } 20 \text{ ENTER } 60$
 $\div + \sin \times$
- TI-81 $22.5 \times \text{SIN} (257 + 20 \div 60)$
 ENTER

Mastery points**Can you**

- Find an approximation to the least nonnegative measure of an angle, given the value of one of the trigonometric functions and the sign of a second for that angle?
- Apply the definitions of the trigonometric functions in appropriate situations?
- Use reference triangles to find the exact values of the remaining trigonometric functions for a given angle, when given the value of one of the trigonometric functions of that angle?

Exercise 2-4

Find the measure of the least nonnegative angle that meets the conditions given in the following problems, to the nearest 0.1° .

- | | | |
|--|--|--|
| 1. $\sin \theta = 0.8251$, $\cos \theta > 0$ | 2. $\cos \theta = -0.1771$, $\sin \theta < 0$ | 3. $\tan \theta = 0.6569$, $\sec \theta > 0$ |
| 4. $\sin \theta = -0.6508$, $\tan \theta > 0$ | 5. $\sec \theta = -1.0642$, $\sin \theta < 0$ | 6. $\csc \theta = -1.3673$, $\tan \theta > 0$ |
| 7. $\tan \theta = -0.0349$, $\csc \theta < 0$ | 8. $\cos \theta = -0.2222$, $\sin \theta > 0$ | 9. $\sin \theta = \frac{3}{8}$, $\cos \theta > 0$ |
| 10. $\sin \theta = \frac{3}{8}$, $\cos \theta < 0$ | 11. $\cot \theta = -5$, $\sin \theta > 0$ | 12. $\tan \theta = -5$, $\sin \theta < 0$ |
| 13. $\cos \theta = -\frac{5}{7}$, $\tan \theta > 0$ | 14. $\cos \theta = -\frac{5}{7}$, $\tan \theta < 0$ | |

In each case (a) draw a representation of angle θ and (b) use a reference triangle to help find the values of the other trigonometric functions. Also, (c) find the reference angle θ' and the least positive value of θ to the nearest 0.1° .

- | | | |
|---|--|--|
| 15. $\sin \theta = \frac{3}{4}$, $\cos \theta > 0$ | 16. $\sin \theta = \frac{4}{5}$, $\cos \theta < 0$ | 17. $\cos \theta = -\frac{1}{2}$, $\tan \theta > 0$ |
| 18. $\cos \theta = -\frac{5}{13}$, $\tan \theta < 0$ | 19. $\sin \theta = 1$ | 20. $\cos \theta = 1$ |
| 21. $\tan \theta = 2$, $\cos \theta < 0$ | 22. $\tan \theta = 3$, $\cos \theta > 0$ | 23. $\csc \theta = -5$, $\sec \theta < 0$ |
| 24. $\csc \theta = -2$, $\sec \theta > 0$ | 25. $\csc \theta = -1$ | 26. $\sec \theta = -1$ |
| 27. $\sin \theta = -\frac{3}{4}$, $\tan \theta > 0$ | 28. $\sin \theta = -\frac{2}{5}$, $\tan \theta < 0$ | 29. $\sec \theta = 4$, $\csc \theta > 0$ |
| 30. $\sec \theta = \sqrt{6}$, $\csc \theta < 0$ | 31. $\cot \theta = \frac{\sqrt{2}}{3}$, $\sin \theta < 0$ | 32. $\cot \theta = \frac{1}{3}$, $\sin \theta > 0$ |
| 33. $\cos \theta = -\frac{5}{13}$, $\sin \theta > 0$ | 34. $\cos \theta = -\frac{3}{\sqrt{10}}$, $\sin \theta < 0$ | 35. $\tan \theta = \frac{7}{2}$, $\sec \theta < 0$ |

36. $\tan \theta = \frac{7}{3}$, $\sec \theta > 0$

37. $\sec \theta = 5$, $\tan \theta > 0$

38. $\sec \theta = 4$, $\tan \theta < 0$

39. $\sin \theta = \frac{1}{\sqrt{5}}$, $\tan \theta < 0$

40. $\sin \theta = \frac{1}{\sqrt{3}}$, $\tan \theta > 0$

Solve the following problems.

41. The point
- $(2, -5)$
- is on the terminal side of
- θ
- .

- Find the exact value of each of the trigonometric functions for θ .
- Find the least nonnegative measure of θ , to the nearest 0.1° .

42. The point
- $(-3, -9)$
- is on the terminal side of
- θ
- .

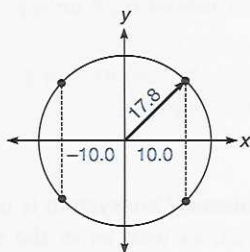
- Find the exact value of each of the trigonometric functions for θ .
- Find the least nonnegative measure of θ , to the nearest 0.1° .

43. A numerically controlled drill is being set up to drill a hole in a piece of steel 6.8 millimeters from the origin at an angle of
- $135^\circ 30'$
- . To the nearest 0.01 millimeter, what are the coordinates of this point?

44. Suppose the hole in problem 43 must be 10.25 inches from the origin at an angle of
- $13^\circ 20'$
- . Find the coordinates of this point to the nearest 0.01 inch.

45. Suppose the hole in problem 43 must be 8.25 centimeters from the origin at an angle of
- -134.4°
- . Find the coordinates of this point to the nearest 0.01 centimeter.

46. A numerically controlled drill must drill four holes on a circle whose center is at the origin with radius 17.8 centimeters, as shown in the diagram. The holes must be drilled wherever on this circle the
- x
- coordinate is
- ± 10.0
- centimeters. Find the
- y
- coordinate and angle (to the nearest
- 0.1°
-) for each of these four holes.



47. Suppose in problem 46 four additional holes must be drilled wherever the
- y
- coordinate is
- ± 15.5
- cm. Find the
- x
- coordinate and angle for each of these holes.

48. A technician is aligning a laser device that is used to cut patterns out of cloth. The device is positioned at an angle of
- 135.20°
- and at a distance 5.50 feet from the origin. What should the
- x
- and
- y
- coordinates be at this point, to the nearest 0.01 foot?

49. A scanning device used in medical diagnosis has a moving part that moves with great precision in a circle around the patient. Assume the
- y
- axis is perpendicular to the top of the table on which the patient lies and the
- x
- axis is at right angles to the length of the table. The diameter of the machine is 4 feet 3.5 inches. Find the coordinates of the moving part when the angle is
- 211.5°
- , to the nearest 0.1 inch.

In problems 50–55 find values of the other five trigonometric functions in terms of u .

- 50.
- $\cos \theta = u$
- and
- θ
- terminates in quadrant I.

- 51.
- $\tan \theta = u$
- and
- θ
- terminates in quadrant I.

- 52.
- $\cos \theta = u$
- and
- θ
- terminates in quadrant III.

- 53.
- $\tan \theta = u$
- and
- θ
- terminates in quadrant III.

- 54.
- $\sin \theta = u + 1$
- and
- θ
- terminates in quadrant I.

- 55.
- $\cos \theta = 1 - u$
- and
- θ
- terminates in quadrant I.

56. In the June 1980 issue of
- Popular Science*
- magazine Mr. R. J. Ransil presented several formulas for calculating saw angles for compound miters. The formulas are:

$$\text{angle } A = 90^\circ - \frac{180^\circ}{\text{number of sides}}$$

$$\tan(\text{arm angle}) = \cot(A) \cdot \sin(\text{slope})$$

$$\sin(\text{tilt angle}) = \cos(A) \cdot \cos(\text{slope})$$

Arm angles and tilt angles are acute.

Calculate the arm angle and tilt angle to the nearest 0.1° for the following numbers of sides and slopes:

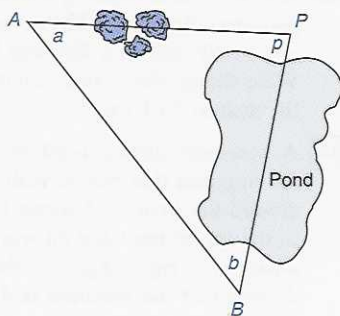
Number of sides	Slope (in degrees)
3	5
5	5
7	25
7	30
6	35
8	35

57. A surveying manual describes how to find distance BP in the figure. The distance AP can be found, but trees prevent measuring angle a . Angle b can be measured, but not distance BP . The manual instructs the surveyor to find BP by solving the following sequence of formulas:

$$\sin p = \frac{AB \sin b}{AP}$$

$$a = 180^\circ - (b + p)$$

$$BP = \frac{AP \sin a}{\sin b}$$



Note that the first formula does not give angle p but only $\sin p$. Also, assume p is acute. Solve the sequence of formulas to compute the distance BP to the nearest 0.1 foot if $AB = 512.4$ feet, $AP = 322.6$ feet, and $b = 28.3^\circ$.

58. Find BP in problem 57 to the nearest 0.1 meter if $AB = 319.2$ meters, $AP = 225.7$ meters, and $b = 31.6^\circ$.

2-5 Radian measure—definitions

The unit circle

The circle with radius one and center at the origin is described by the equation

$$x^2 + y^2 = 1$$

It is called the unit circle. See figure 2-11. Observe that the absolute values of the x - and y -coordinates of any point not on an axis describe the lengths of two sides of a right triangle with hypotenuse of length one. The Pythagorean theorem shows that for these points $x^2 + y^2 = 1$. Those points of the circle that are on an axis also satisfy this equation.

The circumference C of a circle with radius r is the distance around the circle. This distance is found using the relation $C = 2\pi r$. Since the radius r for the unit circle is one, its circumference is $C = 2\pi$ (about 6.28 units).

Note The constant π is approximately 3.14159. It is a much-used number, about which entire books have been written. It is an irrational number, and has been approximated to over a billion digits!³

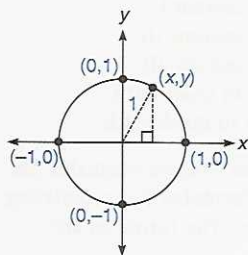


Figure 2-11

Radian measure

A second system of angle measurement is called **radians**. This system is used extensively in engineering and scientific applications, as well as in the calculus. We will use it throughout the rest of this book. To define this system of angle measurement we use the unit circle.

Let θ be an angle in standard position, and let s represent the distance from the point $(1,0)$ along the circumference of the unit circle to the terminal

³An interesting book on π is *A History of π* by Petr Beckmann, Golem Press, Boulder, Colo., 1977. Gregory V. and David V. Chudnovsky of Columbia University calculated 1,011,196,691 digits of π in 1989.

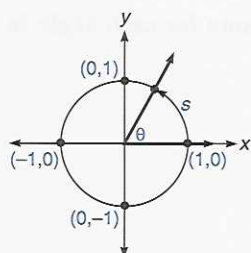


Figure 2-12

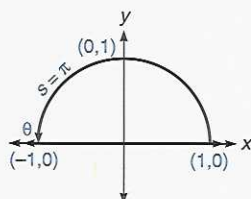


Figure 2-13

side of θ . The distance s is called the **arc length** (see figure 2-12). If the distance is measured in a clockwise direction we say s is positive, and if in a counterclockwise direction s is negative.

We define the radian measure of an angle to be this arc length s .

Radian measure of an angle in standard position

Let θ be an angle in standard position. Let s be the corresponding arc length on the unit circle. Let s be positive if measured in the counterclockwise direction, and negative if measured in the clockwise direction.

Then s is the radian measure of the angle θ .

For example, an angle of degree measure 180° has an arc length that corresponds to half the circumference of the unit circle. Thus, the corresponding radian measure is half of the circumference, or one-half of 2π , which is π . Thus, the radian measure of an angle that corresponds to a rotation of one-half a circle, in the counterclockwise direction, is π (see figure 2-13).

Conversions between radian and degree measure

Since 360° corresponds to a full revolution, and the circumference of the unit circle (2π) also corresponds to a complete revolution about the unit circle, the following relation is true.

$$\frac{\text{arc length } (s)}{\text{circumference } (2\pi)} = \frac{\text{measure of angle in degrees}}{360^\circ}$$

If we multiply each member by 2 we obtain the same true statement, but with smaller denominators of π and 180° .

We use this proportion⁴ to convert between degree and radian measure.

Radian/degree proportion

Let θ be an angle in standard position with degree measure θ° and radian measure s . Then,

$$\frac{s}{\pi} = \frac{\theta^\circ}{180^\circ}$$

The radian measure of an angle is a real number, defined with no units in mind. We often add the word radians after such a measure, but this is not necessary where it is clear that the real number refers to the measure of an angle. Observe that in the radian/degree proportion the ratio of degrees to degrees is unitless also. For example, $\frac{90^\circ}{180^\circ}$ is the same as the unitless ratio $\frac{1}{2}$.

We can describe the measure of an angle in standard position by using degree measure or by stating the arc length to which the angle corresponds on the unit circle (its radian measure). The proportion above shows the relationship between these two systems.

⁴A proportion is a statement of equality between two ratios (fractions).

■ **Example 2-5 A**

Compute the radian or degree measure, given the measure for each angle in degrees or radians.

1. 90°

$$\frac{s}{\pi} = \frac{\theta^\circ}{180^\circ}$$

Radian/degree proportion

$$\frac{s}{\pi} = \frac{90^\circ}{180^\circ}$$

Replace θ° by 90°

$$s = \frac{90(\pi)}{180}$$

Multiply each member by π ; drop the reference to degrees

$$s = \frac{\pi}{2}$$

Simplify the fraction

Thus, 90° corresponds to $\frac{\pi}{2}$ (radians).

2. -210°

$$\frac{s}{\pi} = \frac{\theta^\circ}{180^\circ}$$

Radian/degree proportion

$$\frac{s}{\pi} = \frac{-210^\circ}{180^\circ}$$

Replace θ° with -210°

$$s = \frac{-210(\pi)}{180^\circ} = -\frac{7\pi}{6}$$

Therefore, -210° corresponds to $-\frac{7\pi}{6}$.

3. $\frac{7\pi}{5}$

$$\frac{s}{\pi} = \frac{\theta^\circ}{180^\circ}$$

Radian/degree proportion

$$\frac{\frac{7\pi}{5}}{\pi} = \frac{\theta^\circ}{180^\circ}$$

Replace s by $\frac{7\pi}{5}$

$$\frac{7\pi}{5} \cdot \frac{1}{\pi} = \frac{\theta^\circ}{180^\circ}$$

Division by π is the same as multiplication by $\frac{1}{\pi}$

$$\frac{7}{5} \cdot 180^\circ = \theta^\circ$$

Multiply each member by 180°

$$252^\circ = \theta^\circ$$

$\frac{7\pi}{5}$ (radians) corresponds to 252° .

4. 1

Note that this means $s = 1$ radian.

$$\frac{s}{\pi} = \frac{\theta^\circ}{180^\circ} \quad \text{Radian/degree proportion}$$

$$\frac{1}{\pi} = \frac{\theta^\circ}{180^\circ}$$

$$\frac{180^\circ}{\pi} = \theta^\circ \quad \text{Multiply each member by } 180^\circ$$

$$\frac{180^\circ}{\pi} = \theta^\circ$$

$$57.30^\circ \approx \theta^\circ \quad \text{Decimal approximation to } \frac{180}{\pi}$$

Thus, 1 radian corresponds to $\frac{180^\circ}{\pi}$ or about 57.3° . ■

Note It is useful to remember that 1 radian is a little less than 60° , and that 2π radians exactly equals 360° .

Common radian measures

The unit circle can be very helpful in getting a feeling for radian measure. Those values of radian measure that correspond to quadrantal angles (0° , 90° , 180° , etc.) and to angles with reference angles of 30° , 45° , and 60° are common.

In particular, the following correspondences are useful: $\frac{\pi}{6}$ and 30° , $\frac{\pi}{4}$ and

45° , and $\frac{\pi}{3}$ and 60° . The unit circle can be conveniently marked in terms of

multiples of $\frac{\pi}{6}$ radians (30°) and of multiples of $\frac{\pi}{4}$ radians (45°). This is shown in figure 2-14.

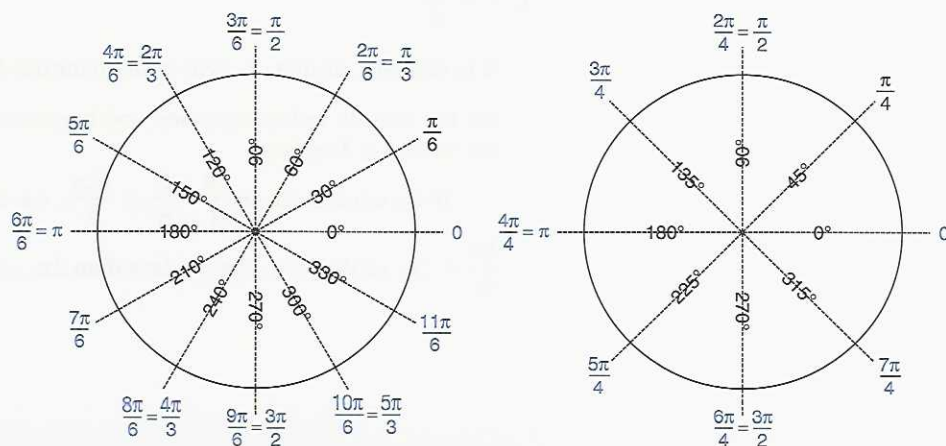


Figure 2-14

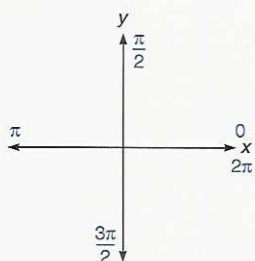


Figure 2-15

Coterminal and reference angles in radian measure

Figure 2-15 shows the smallest positive radian measure of the quadrantal angles, as well as the fact that a full revolution (circle) can be described by 2π radians. Observe that the quadrantal angles 0° , 90° , 180° , 270° , and 360° are, in radians, 0 , $\frac{\pi}{2}$, π , $\frac{3\pi}{2}$, and 2π .

In degree measure, all angles that differ in measure by integer multiples of 360° are coterminal. For radian measure the difference is multiples of 2π . If k is an integer (positive, zero, or negative), then integer multiples of 2π are $k \cdot 2\pi$, or $2k\pi$.

Coterminal angles, radian measure

Two angles α and β are coterminal if $\alpha = \beta + 2k\pi$, k an integer.

Reference angles are found in the same manner as with degree measure (section 2-2) except that 180° becomes π and 360° becomes 2π . If the measure of θ in radians is positive and less than 2π , the following rules give the value of θ' , the reference angle.

Quadrant in which θ terminates	Value of θ' , the reference angle
I	$\theta' = \theta$
II	$\theta' = \pi - \theta$
III	$\theta' = \theta - \pi$
IV	$\theta' = 2\pi - \theta$

Example 2-5 B

For each angle θ find the least positive coterminal angle α (that is, $0 \leq \alpha < 2\pi$) and then find the reference angle θ' .

1. $\theta = \frac{7\pi}{6}$

It is difficult, at first, to deal with quantities like $\frac{7\pi}{6}$. This is because we are not used to radian measure, and because it can be difficult to compare the values of fractions.

If we convert 2π to $\frac{2\pi}{1} \cdot \frac{6}{6} = \frac{12\pi}{6}$, we can see that $\frac{7\pi}{6} < \frac{12\pi}{6}$, or $\frac{7\pi}{6} < 2\pi$, so that θ is already less than 2π , so $\alpha = \theta = \frac{7\pi}{6}$.

To determine the reference angle for θ we must determine in which quadrant $\frac{7\pi}{6}$ terminates. If you cannot see that it terminates in quadrant III then use the following method.

Rewrite the quadrantal angles in terms of denominators of 6, the denominator of $\frac{7\pi}{6}$:

Quadrant	I	II	III	IV
Quadrantal Angle	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
Denominator of 6	$\frac{0}{6}$	$\frac{3\pi}{6}$	$\frac{6\pi}{6}$	$\frac{9\pi}{6}$

Now observe that $\frac{6\pi}{6} < \frac{7\pi}{6} < \frac{9\pi}{6}$ or $\pi < \frac{7\pi}{6} < \frac{3\pi}{2}$, so $\frac{7\pi}{6}$ is in quadrant III.

$$\theta' = \frac{7\pi}{6} - \pi = \frac{7\pi}{6} - \frac{6\pi}{6} = \frac{\pi}{6} \quad \text{In quadrant III, } \theta' = \theta - \pi$$

Thus, the reference angle for $\frac{7\pi}{6}$ is $\frac{\pi}{6}$.

2. $\theta = \frac{11\pi}{3}$

Since $2\pi = \frac{6\pi}{3}$, we see that $\theta > 2\pi$. We must subtract multiples of 2π until we arrive at an angle α , $0 \leq \alpha < 2\pi$.

$$\frac{11\pi}{3} - 2\pi = \frac{11\pi}{3} - \frac{6\pi}{3} = \frac{5\pi}{3}$$

Since $\frac{5\pi}{3} < 2\pi$, the angle α is $\frac{5\pi}{3}$. Thus, $\frac{11\pi}{3}$ is coterminal with $\frac{5\pi}{3}$. To

locate in which quadrant the angle $\frac{5\pi}{3}$ terminates, rewrite $\frac{5\pi}{3}$ as $\frac{10\pi}{6}$ and the quadrantal angles in terms of denominators of 6. (It is not convenient to rewrite the quadrantal angles in terms of denominators of 3.)

Examining the values used in part 1 of this example we see that $\frac{9\pi}{6} < \frac{10\pi}{6} < \frac{12\pi}{6}$ so $\frac{3\pi}{2} < \frac{5\pi}{3} < 2\pi$ and $\frac{5\pi}{3}$ is in quadrant IV. Thus,

$$\theta' = 2\pi - \frac{5\pi}{3} = \frac{6\pi}{3} - \frac{5\pi}{3} = \frac{\pi}{3}.$$

$$3. \theta = -\frac{13\pi}{4}$$

We first add multiples of $2\pi = \frac{8\pi}{4}$ to obtain a positive valued coterminal angle α . It is clear that one multiple of $\frac{8\pi}{4}$ will not give a positive result, so we will add two multiples.

$$-\frac{13\pi}{4} + 2\left(\frac{8\pi}{4}\right) = -\frac{13\pi}{4} + \frac{16\pi}{4} = \frac{3\pi}{4} = \alpha$$

To find a reference angle we will use $\frac{3\pi}{4}$ instead of $-\frac{13\pi}{4}$. We rewrite the quadrantal angles in terms of denominators of 4.

Quadrant	I	II	III	IV
Quadrantal Angle	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
Denominator of 6	$\frac{0}{4}$	$\frac{2\pi}{4}$	$\frac{4\pi}{4}$	$\frac{6\pi}{4}$

Since $\frac{2\pi}{4} < \frac{3\pi}{4} < \frac{4\pi}{4}$, we see that $\frac{\pi}{2} < \frac{3\pi}{4} < \pi$, so $\theta = \frac{3\pi}{4}$ is in quadrant II, and $\theta' = \pi - \frac{3\pi}{4} = \frac{4\pi}{4} - \frac{3\pi}{4} = \frac{\pi}{4}$.

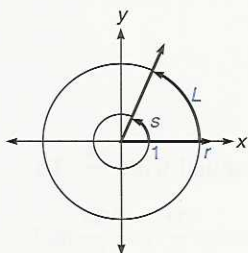


Figure 2-16

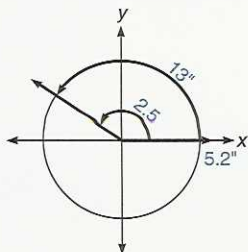
Radian measure and arc length in any circle

A simple relation exists between the radian measure s of an angle θ and arc length L determined by that angle on the circumference of any circle (figure 2-16). Geometry tells us that corresponding parts of similar figures form equal ratios. This means, in this case, that $\frac{s}{1} = \frac{L}{r}$, or $L = rs$. Thus, if s is the radian measure of an angle with vertex at the center of a circle of radius r , and L is the corresponding arc length, then

$$L = rs$$

Thus, the arc length L on any circle equals the product of the radius of the circle and the radian measure of the related angle.

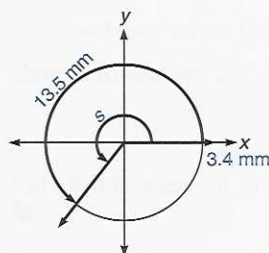
Example 2-5 C



Use the relation $L = rs$ to solve each problem.

- Find the length of the arc determined by a central angle of measure 2.5 radians on a circle of radius 5.2 inches.

$$\begin{aligned} L &= rs \\ L &= 5.2(2.5) && \text{Replace } r \text{ with } 5.2, s \text{ with } 2.5 \\ &= 13 \text{ inches} \end{aligned}$$



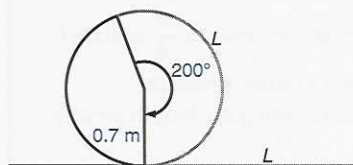
2. Find the measure in radians of the central angle corresponding to an arc length of 13.5 mm on a circle of diameter 6.8 mm.

$$\begin{aligned}
 r &= 3.4 \text{ mm} && \text{One-half the diameter} \\
 L &= rs \\
 13.5 \text{ mm} &= (3.4 \text{ mm})s && \text{Substitute known values} \\
 \frac{13.5}{3.4} &= s \\
 3.97 &\approx s && \text{Rounding to nearest 0.1}
 \end{aligned}$$

Thus, the central angle measures 3.97 radians.

3. A railroad car has wheels with diameter 1.4 m (meters). If the wheels move through an angle of 200° , how far does the train move?

As illustrated, the distance the train will move is the same as the arc length L on the wheel. This length is determined by the central angle of 200° . We will find the measure of the central angle, θ , in radians, then use the relation $L = rs$.



$$\begin{aligned}
 \frac{200^\circ}{180^\circ} &= \frac{s}{\pi} && \frac{\theta^\circ}{180^\circ} = \frac{s}{\pi} \\
 \frac{10}{9} &= \frac{s}{\pi} && \text{Reduce} \\
 \frac{10\pi}{9} &= s && \text{Multiply each member by } \pi \\
 L &= rs \\
 L &= 0.7 \left(\frac{10\pi}{9} \right) && \text{The radius } r \text{ is half the diameter of 1.4 m; } s = \theta \text{ (in radians)} \\
 L &\approx 2.4 \text{ m}
 \end{aligned}$$

Thus, the train moves 2.4 meters when the wheels move through an angle of 200° . ■

Area of a sector of a circle

A sector of a circle is that part enclosed between two radii. See the shaded part of figure 2-17 for an example. The area A of a circle with radius r is determined by the equation $A = \pi r^2$. The area of a sector of a circle is proportional to the measure of the central angle θ determining the sector. Thus, the area of a sector with central angle θ with measure s radians is part of a circle determined by

angle $\times$ area of whole circle

$$\frac{s}{2\pi} \times \pi r^2 = \frac{s}{2} r^2$$

Thus, the area of a sector of a circle of radius r is

$$A_s = \frac{s}{2} r^2$$

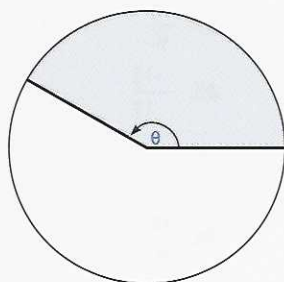


Figure 2-17

Example 2-5 D

Use the formula for the area of a sector of a circle to solve the problem.

Find the area of a sector determined by a central angle of 2 radians in a circle of diameter 8.5''.

The radius r is half the diameter, or 4.25''.

$$A_s = \frac{s}{2}r^2 = \frac{2}{2}(4.25^2) \approx 18.06 \text{ square inches, rounded to the nearest } 0.01 \text{ in.}^2$$

Mastery points**Can you**

- State the equation of the unit circle?
- Convert between degree and radian measure for angles?
- Mark off a unit circle in units of $\frac{\pi}{6}$ radians and in units of $\frac{\pi}{4}$ radians?
- Find the reference angle for an angle given in radian measure?
- Use the relation $L = rs$ to solve problems concerning arc length on any circle?
- Use the relation $A_s = \frac{s}{2}r^2$ to find the area of a sector of a circle?

Exercise 2-5

1. State the algebraic relation (equation) that describes the unit circle.

Convert the following degree measures to radian measures. Leave your answers both in exact form and approximated to two decimal places.

- | | | | | | |
|-----------------|----------------|-----------------|-----------------|------------------|------------------|
| 2. 30° | 3. 45° | 4. 60° | 5. 100° | 6. -200° | 7. -300° |
| 8. -135° | 9. 270° | 10. 750° | 11. 127° | 12. -422° | 13. -305° |

Convert the following radian measures into degree measures. Leave answers both in exact form and approximated to two decimal places.

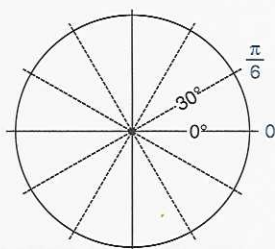
- | | | | | | |
|-----------------------|------------------------|-----------------------|----------------------|-----------------------|----------------------|
| 14. $\frac{5\pi}{2}$ | 15. $\frac{11\pi}{6}$ | 16. $\frac{2\pi}{7}$ | 17. $\frac{3\pi}{5}$ | 18. $\frac{10\pi}{9}$ | 19. $\frac{2\pi}{9}$ |
| 20. $-\frac{5\pi}{3}$ | 21. $-\frac{17\pi}{6}$ | 22. $-\frac{5\pi}{7}$ | 23. $\frac{3}{2}$ | 24. $\frac{11}{6}$ | 25. $-\frac{12}{17}$ |
| 26. 1.5 | 27. 2 | 28. 3.25 | 29. -5 | 30. -6 | |

Find the reference angle for the following angles.

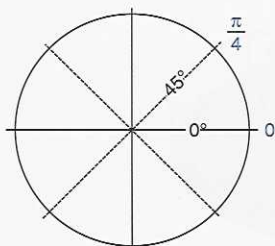
- | | | | | | |
|-----------------------|----------------------|-----------------------|----------------------|-----------------------|-----------------------|
| 31. $\frac{2\pi}{3}$ | 32. $\frac{5\pi}{4}$ | 33. $\frac{11\pi}{6}$ | 34. $\frac{7\pi}{6}$ | 35. $\frac{4\pi}{3}$ | 36. $\frac{5\pi}{6}$ |
| 37. $\frac{3\pi}{4}$ | 38. $\frac{7\pi}{4}$ | 39. $\frac{5\pi}{3}$ | 40. $-\frac{\pi}{4}$ | 41. $-\frac{2\pi}{3}$ | 42. $-\frac{5\pi}{3}$ |
| 43. $-\frac{7\pi}{6}$ | 44. $-\frac{\pi}{6}$ | | | | |

Solve the following problems.

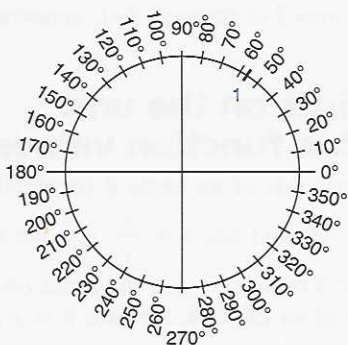
45. The circle is marked off in units of $\frac{\pi}{6}$ (or 30°). Mark each angle shown with its appropriate measure. Reduce fractions.



46. The circle is marked off in units of $\frac{\pi}{4}$ (or 45°). Mark each angle shown with its appropriate measure. Reduce fractions.



47. The circle is marked off in degrees. Also shown is the approximate location of 1 radian, which is approximately 57° . Mark off the approximate locations of 2, 3, 4, 5, and 6 radians.



Find the area of the sector of a circle determined by the given angle and radius. Where necessary round the answer to two decimal places.

55. 4, 7 inches

56. $\frac{\pi}{3}$, 10 millimeters

57. $\frac{3\pi}{5}$, 6 centimeters

58. 2.4, 5 inches

59. $\frac{5}{12}$, 6 inches

60. $\frac{6\pi}{5}$, 22 centimeters

61. 15° , 9 millimeters

62. 135° , 24 inches

48. Find the length of the arc determined by a central angle of 2.1 (radians) on a circle of diameter 10 inches.

49. Find the measure, in radians, of a central angle on a circle of radius 4.5 mm (millimeter) determined by an arc length of 12 mm, to the nearest 0.1 radian.

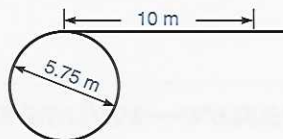
50. Find the length of the arc determined by a central angle of 300° on a circle of diameter 12 mm.

51. Find the length of the arc determined by a central angle of 45° on a circle of radius 8.3 inches, to the nearest 0.1 inch.

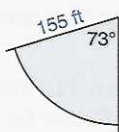
52. Find the measure, in both radians and degrees, of the central angle determined by an arc length of 14.5 mm on a circle with diameter 10.3 mm. Round both answers to the nearest tenth.

53. The diameter of a wheel on an automobile is 32.4 inches. If the wheel moves through an angle of 85° , how far will the car move?

54. The diameter of a wheel that moves the cable of a ski lift is 5.75 meters, as shown in the diagram. Through what angle, in degrees, does the wheel have to move to advance one of the chairs a distance of 10 meters?

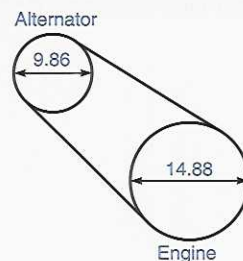


63. Find the radian measure of the central angle necessary to form a sector of area 14.6 cm^2 on a circle of radius 4.85 centimeters. Round the answer to two decimal places.
64. Find the *degree* measure of the central angle necessary to form a sector of area 200 in.^2 on a circle of diameter 50 inches. Round to the nearest 0.1° .
65.  The figure shows a sector of a circle that was painted on the concrete in front of an airport terminal. The radius is now to be extended by 25 feet, to a total of 180 feet. The paint that will cover the unpainted area in the new, larger sector covers about 150 square feet per gallon. How many gallons will it take, to the nearest half gallon, to cover the new, unpainted area?

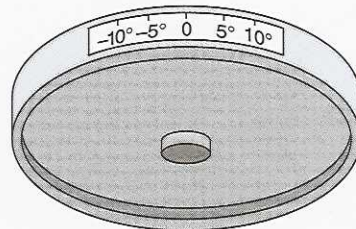


66.  The alternator on an automobile engine is attached by a belt to a wheel on the engine. The wheel on the engine has a diameter of 14.88 cm (see the figure), and the wheel on the alternator has a diameter of

9.86 cm. If the wheel driven by the engine moves through an angle of 2.85 radians, through what angle does the alternator move, to the nearest 0.01 radian?



67. A decal is being made to indicate timing marks on a wheel attached to the front of an engine (see the diagram). The radius of the wheel is 86.6 mm. What should the distance be between the -10° and 10° marks, to the nearest millimeter?



2-6 Radian measure—values of the trigonometric functions

In this section we relate radian measure to the trigonometric functions. The concepts here are the same as seen in sections 2-2 through 2-4, concerning degree measure.

Relationship between points on the unit circle and the trigonometric function values

Recall that if (x, y) is a point on the terminal side of an angle θ (in standard position), and $r = \sqrt{x^2 + y^2}$, then $\sin \theta = \frac{y}{r}$ and $\cos \theta = \frac{x}{r}$. On the unit circle, $r = 1$, so $\sin \theta = y$, and $\cos \theta = x$. Thus, if (x, y) is the point on the unit circle that intersects the terminal side of an angle θ , then $\sin \theta = y$ and $\cos \theta = x$. Figure 2-18 shows this fact.

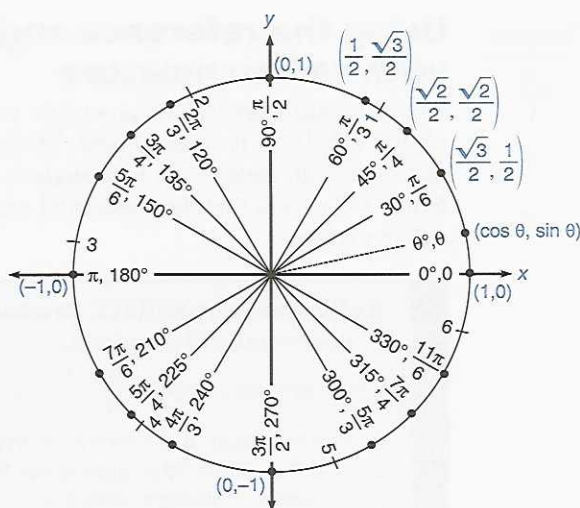


Figure 2-18

This figure is very useful because it shows the degree and radian measure for many common angles with measure between 0° and 360° (0 and 2π in radian measure). The angles shown are either quadrantal or have reference angles of measure 30° ($\frac{\pi}{6}$), 45° ($\frac{\pi}{4}$), or 60° ($\frac{\pi}{3}$). The figure also shows the point on the terminal side of an angle where it meets the unit circle. As stated above, the (x,y) pair at each point on the unit circle is $(\cos \theta, \sin \theta)$ for the corresponding angle. Note that the radian measure is shown for the values 1, 2, 3, . . . , 6 as well as for the multiples of π mentioned above. For example, 2 (radians) is near $\frac{2\pi}{3} \approx 2.1$ (radians), or 120° .

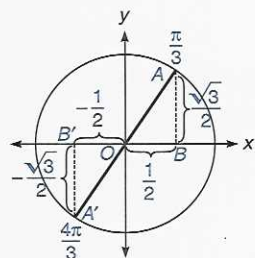


Figure 2-19

Observe that you can find the sine or cosine value for any of the angles shown by observing the symmetries in figure 2-18. For example, the coordinates at $\frac{4\pi}{3}$ must be $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. As seen in figure 2-19, traveling through the origin from one point on the unit circle to another simply changes the signs of the x - and y -coordinates. Thus, $\cos \frac{4\pi}{3}$ is $-\frac{1}{2}$ and $\sin \frac{4\pi}{3}$ is $-\frac{\sqrt{3}}{2}$.

Similarly, the coordinates at $\frac{5\pi}{6}$ (figure 2-18) must have the same y -coordinate as at $\frac{\pi}{6}$, but the opposite of the x -coordinate. Thus, the coordinates there must be $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$, and from this point we know that $\sin \frac{5\pi}{6} = \sin 150^\circ = y = \frac{1}{2}$, $\cos \frac{5\pi}{6} = \cos 150^\circ = -\frac{\sqrt{3}}{2}$.

	Sine	Cosine	Tangent
$\frac{\pi}{6}, 30^\circ$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}, 45^\circ$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}, 60^\circ$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Table 2-2

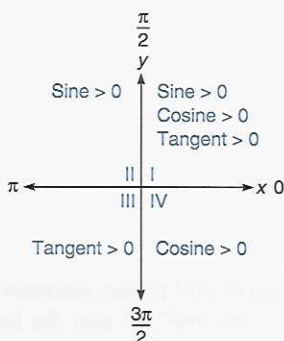


Figure 2-20

Example 2-6 A

Using the reference angle/ASTC procedure with radian measure

The reference angle/ASTC procedure (section 2-3) can also be used instead of figure 2-18. It is restated here. Table 2-2 is the same as table 2-1 except that the radian measure of each angle is included. Figure 2-20 is the same as figure 2-5 except that the quadrantal angles less than 2π (360°) are shown in radian measure.

Reference angle/ASTC procedure for radian measure

To find the value of a trigonometric function for a nonquadrantal angle

whose reference angle is $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$:

1. Find the value of the reference angle.
2. Find the value of the appropriate trigonometric function for the reference angle from table 2-2.
3. Determine the sign of this value using the ASTC rule (figure 2-20).

Use table 2-2 and the ASTC rule to find the exact value of each expression.

1. $\cos \frac{7\pi}{6}$

Step 1: $\theta' = \frac{\pi}{6}$

This was shown in part 1 of example 2-5 B

Step 2: $\cos \theta' = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ Table 2-2

Step 3: $\cos \frac{7\pi}{6} < 0$ ASTC rule; $\cos \theta < 0$ in quadrant III

Thus, $\cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$.

2. $\sin \frac{11\pi}{3}$

Step 1: $\theta' = \frac{\pi}{3}$

Shown in part 2 of example 2-5 B

Step 2: $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ Table 2-2

Step 3: $\sin \frac{11\pi}{3} < 0$ ASTC rule; $\sin \theta < 0$ in quadrant IV

Thus, $\sin \frac{11\pi}{3} = -\frac{\sqrt{3}}{2}$ and therefore $\sin \frac{11\pi}{3} = -\frac{\sqrt{3}}{2}$.

$$3. \cot\left(-\frac{13\pi}{4}\right)$$

$$\cot\left(-\frac{13\pi}{4}\right) = \frac{1}{\tan\left(-\frac{13\pi}{4}\right)}$$

In part 3 of example 2-5 B, we saw that $-\frac{13\pi}{4}$ and $\frac{3\pi}{4}$ are coterminal,

and the reference angle for either is $\frac{\pi}{4}$, so

$$\tan\left(-\frac{13\pi}{4}\right) = \tan \frac{3\pi}{4}$$

We proceed to find $\tan \frac{3\pi}{4}$:

Step 1: The reference angle for $\frac{3\pi}{4}$ is $\frac{\pi}{4}$.

Step 2: $\tan \frac{\pi}{4} = 1$

Step 3: $\frac{3\pi}{4}$ terminates in quadrant II, where the tangent function is negative, so $\tan \frac{3\pi}{4} < 0$.

Thus, $\tan \frac{3\pi}{4} = -1$.

Now we can finish the problem.

$$\cot\left(-\frac{13\pi}{4}\right) = \frac{1}{\tan\left(-\frac{13\pi}{4}\right)} = \frac{1}{-1} = -1$$

Calculators are programmed to accept angle input in radian measure. All scientific calculators have a key, often marked **DRG** or **MODE** to tell the calculator to accept angles in radian measure. On the TI-81 use the **MODE** key to select the mode display. Then use the four cursor keys to darken Rad, and use **ENTER** to change to radian mode. Use **QUIT** (**2nd** **CLEAR**) to exit the mode display.

Thus, for angles that are not coterminal with those in figure 2-18 we use the calculator, in radian mode. This is illustrated in example 2-6 B.

■ Example 2-6 B

Find the required value with a calculator; round the answer to four decimal places.

Make sure the calculator is in radian mode.

1. $\sin 1.2$

1.2 Display

1.2

$$\sin 1.2 \approx 0.9320$$

2. $\cot(-0.7)$

$$\cot(-0.7) = \frac{1}{\tan(-0.7)}$$

.7 Display

.7

$$\cot(-0.7) \approx -1.1872$$

As with degrees, we need to be able to find an angle in radians when given the value of one of the trigonometric functions for that angle, and information about the quadrant. As in section 2-3 we will restrict ourselves to the least (smallest value) nonnegative solution to trigonometric equations.

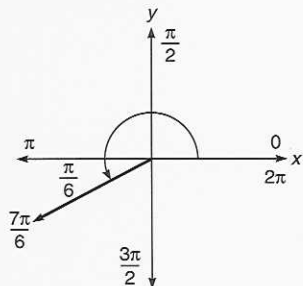
Since a reference angle is acute (between 0 and $\frac{\pi}{2}$ radians), and since the values of all the trigonometric functions are positive for acute angles (quadrant I, ASTC rule) we *always use a nonnegative value to find a reference angle*. This is illustrated in the following example.

In section 2-5 we saw formulas that give θ' if $0 < \theta < 2\pi$. If these are solved for θ , we obtain formulas for finding θ given θ' and the quadrant in which θ terminates.

Relationship between θ and θ' if $0 < \theta < 2\pi$

θ in Quadrant I:	$\theta' = \theta$	$\theta = \theta'$
θ in Quadrant II:	$\theta' = \pi - \theta$	$\theta = \pi - \theta'$
θ in Quadrant III:	$\theta' = \theta - \pi$	$\theta = \pi + \theta'$
θ in Quadrant IV:	$\theta' = 2\pi - \theta$	$\theta = 2\pi - \theta'$

Example 2-6 C



Find the least nonnegative value of θ , in radians. Round to two decimal places if necessary.

1. $\sin \theta = -\frac{1}{2}$, $\cos \theta < 0$

$$\theta' = \sin^{-1} \frac{1}{2}$$

Always use a nonnegative value to find a reference angle

$$\theta' = \frac{\pi}{6}$$

Table 2-2

θ is in quadrant III, since $\sin \theta < 0$ and $\cos \theta < 0$.

$$\theta = \pi + \frac{\pi}{6}$$

In quadrant III $\theta = \pi + \theta'$

$$\theta = \frac{7\pi}{6}$$

$$\pi + \frac{\pi}{6} = \frac{6\pi}{6} + \frac{\pi}{6}$$

2. $\cos \theta = -0.8$, $\tan \theta < 0$

$$\theta' = \cos^{-1} 0.8$$

Use the nonnegative value

$$\theta' \approx 0.64 \text{ radians}$$

$$0.8 \quad \boxed{\cos^{-1}} \quad (\text{in radian mode.})$$

$$\boxed{0.6435011088}$$

$$\boxed{\text{TI-81}} \quad \boxed{\text{COS}^{-1}} \quad .8 \quad \boxed{\text{ENTER}}$$

θ is in quadrant II since $\cos \theta < 0$, $\tan \theta < 0$.

$$\theta = \pi - \theta' \approx 2.50 \text{ radians}$$

Solutions to trigonometric equations

In sections 1-4 and 2-3 we examined solutions to equations involving the trigonometric ratios and functions in terms of degrees. Chapter 5 treats this subject in more depth, but this is a very important topic, so we revisit it here.

In the problems in example 2-6 D there are an unlimited number of answers. We will look for one basic answer in each case. By limiting the problems to practically all nonnegative values we can almost always find answers in the first quadrant in a very straightforward way.

The objective here is not to completely solve these equations, but to get used to the algebra involved in solving trigonometric equations. These equations use the following facts, which are presented in depth in chapters 4 and 5:

if $\sin \theta = k$, then one solution is $\theta = \sin^{-1} k$

if $\cos \theta = k$, then one solution is $\theta = \cos^{-1} k$

if $\tan \theta = k$, then one solution is $\theta = \tan^{-1} k$

We will see some equations that use the **zero product property**: If a product is zero, then at least one factor is zero. For example, if

$$\sin x(\sin x - 1) = 0$$

then either

$$\sin x = 0$$

or

$$\sin x - 1 = 0$$

Example 2-6 D

Find a solution to each equation, in radians. Round to two decimal places when necessary.

1. $4 \cos x = 1$

$$4 \cos x = 1$$

$$\cos x = \frac{1}{4}$$

$$x = \cos^{-1} \frac{1}{4} \approx 1.32 \text{ (radians)}$$

2. $3 \sin 2x = 2$

$$3 \sin 2x = 2$$

$$\sin 2x = \frac{2}{3}$$

$$2x = \sin^{-1} \frac{2}{3}$$

$$x = \frac{1}{2} \sin^{-1} \frac{2}{3}$$

$$x \approx 0.36 \text{ (radians)}$$

Divide both members by 3

$$1 \div 2 \times (2 \div 3) \sin^{-1} =$$

$$\text{TI-81 } (1 \div 2) \text{ SIN}^{-1} (2 \div$$

$$3) \text{ ENTER}$$

3. $\tan \frac{x}{2} = 1.5$

$$\tan \frac{x}{2} = 1.5$$

$$\frac{x}{2} = \tan^{-1} 1.5$$

$$x = 2 \tan^{-1} 1.5$$

$$x \approx 1.97 \text{ (radians)}$$

4. $(2 \sin \theta - 1)(\sin \theta + 1) = 0$

This is a product that equals zero, so we apply the zero product property.

$$2 \sin \theta - 1 = 0 \quad \text{or} \quad \sin \theta + 1 = 0$$

$$2 \sin \theta = 1 \quad \text{or} \quad \sin \theta = -1$$

$$\sin \theta = \frac{1}{2} \quad \theta = \frac{3\pi}{2}$$

$$\theta = \sin^{-1} \frac{1}{2}$$

$$\theta = \frac{\pi}{6}$$

Thus, there are two solutions, $\frac{\pi}{6}$ and $\frac{3\pi}{2}$.

Mastery points

Can you

- Find the exact value of a trigonometric function for an angle whose reference angle is $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$, as well as quadrantal angles?
- Find approximate values of the trigonometric functions with a calculator when the angle is given in radians?
- Find a solution, in radians, to certain trigonometric equations?

Exercise 2-6

Find the exact function values for the following angles.

- | | | | | |
|--|--------------------------|--|---------------------------------------|---------------------------------------|
| 1. $\sin \frac{2\pi}{3}$ | 2. $\tan \frac{5\pi}{4}$ | 3. $\cos \frac{11\pi}{6}$ | 4. $\tan \frac{7\pi}{6}$ | 5. $\cos \frac{4\pi}{3}$ |
| 6. $\sin \frac{5\pi}{6}$ | 7. $\sin \frac{3\pi}{4}$ | 8. $\cos \frac{7\pi}{4}$ | 9. $\tan \frac{5\pi}{3}$ | 10. $\sin\left(-\frac{\pi}{4}\right)$ |
| 11. $\tan\left(-\frac{2\pi}{3}\right)$ | 12. $\sec(-\pi)$ | 13. $\sin\left(-\frac{7\pi}{6}\right)$ | 14. $\cos\left(-\frac{\pi}{6}\right)$ | |

Find the following function values where the angle is given in radian measure. Round your answers to four decimal places.

- | | | | | | |
|----------------|----------------|----------------|----------------|----------------|----------------|
| 15. $\sin 0.9$ | 16. $\cos 1.1$ | 17. $\tan 0.5$ | 18. $\sec 1.4$ | 19. $\csc 0.7$ | 20. $\cot 1.5$ |
| 21. $\sin 2.3$ | 22. $\cos 3.5$ | 23. $\tan 4.1$ | 24. $\sec 5.2$ | 25. $\csc 2.5$ | 26. $\cot 1.9$ |

Find the least nonnegative value of θ , in radians. Round to two decimal places if necessary.

- | | | |
|---|---|--|
| 27. $\cos \theta = -\frac{1}{2}, \tan \theta > 0$ | 28. $\tan \theta = \sqrt{3}, \sin \theta < 0$ | 29. $\sin \theta = -\frac{\sqrt{3}}{2}, \tan \theta < 0$ |
| 30. $\sec \theta = \frac{2}{\sqrt{3}}, \sin \theta > 0$ | 31. $\cot \theta = -1, \sec \theta > 0$ | 32. $\tan \theta = -\sqrt{3}, \cos \theta > 0$ |
| 33. $\csc \theta = -2, \cos \theta > 0$ | 34. $\sin \theta = -0.5624, \tan \theta > 0$ | 35. $\tan \theta = -2.5, \csc \theta > 0$ |
| 36. $\cot \theta = -0.3, \sin \theta > 0$ | 37. $\cos \theta = -0.885, \tan \theta > 0$ | 38. $\sin \theta = -0.2258, \cos \theta > 0$ |
| 39. $\csc \theta = -3, \sec \theta < 0$ | | |

Find one solution to the following equations, in radians. Round to two decimal places if necessary.

- | | | | |
|--|---------------------------------|---|----------------------------------|
| 40. $4 \cos 3\theta = 2$ | 41. $3 \sin 2\theta = 1$ | 42. $\frac{1}{3} \tan 2\theta = 1$ | 43. $2 \tan \theta = 5$ |
| 44. $2 \sin 3\theta = \sqrt{3}$ | 45. $\sin \frac{\theta}{2} = 1$ | 46. $2 \sec 3\theta = 6$ | 47. $\sin 3\theta = \frac{1}{2}$ |
| 48. $(2 \sin \theta - 1)(\sin \theta - 1) = 0$ | | 49. $\cos \theta(2 \cos \theta - 1) = 0$ | |
| 50. $\tan \theta(\tan \theta - 1) = 0$ | | 51. $(2 \sin \theta - \sqrt{3})(\sin \theta - 1) = 0$ | |

52. In a certain series circuit the applied voltage
- V
- in volts is determined by the function

$$V = 200 \sin(35t + 1)$$

where t represents time in milliseconds and the expression $35t + 1$ is in radians. Compute V to the nearest 0.1 volt for the following values of t :

- a. 0 b. 0.1 c. 0.8 d. 1

53. The position
- d
- at the end of a spring, under certain initial conditions, as a function of time
- t
- in seconds, is

$$d = \frac{1}{3} \cos 8t - \frac{1}{4} \sin 8t$$

Compute d for the following values of t :

- a.
- $\frac{1}{8}$
- b.
- $\frac{1}{4}$

54. An equation that arises in finding the trajectory of a rocket is

$$r = \frac{p}{1 + e \cos(s - C)}$$

Find r if $p = 200$, $e = 1.5$, $C = 0.5$, and

- a.
- $s = 1$
- b.
- $s = 1.25$

55. In interpreting an electrocardiogram a cardiologist obtains values
- a
- and
- b
- from the heights of certain peaks, and the value of angle
- θ
- , which depends on where the electrodes are attached to the patient. Then the value
- V
- is calculated from the expression

$$V = \frac{\sqrt{a^2 + b^2 - 2ab \cos \theta}}{\sin \theta}$$

The value of V helps the cardiologist diagnose specific heart abnormalities. Compute V if $a = 6.2$ cm, $b = 3.5$ cm, and $\theta = 2.6$ (radians). Round the answer to two decimal places. The units will be centimeters.

56. If a 100-pound force pushes on an object at an angle
- θ
- that is measured between the direction of the force and the direction of motion of the object, then the amount of force actually moving the object is given by the function

$$f(\theta) = 100 \cos \theta$$

Find the force f for the following values of θ (all in radians), to one decimal place:

- a. 0.2 b. 0.4 c. 0.6 d. 1

57. An equation that can be used to compute $\sin x$, if x is in radians, is called the *Maclaurin series* for the sine function. It is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots,$$

where

$$3! = 1 \cdot 2 \cdot 3 = 6,$$

$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120,$$

$$7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5,040, \text{ etc.}$$

($n!$ is read “ n factorial” and is defined as $1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n$.)

Although the Maclaurin series goes on forever, good accuracy is obtained by using the first few terms. Use the first four of the five terms shown here to compute approximations to

- a. $\sin 0.1$ b. $\sin 0.5$ c. $\sin 1$ d. $\sin \frac{\pi}{6}$

Check the results with the sine key of a calculator. (Some computers use a method similar to this for computing the trigonometric functions.)

58. (See problem 57.) The Maclaurin series for the cosine function, if x is in radians, is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

Use the first four of the five terms shown above to calculate approximations to

- a. $\cos 0.8$ b. $\cos 1$ c. $\cos 1.3$
d. Approximate $\cos 10^\circ$ by first converting 10° to radians.

Chapter 2 summary

- A function is a set of ordered pairs having the property that no first element of the ordered pairs repeats.
- A function is one to one if no second element of the ordered pairs repeats.
- A one-to-one function f has an inverse function f^{-1} .
- An angle in standard position** is formed by two rays, one of which always lies on the nonnegative portion of the x -axis. This ray is called the initial side. The second ray is called the terminal side. It may be in any quadrant or along any axis.
- Two angles α and β are said to be coterminal if $\alpha = \beta + k(360^\circ)$, k an integer.
- The trigonometric functions** Let θ be an angle in standard position, and let (x, y) be any point on the terminal side of the angle, except $(0, 0)$. Let $r = \sqrt{x^2 + y^2}$ be the distance from the origin to the point. Then

$$\sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r}, \quad \tan \theta = \frac{y}{x}$$

$$\csc \theta = \frac{r}{y}, \quad \sec \theta = \frac{r}{x}, \quad \cot \theta = \frac{x}{y}$$

- Tangent/cotangent identities**

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}.$$

The ASTC rule

In quadrant I, All the trigonometric functions are positive.

In quadrant II, the Sine function is positive.

In quadrant III, the Tangent function is positive.

In quadrant IV, the Cosine function is positive.

- Angles whose degree measures are integer multiples of 90° , such as $0^\circ, \pm 90^\circ, \pm 180^\circ, \pm 270^\circ$, etc., are called quadrantal angles.
- A reference angle for a nonquadrantal angle is the acute angle formed by the terminal side of the angle and the x -axis.
- If $0^\circ < \theta < 360^\circ$, then the following relate the reference angle θ' and the angle θ .

θ in Quadrant I:	$\theta' = \theta$	$\theta = \theta'$
θ in Quadrant II:	$\theta' = 180^\circ - \theta$	$\theta = 180^\circ - \theta'$
θ in Quadrant III:	$\theta' = \theta - 180^\circ$	$\theta = 180^\circ + \theta'$
θ in Quadrant IV:	$\theta' = 360^\circ - \theta$	$\theta = 360^\circ - \theta'$
- The absolute value of a trigonometric function for any angle is the same as the trigonometric ratio for the corresponding reference angle.

- **Reference angle/ASTC procedure** To find the exact value of a trigonometric function for a nonquadrantal angle whose reference angle is 30° (or $\frac{\pi}{6}$), 45° (or $\frac{\pi}{4}$), or 60° (or $\frac{\pi}{3}$):

1. Find the value of the reference angle.
2. Find the value of the appropriate trigonometric ratio for the reference angle from the following table.
3. Determine the sign of this value using the ASTC rule.

	Sine	Cosine	Tangent
$30^\circ, \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45^\circ, \frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60^\circ, \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

- Solutions to simple trigonometric equations:
 - if $\sin \theta = k$, then one solution for θ is $\theta = \sin^{-1} k$
 - if $\cos \theta = k$, then one solution for θ is $\theta = \cos^{-1} k$
 - if $\tan \theta = k$, then one solution for θ is $\theta = \tan^{-1} k$
- Finding the least nonnegative measure of an angle from a trigonometric function value and information about a quadrant.
 1. If necessary use the ASTC rule to determine the quadrant for the terminal side of the angle.
 2. Use $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ to find θ' . Use the absolute value of the given trigonometric function value.
 3. Apply θ' to the correct quadrant to determine the value of θ .
- A **reference triangle** is a right triangle with one leg on the x-axis and one leg parallel to the y-axis.
- The circle with radius one and center at the origin is described by the equation $x^2 + y^2 = 1$. It is called the unit circle.
- Let θ be an angle in standard position. Let s be the corresponding arc length on the unit circle. Let s be positive if measured in the counterclockwise direction, and negative if measured in the clockwise direction. Then s is the radian measure of the angle θ .

- **Radian/degree proportion** Let θ be an angle in standard position with degree measure θ° and radian measure s .

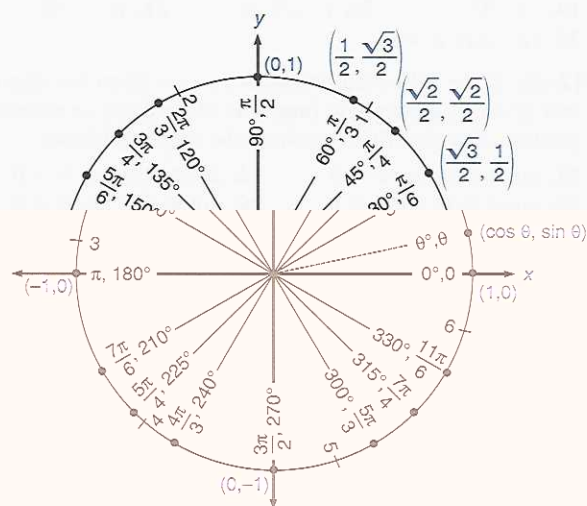
$$\text{Then } \frac{s}{\pi} = \frac{\theta^\circ}{180^\circ}.$$

- Relationship between an angle θ , $0 < \theta < 2\pi$, and θ' , its reference angle in radian measure.

Quadrant in which θ terminates

I	$\theta' = \theta$	$\theta = \theta'$
II	$\theta' = \pi - \theta$	$\theta = \pi - \theta'$
III	$\theta' = \theta - \pi$	$\theta = \pi + \theta'$
IV	$\theta' = 2\pi - \theta$	$\theta = 2\pi - \theta'$

- If s is the radian measure of an angle with vertex at the center of a circle of radius r , and L is the corresponding arc length, then $L = rs$.
- The area of a sector of a circle of radius r is $A_s = \frac{s}{2}r^2$.
- If (x, y) is the point on the unit circle that intersects the terminal side of an angle θ , then $\sin \theta = y$ and $\cos \theta = x$.



Chapter 2 review

[2-1] In each of the following sets of ordered pairs:

- a. Determine if the set is a function.
- b. If a function, state the domain and range.
- c. If a one-to-one function, state the inverse function.

1. $\{(1,7), (4,5), (6,7), (7,10)\}$
2. $\{(-2,-3), (-1,1), (1,2), (2,5)\}$
3. $\{(1,3), (2,5), (2,9), (6,10)\}$

In each of the following problems a rule that describes a function is given. Form the ordered pairs that this function contains for the following domain elements:

a. -1 b. 0 c. $\sqrt{5}$ d. $\frac{1}{3}$

4. $f(x) = 4 - 2x$

5. $f(x) = 3x^2 - 2x + 3$

6. $f(x) = x^4 - 5$

7. $f(x) = \frac{3x}{x-1}$

[2-2] In the following exercises draw the initial side and terminal side of the given angle. Also, state the measure of the smallest nonnegative angle that is coterminal with the angle.

- | | | |
|---------------------|---------------------|---------------------|
| 8. 465° | 9. -40.25° | 10. -270° |
| 11. $1,800.6^\circ$ | 12. $132^\circ 18'$ | 13. $547^\circ 26'$ |
| 14. 429.3° | 15. $-0^\circ 15'$ | |

In the following problems you are given a point that lies on the terminal side of an angle in standard position. In each case draw a representation of the least positive angle that has the point on its terminal side and compute all six trigonometric functions for the angle.

- | | | |
|-----------------------|----------------------|----------------------|
| 16. $(5, -12)$ | 17. $(-5, 8)$ | 18. $(0, -3)$ |
| 19. $(2, \sqrt{5})$ | 20. $(-\sqrt{2}, 3)$ | 21. $(1, -\sqrt{6})$ |
| 22. $(a, -2a), a > 0$ | | |

[2-3] In the following problems you are given the sign of two of the trigonometric functions of an angle in standard position. State in which quadrant the angle terminates.

- | | |
|--|--|
| 23. $\sin \theta < 0, \cos \theta < 0$ | 24. $\sec \theta < 0, \tan \theta > 0$ |
| 25. $\cos \theta > 0, \tan \theta < 0$ | 26. $\cot \theta < 0, \csc \theta < 0$ |
| 27. $\csc \theta > 0, \cot \theta < 0$ | 28. $\tan \theta < 0, \cos \theta < 0$ |
| 29. $\tan \theta > 0, \csc \theta < 0$ | 30. $\sec \theta > 0, \csc \theta < 0$ |

In the following problems you are given the degree measure of an angle in standard position. For each indicate the measure of the reference angle θ' .

- | | | |
|--------------------|---------------------|---------------------|
| 31. 46.3° | 32. $323^\circ 16'$ | 33. $421^\circ 48'$ |
| 34. -22.7° | 35. -248.7° | 36. -105.15° |
| 37. 242.57° | | |

In the following problems you are asked to find a trigonometric function value for an angle. If the reference angle is 30° , 45° , or 60° , give the exact answer. Otherwise find the required value to four decimal places.

- | | |
|----------------------------|----------------------------|
| 38. $\sin 213.4^\circ$ | 39. $\cos 240^\circ$ |
| 40. $\tan(-114.6^\circ)$ | 41. $\csc 870^\circ$ |
| 42. $\cot 212.3^\circ$ | 43. $\tan 213.9^\circ$ |
| 44. $\sec 300^\circ$ | 45. $\cos(-133^\circ 20')$ |
| 46. $\sin(-293^\circ 40')$ | |

[2-4] In the following problems you are given a trigonometric function value of an angle in standard position, along with the sign of one of the other function values. Find the

measure of the smallest nonnegative angle that meets these conditions, to the nearest 0.1° .

47. $\sin \theta = 0.3251, \cos \theta < 0$
 48. $\cos \theta = -0.7771, \sin \theta > 0$
 49. $\tan \theta = 0.6306, \sec \theta < 0$
 50. $\sin \theta = -0.9088, \tan \theta < 0$
 51. $\sec \theta = -2.0642, \sin \theta > 0$
 52. $\cot \theta = 4.1046, \sin \theta < 0$
 53. $\sin \theta = \frac{\sqrt{3}}{2}, \cos \theta < 0$
 54. $\tan \theta = -2, \sin \theta < 0$
 55. $\cos \theta = -\frac{5}{13}, \tan \theta < 0$

In the following problems you are given the value of one trigonometric function and the sign of another function of an angle in standard position. Draw a representation of the least positive angle that meets these conditions and use it. Compute the exact value of the remaining five trigonometric functions.

- | | |
|--|---|
| 56. $\sin \theta = \frac{4}{5}, \cos \theta > 0$ | 57. $\cos \theta = -\frac{1}{2}, \tan \theta > 0$ |
| 58. $\cos \theta = -\frac{5}{16}, \tan \theta > 0$ | 59. $\cos \theta = \frac{1}{4}, \cot \theta < 0$ |
| 60. $\tan \theta = -2, \cos \theta < 0$ | 61. $\csc \theta = -4, \sec \theta > 0$ |
| 62. $\sec \theta = 6, \csc \theta < 0$ | 63. $\cot \theta = 2, \sin \theta < 0$ |
| 64. $\cot \theta = \frac{2}{7}, \sin \theta < 0$ | 65. $\tan \theta = \frac{7}{2}, \sec \theta < 0$ |
| 66. $\tan \theta = 0, \cos \theta > 0$ | 67. $\sin \theta = 0.25, \cos \theta < 0$ |

68. $\sin \theta = z$ and θ terminates in quadrant II. Find $\tan \theta$ in terms of z .

69. $\tan \theta = z$ and θ terminates in quadrant IV. Find $\cos \theta$ in terms of z .

70. In a certain electrical circuit the instantaneous voltage E (in volts) is found by the formula

$$E = 120 \sin(\theta + 15^\circ)$$

Compute E to the nearest 0.01 volt for the following values of θ :

- | | | | |
|----------------|----------------|-----------------|-----------------|
| a. 0° | b. 45° | c. 84.2° | d. -200° |
| e. -15° | f. -45° | | |

71. For a surveyor to locate a point by measuring an angle at one station and a distance from another one, the distance BP must be found by solving the following sequence of formulas:

$$\sin p = \frac{AB \sin b}{AP} \quad (0^\circ < p < 90^\circ)$$

$$a = 180^\circ - (b + p)$$

$$BP = \frac{AP \sin a}{\sin b}$$

Compute the distance BP to the nearest 0.1 meter if $AB = 211.5$ meters, $AP = 185.7$ meters, and $b = 29.6^\circ$.

72. A machinist is setting up a numerically controlled drill. The drill must drill a hole in a piece of steel 9.0 millimeters from the origin at an angle of $125^\circ 30'$. To the nearest 0.1 millimeter, what are the coordinates of this point?
73. Suppose the hole of problem 72 must be 8.075 inches from the origin at an angle of 10.2° . Find the coordinates of this point to the nearest 0.1 inch.
74. A technician is aligning a laser device used to cut patterns from cloth, and positions the device at an angle of -42.3° and a distance of 6.90 feet from the origin. What should the x - and y -coordinates be at this point to the nearest 0.1 foot?

[2–5] Convert the following degree measures into radian measures. Leave your answers both in exact form and approximated to two decimal places.

75. 120° 76. -215° 77. 430°

Convert the following radian measures into degree measures. Leave your answers both in exact form and approximated to two decimal places.

78. $\frac{7\pi}{2}$ 79. $\frac{11\pi}{3}$ 80. $-\frac{5\pi}{7}$
 81. 2.5 82. -4.2

83. Find the length of the arc determined by a central angle of 3.9 (radians) on a circle of diameter 6.3 inches, to the nearest 0.1 inch.
84. Find the measure, in radians, of a central angle on a circle of radius 8.8 mm determined by an arc length of 20 mm, to the nearest 0.1 radian.
85. Find the length of the arc determined by a central angle of 150° on a circle of diameter 15 mm.
86. The diameter of a wheel on an automobile is 30 inches. If the wheel moves through an angle of 385° , how far will the car move?

Find the area of the sector determined by each of the following angles and radii. Give both the exact answer and a two-decimal-place approximation.

87. 30° , 9 inches 88. 240° , 8 mm
 89. $\frac{11}{12}$, 6 mm 90. $\frac{2\pi}{5}$, 7 inches

[2–6] Find the following function values where the angle is given in radian measure. Round your answer to four decimal places.

91. $\sin 1.9$ 92. $\sec 2.4$ 93. $\tan 4.5$

Find the exact function values for the following angles.

94. $\sin \frac{5\pi}{3}$ 95. $\tan \frac{3\pi}{4}$ 96. $\cos \frac{7\pi}{6}$
 97. $\cos \left(-\frac{\pi}{6} \right)$ 98. $\tan \left(-\frac{4\pi}{3} \right)$ 99. $\sin \left(-\frac{5\pi}{6} \right)$
 100. $\sec \left(-\frac{\pi}{4} \right)$

Find the least nonnegative value of θ in radians. Round to two decimal places if necessary.

101. $\sin \theta = -\frac{1}{2}$, $\cos \theta > 0$
 102. $\tan \theta = -1.82$, $\sin \theta > 0$

Find one solution to the following equations, in radians. Round to two decimal places if necessary.

103. $2 \sin \theta = 0.84$ 104. $3 \tan 2\theta = 5$
 105. $(2 \cos \theta - 1)(\cos \theta - 1) = 0$

106. The position d at the end of a spring, under certain initial conditions, as a function of time t in seconds, is

$$d = \frac{1}{3} \cos 8t - \frac{1}{4} \sin 8t$$

Compute d if $t = \frac{1}{12}$.

Chapter 2 test

1. Draw the initial side and terminal side of the given angle. Also, state the measure of the least nonnegative angle that is coterminal with the given angle.
 a. 665° b. -417°

In the following problems you are given a point that lies on the terminal side of an angle in standard position. In each case draw a representation of the least positive angle that has the point on its terminal side and compute all six trigonometric function values for the angle (leave answers exact).

2. $(6, -12)$ 3. $(-2, -6)$ 4. $(\sqrt{2}a, a)$, $a > 0$

In the following problems you are given the value of one trigonometric function and the sign of another function of an angle in standard position. (a) Draw a representation of the least positive angle that meets these conditions. (b) Use a reference triangle to compute the exact value of the remaining five trigonometric functions. (c) Use one of the function values to find the least positive measure of the angle to the nearest 0.1° .

5. $\sin \theta = -\frac{1}{6}$, $\cos \theta > 0$ 6. $\tan \theta = 8$, $\cos \theta < 0$

In the following problems you are given the degree measure of an angle in standard position. For each indicate the measure of the reference angle θ' .

7. 246.2° 8. -55.6°

In the following problems you are asked to find a trigonometric function value for an angle. Find the required value to four decimal places.

9. $\sin 116.4^\circ$ 10. $\tan(-14.8^\circ)$ 11. $\csc 115^\circ 20'$

In the following problems you are given a trigonometric function value of an angle in standard position, along with the sign of one of the other function values. Find the measure of the smallest nonnegative angle that meets these conditions to the nearest 0.1° .

12. $\sin \theta = -0.2961$, $\cos \theta < 0$

13. $\sec \theta = -2.0642$, $\sin \theta > 0$

14. In a certain electrical circuit the instantaneous current I is found by the formula $I = 5.4 \cos(\theta - 25^\circ)$. Compute I to the nearest 0.1 ampere for $\theta = 45^\circ$.

15. The arm of an industrial robot is positioned at 22.6 centimeters from the origin at an angle of 261.42° . To the nearest 0.1 centimeter, what are the coordinates of this point?

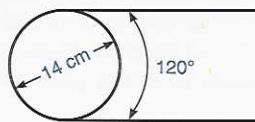
16. To align a precision laser that is part of an optical bench, a technician points the device at a test point with coordinates $(-211.5, 620.0)$ (inches). Assuming the bench is coordinatized in the usual way, with the laser at the origin, (a) what angle should the laser's indicator show to the nearest 0.1° , and (b) how far from the origin is the test point?

17. Convert 415° into radian measure. Leave your answer both in exact form and approximated to two decimal places.

18. Convert $\frac{7\pi}{12}$ into degree measure.

19. Find the length of the arc subtended by a central angle of 5.0 (radians) on a circle of diameter 8.2 inches, to the nearest 0.1 inch.

20. The diameter of a wheel on a pulley is 14 cm (see the diagram). If the wheel moves through an angle of 120° , how far will the belt that the wheel drives move?



21. Find a four-decimal-place approximation to $\csc 2.5$.

22. Find the exact value of $\cos \frac{5\pi}{6}$.

23. Find the area of the sector determined by a central angle of $\frac{\pi}{3}$ (radians) in a circle of diameter 32 mm, to the nearest 0.01 mm^2 .

24. Let $f = \{(2, -3), (3, 5), (5, 6), (10, 12)\}$.

a. Is f a function?

b. If so, is it one to one?

c. Does f have an inverse? If so, state it.

25. If the function f is described by the rule $f(x) = x^2 - 3x + 5$, state the ordered pair that is an element of f when the domain element is (a) -4 and (b) $\sqrt{2}$.

Find the least nonnegative value of θ in radians. Round to two decimal places if necessary.

26. $\sin \theta = -\frac{1}{3}$, $\cos \theta > 0$

27. $\tan \theta = \sqrt{3}$, $\sin \theta < 0$

Find one solution to the following equations, in radians. Round to two decimal places if necessary.

28. $\frac{1}{2} \cos \theta = 0.20$

29. $2 \tan 3\theta = 4.2$

30. $\sin \theta (2 \sin \theta - 1) = 0$

31. The position d at the end of a spring, under certain initial conditions, as a function of time t in seconds, is $d =$

$\frac{1}{3} \cos 4t - \frac{1}{4} \sin 4t$. Compute d if $t = \frac{\pi}{16}$.



Properties of the Trigonometric Functions

To understand many applications of the trigonometric functions it is necessary to have a good understanding of the properties of these functions. As with any functions, we can gain a great deal of knowledge from their graphs. There are many graphing calculators and computer programs available today that are capable of graphing functions defined from equations. We will illustrate the graphing of trigonometric functions by two methods:

1. obtaining important information about the graph, then using this information to sketch the graph by hand, and
2. using a graphing calculator.

We use the TI-81 graphing calculator to illustrate using a graphing calculator. The process is practically the same with another brand or model.

Before going further, we show some basics of using the TI-81 graphing calculator. The reader can jump to section 3-1 if not using a graphing calculator.

3-0 TI-81 graphing basics

RANGE

Xmin=-10
Xmax=10
Xscl=1
Ymin=-10
Ymax=10
Yscl=1
Xres=1

Table 3-1

Setting the range for the screen

Graphing calculators have a way to describe which part of the coordinate plane will be displayed. It is called setting the RANGE. Using the **RANGE** key shows a display similar to that in table 3-1. The Xmin and Xmax values refer to the range of x values which will be displayed. The Ymin and Ymax values refer to the range of y values which will be displayed. The Xscl and Yscl values refer to the tick marks which will appear on the screen. The Xres refers to the number of x values which will be calculated. It should be left at 1.

Throughout the text we will show the Xmin, Xmax, Xscl, Ymin, Ymax and Yscl values, in this order, in a box labeled RANGE. For the values shown in figure 3-1 we would write **RANGE -10,10,1,-10,10,1**. This omits the value for Xres, which we will assume is 1.

By entering numeric values and using the **ENTER** key to move down the list, the values in the RANGE can be changed. Note that to obtain a negative number the **(-)** (change sign) key is used, not the **-** (subtract) key.

Figure 3-1 shows the screen appearance for various settings of Xmax, Xmin, Ymax, and Ymin. Xscl and Yscl are 1 except where labeled Yscl=3 and Xscl=2. After setting these values with the **RANGE** key, use the **GRAPH** key to show the screen. Using the **CLEAR** button readies the calculator for numeric calculations again. The settings in part (a) of figure 3-1 are the “standard” settings, obtained by selecting **ZOOM** 6.

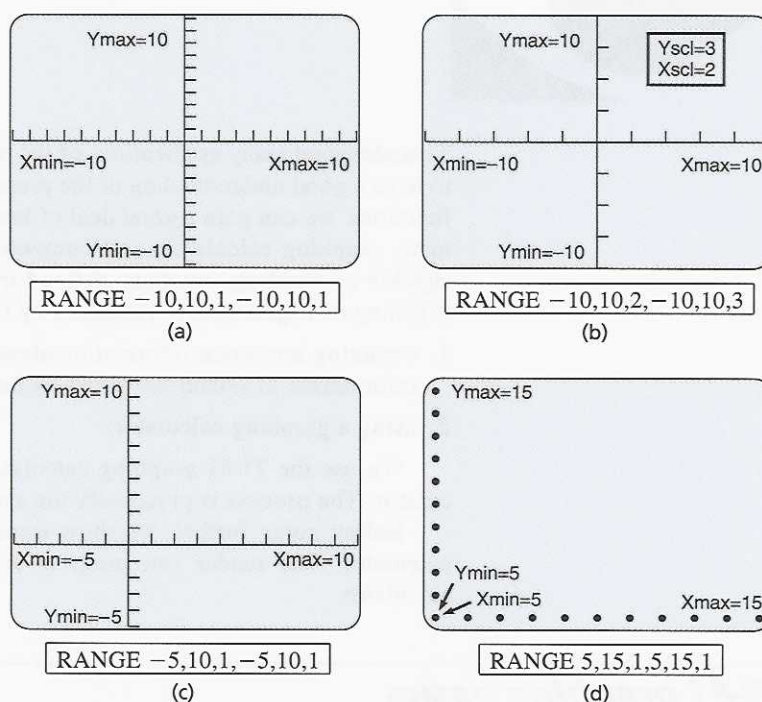


Figure 3-1

Observe that the distance between units are not the same on the screen. The calculator automatically makes horizontal units 1.5 times as long as vertical units. To have horizontal and vertical distances the same, use the **ZOOM** function, where option 5 says **SQUARE**. This makes the screen use the same scale for distance vertically and horizontally by changing the values of Xmin and Xmax. In most cases, having equal horizontal and vertical scales will not be important.

When graphing trigonometric functions **ZOOM** 7 (Trig) can be useful. It sets Range settings to

$$\text{RANGE } -6.28 (-2\pi), 6.28 (2\pi), 1.57 \left(\frac{\pi}{2}\right), -3, 3, .25$$

Graphing an equation in which y is described in terms of x

If an equation describes values for a variable y in terms of a variable x , the graphing calculator can be used to view the graph of the equation.

■ Example 3-0 A

x	$y (2x - 3)$
-3	-9
-2	-7
-1	-5
0	-3
1	-1
2	1
3	3

Graph each equation.

1. Graph $y = 2x - 3$

This could be done without a graphing calculator with practically no knowledge of graphing by a table of values, by letting x take on many values, such as $-3, -2, -1, 0, 1, 2, 3$, etc., and computing y for each one. In fact, this table is shown here. The y -values are computed by computing $2x - 3$ for the given x -value. Each pair of values for x and y represents an ordered pair (x, y) (we always write the x -value first). If we plot enough of these values in a coordinate system we start to see a picture emerge. In this case it is a straight line.

Of course the point of this section is to have the calculator automatically calculate the x - and y -values and plot them. Assuming the standard RANGE settings (obtained by **ZOOM** 6) proceed as follows to obtain the graph:

Y=

Allows us to enter up to four equations.

2

X|T

The variable x .

-

3

The display looks like

$$:Y_1=2X-3$$

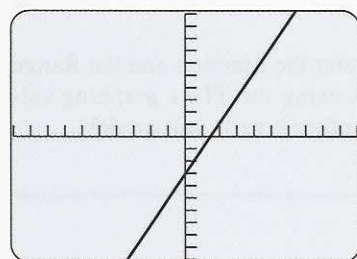
$$:Y_2=$$

$$:Y_3=$$

$$:Y_4=$$

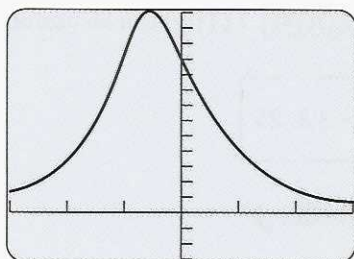
GRAPH

Note If there are any equations already entered for Y_1 use the **CLEAR** key before entering the equation. If there are any extra equations entered for Y_2, Y_3 , or Y_4 , move down with the down arrow key **▽** to that equation and use the **CLEAR** key to clear that entry. Use **ZOOM** 6 to obtain the standard Range settings. The figure shows what the display will look like.



Y= 2 **X|T** **-** 3,

RANGE -10,10,1,-10,10,1



Y= (X|T x² +
X|T + 1) x⁻¹,
RANGE -3,3,1,-.3,1.3,.1

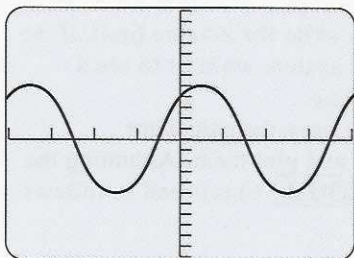
2. Graph $y = \frac{1}{x^2 + x + 1}$

The following steps would produce a graph similar to that shown in the figure.

Steps	Explanation
RANGE	Enter the x- and y-axis limits.
(-) 3 ENTER	Xmin becomes -3.
3 ENTER	Xmax becomes 3.
1 ENTER	Xscl becomes 1.
(-) .3 ENTER	Ymin becomes -0.3.
1.3 ENTER	Ymax becomes 1.3.
.1 ENTER	Yscl becomes 0.1.

The x^{-1} key is used to define a reciprocal (something divided into one).

Y= (X|T x² + X|T + 1) x⁻¹ GRAPH



Y= SIN X|T +
COS X|T
RANGE -6.28,6.28,1.57,-3,3,.25

3. $y = \sin x + \cos x$

Make sure the calculator is in radian mode (with the **MODE** key).

Steps	Explanation
Y= CLEAR	Enter graphing mode. Remove the previous function.
SIN X T + COS X T	
ZOOM 7	Select standard settings for trigonometric functions. The graphing begins automatically after ZOOM 7 is selected.

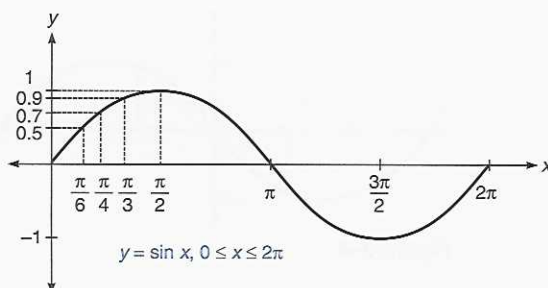
In the remainder of this text we show how to enter the function and the Range settings for each graph, assuming the reader is using the TI-81 graphing calculator. The steps are practically the same for other brands and models.

3-1 Graphs and properties of the sine, cosine, and tangent functions

Graph of the sine function

The graph of $y = \sin x$ for x between 0 and 2π is shown in figure 3-2. This graph can be obtained by plotting points for various values of x . For example, the points for the following table of values are shown in the figure.

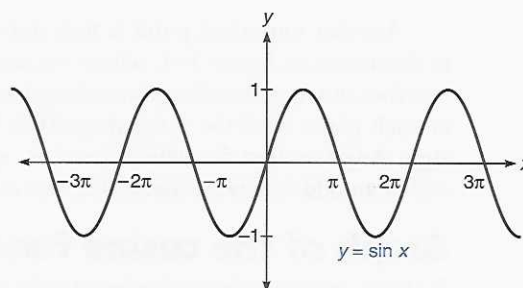
x (radians)	$\sin x$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$ (0.7)
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$ (0.9)
$\frac{\pi}{2}$	1



Y= SIN X T RANGE 0,6.28,1.57,-1.5,1.5,1

Figure 3-2

The graph in figure 3-2 repeats itself, as shown in figure 3-3, for other values of x because for values of x greater than 2π or less than 0 we have angles that are coterminal with values we have already plotted. Thus, every 2π units we find that the part of the graph that was shown in figure 3-2 is repeated. If we memorize the graph in figure 3-2, we can use it to reproduce the graph in figure 3-3.



Y= SIN X T RANGE -10,10,3.14,-1.5,1.5,1

Figure 3-3

The repetitious nature of the sine function can be described with the identity

$$\sin x = \sin(x + k \cdot 2\pi), k \text{ any integer}$$

We say that the sine function is 2π -periodic, or is periodic with period 2π .

Note Any function that repeats the same pattern over and over is said to be **periodic**. The period is the length of the shortest pattern that produces the function when repeated. Algebraically, a function f is p -periodic if there is a number p , $p > 0$, such that

$$f(x + p) = f(x)$$

for all x in the domain, and p is the smallest such number.

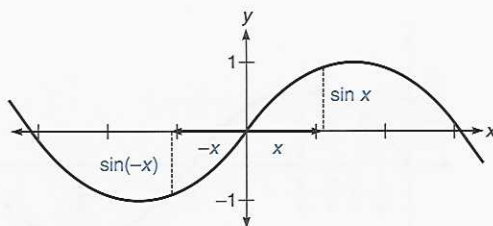


Figure 3-4

Several observations can be made by looking at the graph of $y = \sin x$ in figure 3-4. The domain of the sine function is all real numbers; that is, x can be any real number. The range of the sine function is restricted to the values between and including -1 to 1 . In other words, if R represents the collection of all real numbers, then:

Domain_{sine}: R

and

Range_{sine}: $-1 \leq y \leq 1$

Note The domain is verbalized as "all real numbers"; the range is verbalized as "all real numbers y having the property that $y \geq -1$ and $y \leq 1$."

Another important point is that $\sin(-x) = -(\sin x)$ for any value x . This is illustrated in figure 3-4, where we see that if we go equal distances in the positive and negative directions along the x -axis, the value of the sine function at each place is of the same magnitude (absolute value) but of the opposite sign. Any function for which $f(-x) = -f(x)$ is true for all x in its domain is called an odd function; therefore, *sine is an odd function*.

Graph of the cosine function

Plotting various values of ordered pairs (x, y) , where $y = \cos x$, and then connecting them with a smooth curve produces the graph shown in figure 3-5 for x between and including 0 and 2π . For example, we know that $\cos 0 = 1$, $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\cos \frac{\pi}{3} = \frac{1}{2}$, $\cos \pi = -1$, etc. These values are shown in figure 3-5.

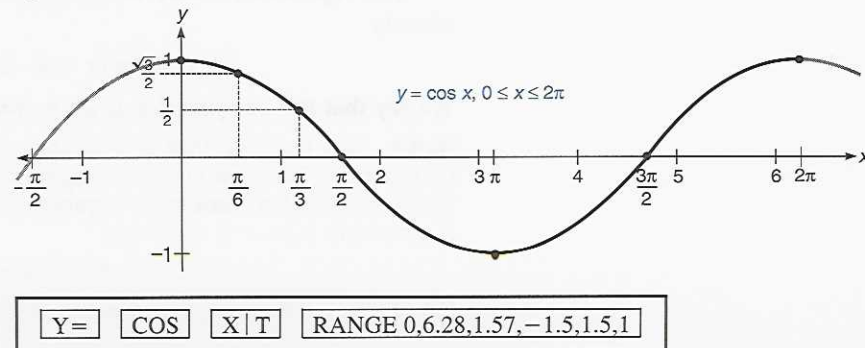


Figure 3-5

Just as with the sine function, the cosine function repeats after 2π units. The identity that states this algebraically is $\cos x = \cos(x + k \cdot 2\pi)$, k any integer. As with the sine function, we also say that *the cosine function is 2π -periodic*.

The graph of $y = \cos x$ is shown in figure 3-5. If we memorize the graph in figure 3-5, we can use it to reproduce the graph in figure 3-6.

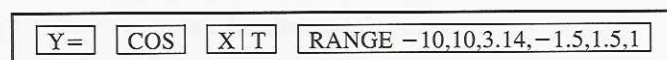
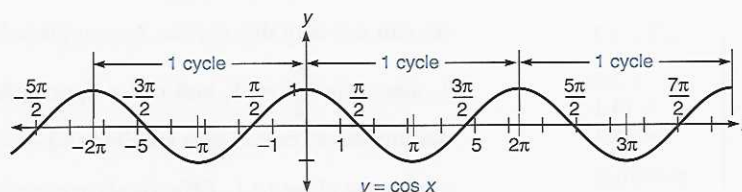


Figure 3-6

Several observations can be made by looking at the graph of $y = \cos x$ in figure 3-6. The domain of the cosine function is all real numbers; that is, x can be any real number. The range of the cosine function is restricted to the values between and including -1 to 1 . These are, of course, the same as the domain and range of the sine function.

Domain_{cosine}: R

and

Range_{cosine}: $-1 \leq y \leq 1$

We also see that $\cos(-x) = \cos x$. This is illustrated in figure 3-7, where we see that if we go equal distances in the positive and negative directions along the x -axis, the value of the cosine function at each place is the same value. Any function for which $f(-x) = f(x)$ is true for all x in its domain is called an even function; therefore, *cosine is an even function*.

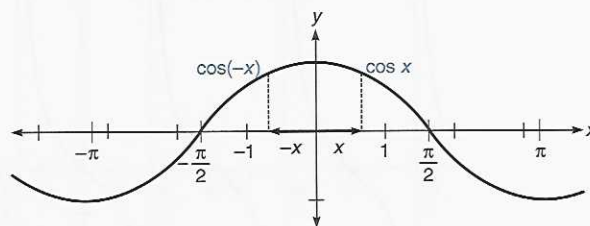


Figure 3-7

Finally, *the graphs of the sine and cosine functions have exactly the same shape*; either one becomes the other if it is shifted right or left a suitable amount. The smallest such amount is $\frac{\pi}{2}$, which is described in the statement

that $\sin\left(x + \frac{\pi}{2}\right) = \cos x$. This statement is proved in chapter 5.

x	$\tan x$
0	0
$\frac{\pi}{6} \approx 0.52$	$\frac{\sqrt{3}}{3} \approx 0.6$
$\frac{\pi}{4} \approx 0.79$	1
$\frac{\pi}{3} \approx 1.05$	$\sqrt{3} \approx 1.7$
1.25	≈ 3.0
1.50	≈ 14.1
1.55	≈ 48.1
$\frac{\pi}{2} \approx 1.57$	undefined

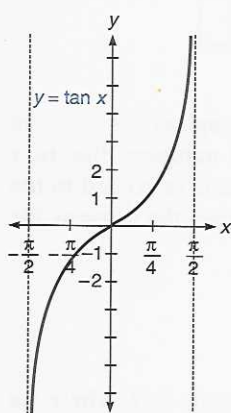


Figure 3-8

Graph of the tangent function

To obtain the graph of the tangent function, we also compute values and plot points. Some values for x between 0 and $\frac{\pi}{2}$ are shown in the table. Observe that as x gets closer to $\frac{\pi}{2}$, $\tan x$ gets larger. If we recall that $\tan x = \frac{\sin x}{\cos x}$, we can see why this is true. As x approaches $\frac{\pi}{2}$ (from below), $\sin x$ approaches 1, since $\sin \frac{\pi}{2} = 1$, and $\cos x$ approaches 0, since $\cos \frac{\pi}{2} = 0$. Now as the denominator, $\cos x$, gets smaller and smaller we divide it into values of $\sin x$, which are close to 1. Effectively we are calculating $\frac{1}{\cos x}$, or the reciprocal of $\cos x$. The smaller the absolute value of a number, the larger is its reciprocal. For example, the reciprocal of $\frac{1}{100}$ is 100, and of $\frac{1}{10,000}$ is 10,000. Thus, as $\cos x$ gets smaller, $\frac{1}{\cos x}$ gets larger and larger, and so $\tan x = \frac{\sin x}{\cos x}$ gets larger and larger. Since $\cos \frac{\pi}{2} = 0$, $\tan \frac{\pi}{2}$ is not defined. For the same reason, $\tan\left(-\frac{\pi}{2}\right)$ is not defined either. The graph of $y = \tan x$ is shown for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ in figure 3-8. It can be proved that the tangent function is π -periodic; that is, for any x , $\tan x = \tan(x + k\pi)$, k any integer. The actual proof will be an exercise in chapter 5. This π -periodic property means that the graph of the tangent function repeats every π units. More of the graph of $y = \tan x$ is shown in figure 3-9. The vertical dashed lines indicate values

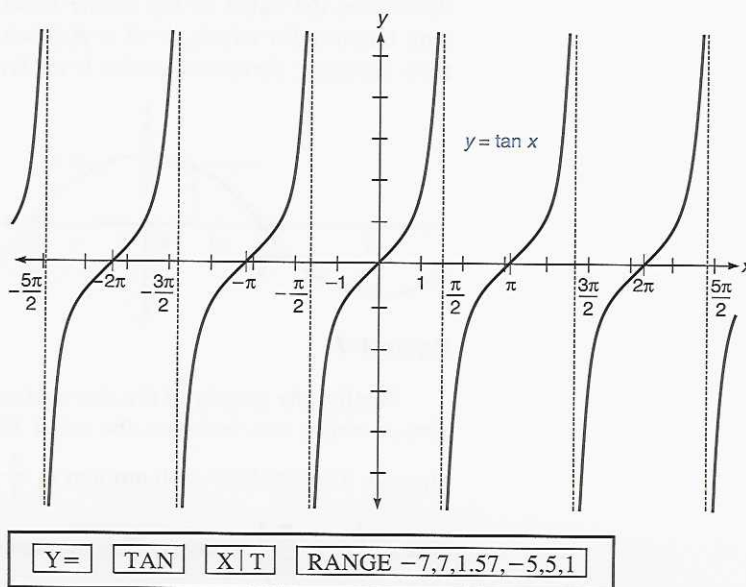


Figure 3-9

of x for which the tangent function is not defined. Since $\tan x = \frac{\sin x}{\cos x}$, we know that the tangent function is not defined wherever $\cos x = 0$. The vertical dashed lines are called **asymptotes** of the tangent function and occur wherever $\cos x = 0$.

The domain of the tangent function is all values of x except where $\cos x = 0$, and the range is all values of y .

$$\text{Domain}_{\text{tangent}}: x \neq \frac{\pi}{2} + k\pi, k \text{ any integer}$$

$$\text{Range}_{\text{tangent}}: R$$

It will also be an exercise to show that *the tangent function is an odd function*; that is, $\tan(-x) = -(\tan x)$ for any x in its domain.

Table 3-2 summarizes the properties of the three functions we have examined.

Function	Domain	Range	Period
$y = \sin x$	R	$-1 \leq y \leq 1$	2π
$y = \cos x$	R	$-1 \leq y \leq 1$	2π
$y = \tan x$	$x \neq \frac{\pi}{2} + k\pi$	R	π

$$\sin(-x) = -(\sin x) \text{ (odd)}$$

$$\cos(-x) = \cos x \text{ (even)}$$

$$\tan(-x) = -(\tan x) \text{ (odd)}$$

Table 3-2

We can use the odd-even properties of these functions to simplify some computations.

■ Example 3-1 A

1. Find $\tan\left(-\frac{5\pi}{6}\right)$.

Since tangent is an odd function, we know that

$$\tan\left(-\frac{5\pi}{6}\right) = -\left(\tan \frac{5\pi}{6}\right)$$

$\frac{5\pi}{6}$ terminates in quadrant II, so its reference angle is $\pi - \frac{5\pi}{6} = \frac{\pi}{6}$, and

$\tan \frac{\pi}{6} = \frac{\sqrt{3}}{3}$. Also, the tangent function is negative in quadrant II, so

$$\tan \frac{5\pi}{6} = -\frac{\sqrt{3}}{3}.$$

$$\text{Therefore, } -\left(\tan \frac{5\pi}{6}\right) = -\left(-\frac{\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3} \text{ and } \tan\left(-\frac{5\pi}{6}\right) = \frac{\sqrt{3}}{3}.$$

2. Find
- $\cos(-210^\circ)$
- .

Since the cosine function is an even function, $\cos(-210^\circ) = \cos 210^\circ$.

The reference angle for 210° is 30° , and $\cos 30^\circ = \frac{\sqrt{3}}{2}$. Since 210° terminates in quadrant III where the cosine function is negative, $\cos 210^\circ = -\frac{\sqrt{3}}{2}$. Therefore, $\cos(-210^\circ) = -\frac{\sqrt{3}}{2}$. ■

Mastery points

Can you

- Sketch the graphs of the sine, cosine, and tangent functions?
- State the domain, range, and period of the sine, cosine, and tangent functions?
- Use the odd-even properties to compute the values of $\sin x$, $\cos x$, and $\tan x$ for negative values of x ?

Exercise 3-1

- Sketch the graphs of
 - $y = \sin x$
 - $y = \cos x$
 - $y = \tan x$
- Using the graph of $y = \sin x$ as a guide, describe all values of x for which $\sin x$ is
 - 1
 - 1
 - 0
- Using the graph of $y = \tan x$ as a guide, describe all values of x for which $\tan x$ is 0.
- From memory, or using their graphs as an aid, state the domain, range, and period of each of the functions sine, cosine, and tangent.
- Using the graph of $y = \cos x$ as a guide, describe all values of x for which $\cos x$ is
 - 1
 - 1
 - 0

Use the appropriate property, odd or even, to simplify the computation of the exact value of the (a) sine, (b) cosine, and (c) tangent functions for the following values.

6. $-\frac{\pi}{3}$

7. $-\frac{\pi}{6}$

8. -45°

9. $-\frac{5\pi}{3}$

In the text we stated that a function f is odd if $f(-x) = -f(x)$ for all x in its domain and is even if $f(-x) = f(x)$ for all x in its domain. An algebraic example of an odd function is $f(x) = x^3$, since

$$\begin{aligned} f(-x) &= (-x)^3 \\ &= -x^3 \\ &= -f(x) \end{aligned}$$

Thus, to illustrate this point, again using $f(x) = x^3$, we can see that $f(-2) = -8$, $f(2) = 8$, and so $f(-2) = -f(2)$.

An example of an even function is $f(x) = x^2$, since we can show that $f(-x) = f(x)$.

$$\begin{aligned} f(-x) &= (-x)^2 \\ &= x^2 \\ &= f(x) \end{aligned}$$

Some functions are neither odd nor even, such as $f(x) = x - 3$, since $f(-x) = -x - 3$, but $-f(x) = -(x - 3) = -x + 3$, so $f(-x)$ is neither $f(x)$ nor $-f(x)$, as we see when we compare

$$\begin{aligned} f(x) &= x - 3 \\ f(-x) &= -x - 3 \\ -f(x) &= -x + 3 \end{aligned}$$

Compute $f(-x)$ and $-f(x)$ for each of the following functions, and state whether the function is odd, even, or neither.

- | | | | |
|--------------------------|--------------------------|--------------------------------|----------------------------------|
| 10. $f(x) = x$ | 11. $f(x) = 3x$ | 12. $f(x) = 3x^2$ | 13. $f(x) = -x^2$ |
| 14. $f(x) = 2x^4 - 4x^2$ | 15. $f(x) = 3x^2 - 2x^4$ | 16. $f(x) = 2x^3 - 4x$ | 17. $f(x) = 3x - 2x^3$ |
| 18. $f(x) = 3 \sin x$ | 19. $f(x) = 2 \cos x$ | 20. $f(x) = \frac{x^2 - 1}{4}$ | 21. $f(x) = \frac{x^5 - x^3}{x}$ |
| 22. $f(x) = \sin^2 x$ | 23. $f(x) = \tan x$ | 24. $f(x) = \sin x + \cos x$ | 25. $f(x) = \frac{\sin x}{x}$ |
- (Hint: Rewrite as $\frac{\sin x}{\cos x}$.)

3-2 Graphs and properties of the reciprocal functions

To graph the reciprocal functions cosecant, secant, and cotangent, we can use the graphs of the sine, cosine, and tangent functions as our guide.

The graph of the cosecant function

Remember that $\csc x = \frac{1}{\sin x}$. Thus, to graph $y = \csc x$ we first graph $y =$

$\sin x$, and then examine the reciprocal values. Figure 3-10 shows the graph of $y = \sin x$ for $0 \leq x \leq 2\pi$, as well as dashed lines that represent the cosecant function values for these values of x .

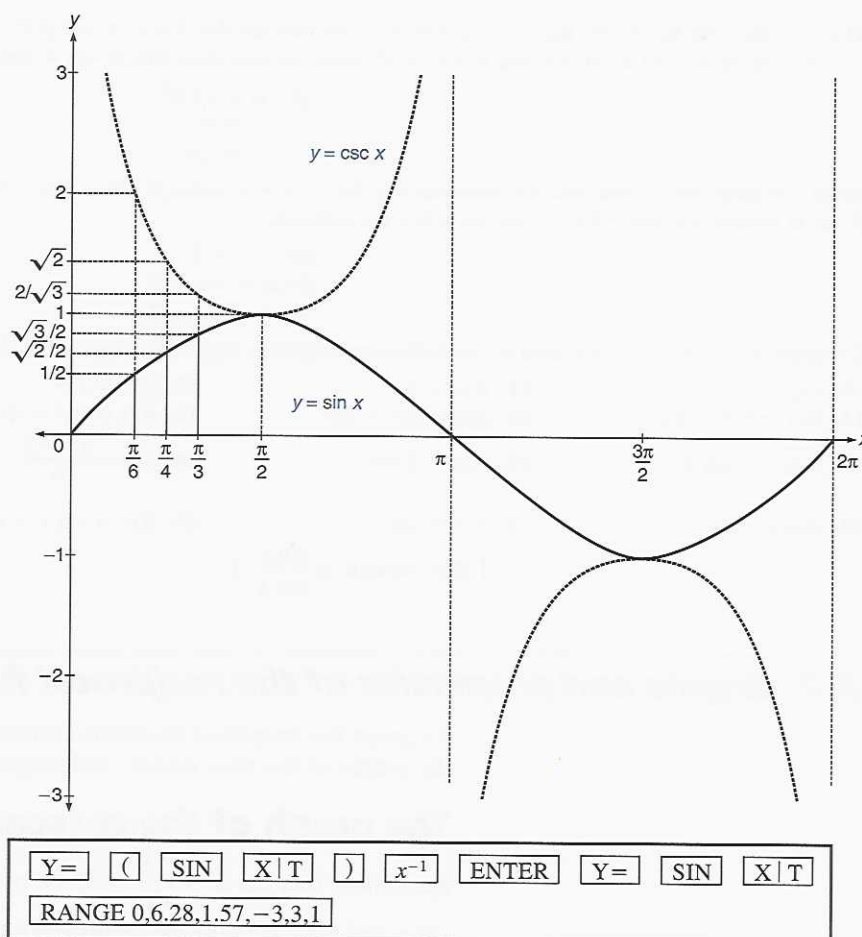


Figure 3-10

x	$\sin x$	$\csc x$
$\frac{\pi}{6}$	$\frac{1}{2}$	2
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$ (1.4)
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{2\sqrt{3}}{3}$ (1.2)
$\frac{\pi}{2}$	1	1

To see that the dashed lines represent the values for the cosecant, consider the table, which shows both the sine and cosecant values for selected values of x . These selected values are also illustrated in figure 3-10.

Referring to figure 3-10 and the table, we see that as x increases from $\frac{\pi}{6}$ to $\frac{\pi}{2}$, $\sin x$ increases from $\frac{1}{2}$ to 1, and the reciprocal values, $\csc x$, decrease from 2 to 1. Also, as $\sin x$ decreases in absolute value (i.e., gets closer to the x -axis), the reciprocal gets larger in absolute value. Wherever $\sin x$ is 1 or -1 , so is its reciprocal value, $\csc x$. Wherever $\sin x$ approaches 0, the absolute value of $\csc x$ approaches infinity (gets larger and larger). Note that if $\sin x$ approaches 0 through positive values, its reciprocal gets larger and larger, and

wherever $\sin x$ approaches 0 through negative values, $\csc x$ becomes larger and larger in *absolute value*, although it is negative.

To graph $y = \csc x$, we can rely on the graph of $y = \sin x$ for our guide. The steps are:

1. Graph $y = \sin x$.
2. Wherever $\sin x$ is $+1$ or -1 , so is $\csc x$.
3. Wherever $\sin x$ is 0, draw vertical dashed lines (asymptotes).
4. As $\sin x$ approaches 0, draw $\csc x$ getting greater and greater in absolute value, positive or negative depending on the sign of the sine function.

The graph of $y = \csc x$ is shown in figure 3-11; the graph of $y = \sin x$ is shown as a dashed line.

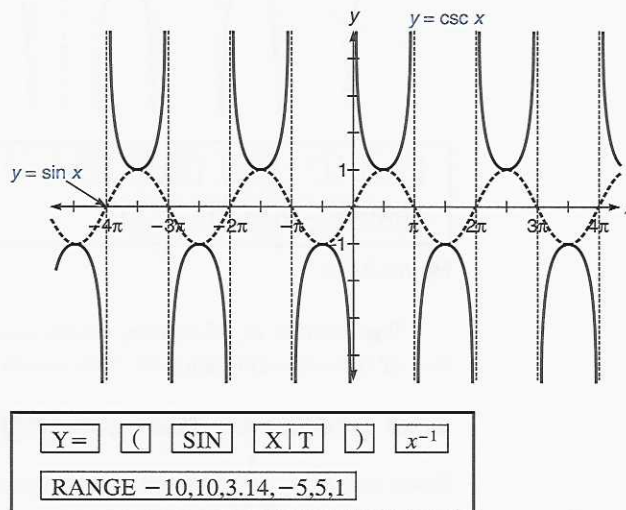


Figure 3-11

Note that the domain of the cosecant function is all x except where $\sin x = 0$, and the range is all y greater than or equal to 1 in absolute value. This range reflects the fact that since $\sin x$ is less than or equal to 1 in absolute value, $\frac{1}{\sin x}$ must be greater than or equal to 1 in absolute value. Also, the cosecant function is 2π -periodic, just as the sine function is.

The graph of the secant function

The graph of $y = \sec x$ is analyzed in the same manner as the graph of $y = \csc x$, except that we are considering $y = \frac{1}{\cos x}$ instead of $y = \frac{1}{\sin x}$.

Thus, to graph $y = \sec x$, we can rely on the graph of $y = \cos x$ for our guide. The steps are the same as when graphing the cosecant function, except that the cosine function is our guide. This produces the graph of $y = \sec x$, which is shown along with the graph of $y = \cos x$ in figure 3-12.

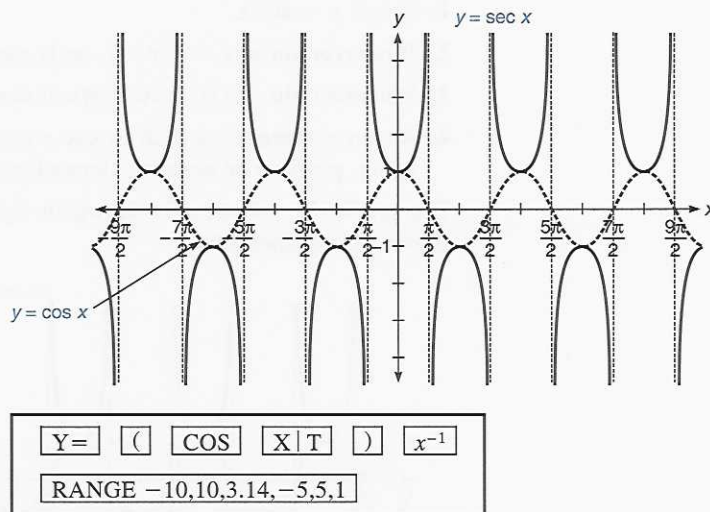


Figure 3-12

The domain is, of course, where $\cos x \neq 0$, and the range is the same as that of the cosecant function. The secant function is also 2π -periodic.

The graph of the cotangent function

Since $\cot x = \frac{1}{\tan x}$ except where $\tan x = 0$, we can obtain the graph of $y = \cot x$ by analyzing the graph of $y = \tan x$ as we did previously for the other reciprocal functions. The graphs of both $y = \tan x$ and $y = \cot x$ are shown in figure 3-13. Note that wherever $\tan x$ is 0, we have a vertical asymptote, and wherever $\tan x$ approaches infinity or negative infinity, $\cot x$ approaches 0. This should make sense, since as a quantity gets greater and greater in absolute value, its reciprocal will get smaller and smaller in absolute value. Note that $y = \cot x$ is π -periodic as is the tangent function, its domain is all x except where $\sin x = 0$ (since $\cot x = \frac{\cos x}{\sin x}$), and its range is all real numbers, as is the range of the tangent function.

Note One way to remember where the vertical asymptotes are for the cotangent function is to sketch the graph of the sine function. Wherever $\sin x$ is 0, $\cot x$ does not exist, and instead "goes to infinity or negative infinity." This is where we draw the vertical asymptotes.

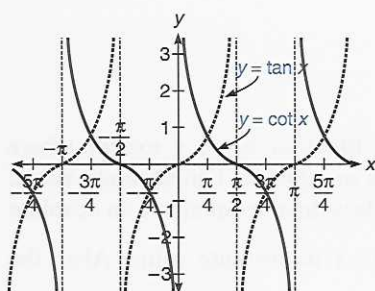


Figure 3-13

The graph of $y = \cot x$ is shown in figure 3-14.

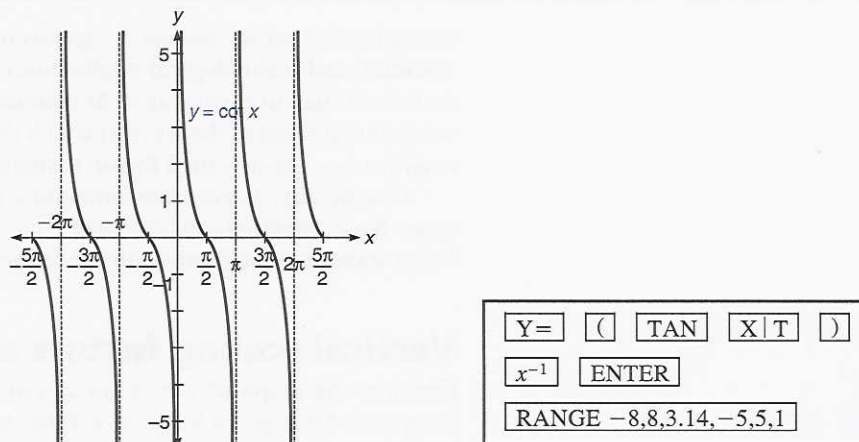


Figure 3-14

The properties of the three reciprocal trigonometric functions are summarized in table 3-3, where k is any integer.

Function	Domain	Range	Period
$y = \csc x$	$x \neq k\pi$	$ y \geq 1$	2π
$y = \sec x$	$x \neq \frac{\pi}{2} + k\pi$	$ y \geq 1$	2π
$y = \cot x$	$x \neq k\pi$	R	π

Table 3-3

Mastery points

Can you

- Sketch the graphs of the reciprocal functions?
- State the domain, range, and period of the reciprocal functions?

Exercise 3-2

1. Sketch the graphs of the three reciprocal trigonometric functions.
2. State the domain, range, and period for each of the three reciprocal trigonometric functions.
3. Use the identity $\csc x = \frac{1}{\sin x}$ to show that cosecant is an odd function.
4. Use the identity $\sec x = \frac{1}{\cos x}$ to show that secant is an even function.
5. Use the identity $\cot x = \frac{1}{\tan x}$ to show that cotangent is an odd function.

3-3 Linear transformations of the sine and cosine functions

In section 3-1 we developed the graphs of the sine and cosine functions. Most scientific and technological applications of these functions require that they be transformed in some way to fit measured or theoretical data. In this section we examine some of the ways in which this can be done. In particular, we will examine operations called **linear transformations**.

Graphically, linear transformations are operations that move a graph in some fixed direction, or “squeeze” or “expand” the graph uniformly. The linear transformations we examine are scaling factors and translations.

Vertical scaling factors and translations

Consider the graph of $y = 3 \sin x$, and what this equation tells us to do to compute y for a given value of x . First, we are to compute the value of $\sin x$, and then multiply this value by 3. Thus, for the same value of x , the expression $3 \sin x$ will be 3 times greater (in absolute value) than the expression $\sin x$. If we then compare the graph of $y = 3 \sin x$ with the graph of $y = \sin x$, the y -values of the first must be 3 times greater than the y -values of the second. See figure 3-15.

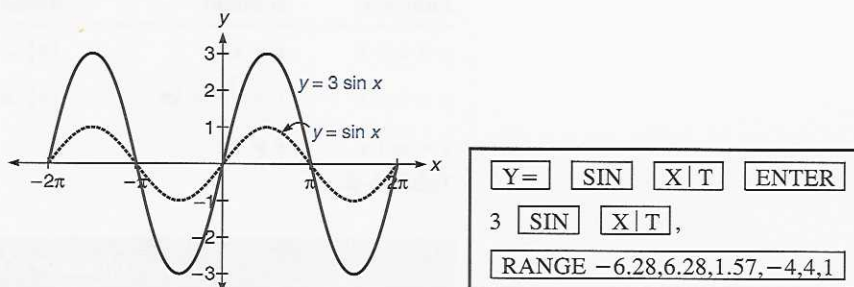
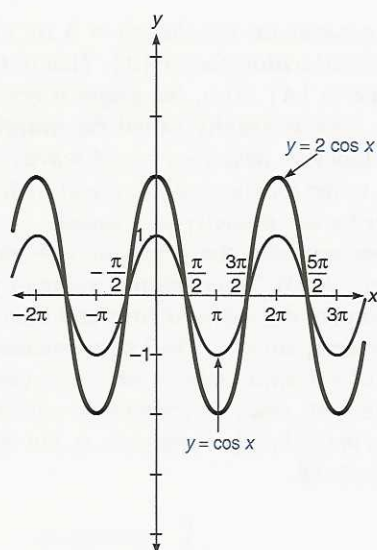


Figure 3-15

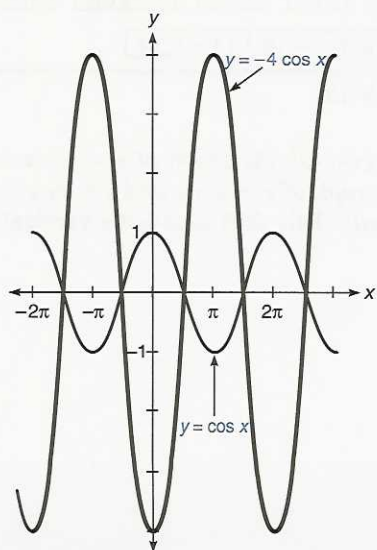
For the same reasons, the graph of $y = 2 \cos x$ is the same as that of $y = \cos x$, except that it reaches a magnitude of 2 instead of 1. See figure 3-16. Except for this vertical change in scale, the graph is the same as that of $y = \cos x$.



Y=	COS	X T	ENTER	2
COS	X T	,		
RANGE -3.25,8,1.57,-3,3,1				

Figure 3-16

If the coefficient of the sine or cosine value is negative, a reflection about the horizontal axis occurs. Consider, for example, the graph of $y = -4 \cos x$ compared to the graph of $y = \cos x$. To compute a y -value in $y = -4 \cos x$, we first compute $\cos x$ and then multiply this value by -4 . Multiplying by a negative value changes the sign of the value being multiplied. Thus, whenever y is negative in the graph $y = \cos x$, the y in $y = -4 \cos x$ is positive and scaled (multiplied) by a factor of 4. Also, whenever y is positive in the graph $y = \cos x$, the y in $y = -4 \cos x$ is negative and scaled by a factor of 4. See figure 3-17.

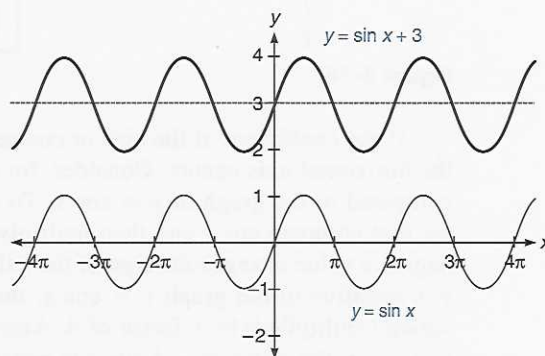


Y=	COS	X T	ENTER	
(-)	4	COS	X T	,
RANGE -3.25,8,1.57,-5,5,1				

Figure 3-17

In general, the graphs of $y = A \sin x$ and $y = A \cos x$ are scaled vertically by a vertical scaling factor $|A|$. That is, the magnitude of the graph is changed from one to $|A|$. Also, the graph is reflected about the horizontal (x) axis if $A < 0$. $|A|$ is usually called the **amplitude** of the function. If the sine or cosine function describes sound waves in the air, then the amplitude corresponds to the loudness of the sound; indeed, we often use the word amplitude to describe this property of a sound.

Now consider the graph of $y = \sin x + 3$ (not to be confused with $y = \sin(x + 3)$). To compute a value of y for a given x in $y = \sin x + 3$, we first compute the value of $\sin x$ and then add 3 to this value. This means that for a given x , $\sin x + 3$ is 3 units greater than $\sin x$. Thus, if we compare the graphs of $y = \sin x$ and $y = \sin x + 3$, the second graph must be 3 units higher than the first, since to compute a y in the second equation we do the same thing as in the first (compute $\sin x$), but then add 3. This vertical shift is shown in figure 3-18.



Y=	SIN	X T	ENTER	SIN	X T	+	3,
RANGE -4,10,3.14,-2.5,1							

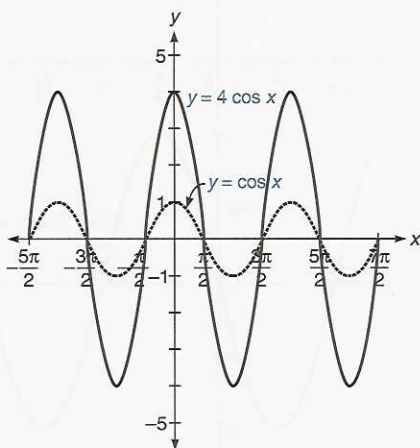
Figure 3-18

In general, the graph of $y = \sin x + D$ and $y = \cos x + D$, is the same as the graph of $y = \sin x$ and $y = \cos x$, respectively, but shifted up or down $|D|$ units. This shift is called a **vertical translation**.

Example 3-3 A

1. Graph $y = 4 \cos x$.

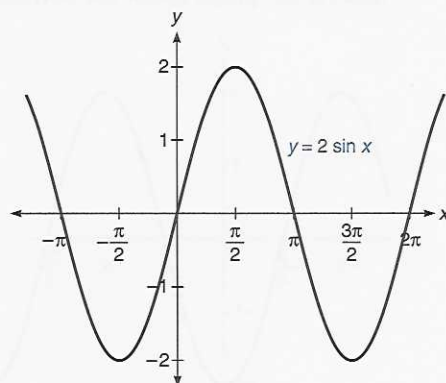
This is the same as the graph of $y = \cos x$, except scaled vertically by a factor of 4. Thus, the amplitude is 4. This is shown in the figure.



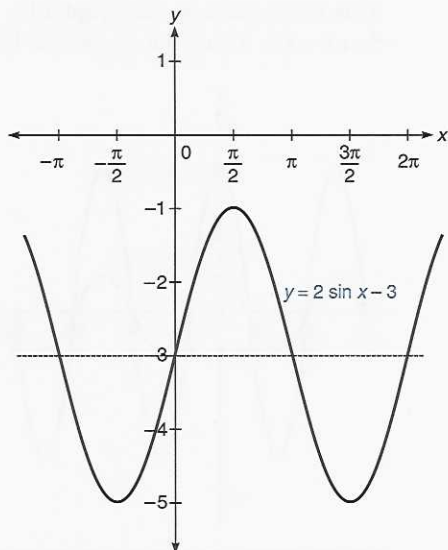
Y=	4	COS	X T	, RANGE -8,12,1.57,-5.5,1
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2. Graph $y = 2 \sin x - 3$.

The 2 affects the amplitude of the graph, and the -3 causes a vertical translation down, because we are subtracting values from $2 \sin x$. In the first figure we draw a sine curve with amplitude 2.



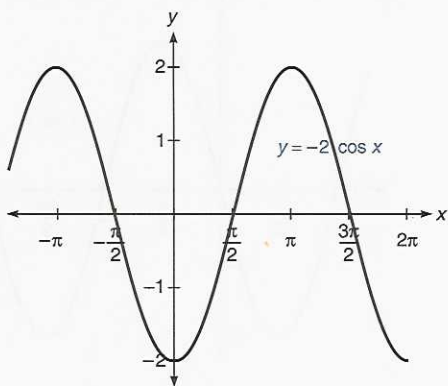
In the second figure we show the same curve translated down 3 units.



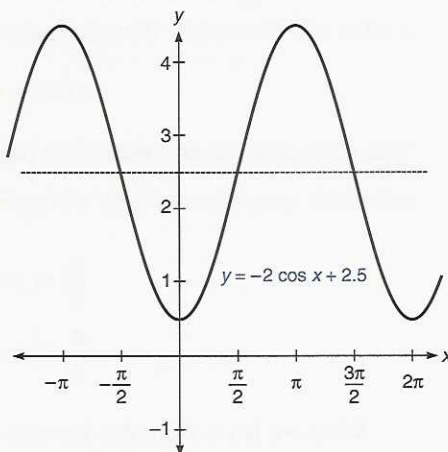
Y=	2	SIN	X T	-	3	ENTER	(-)	3,
RANGE -4,7,1.57,-6,1,1								

3. Graph $y = -2 \cos x + 2.5$.

We first graph $y = -2 \cos x$; the amplitude is $|-2| = 2$, and the -2 reflects the graph about the horizontal axis. See the first figure.



Now we shift this graph up 2.5 units. See the second figure.



Y=	(-)	2	COS	X T	+	2.5	ENTER	2.5,
RANGE -4,7,1.57,-1,5,1								

Horizontal scaling factors and translations

We have seen how to transform the graph of a sine or cosine function vertically. It is just as important to be able to do this horizontally.

The **argument** of a function is the expression to be used as the domain element when doing computations. In $y = \sin x$ or $y = \cos x$, the x is the argument of the function. In $y = \sin 3x$, the expression $3x$ is the argument. In $y = \cos(x - 4)$ the expression $x - 4$ is the argument. In $y = 2 \sin 4x - 3$, $4x$ is the argument. The argument is the quantity we “take the sine or cosine of.”

Now consider what we know about the sine and cosine functions. As the argument goes from 0 to 2π , each of these functions produces the graph shown in figure 3-19. We call the portion of each graph shown in figure 3-19 the **basic sine cycle** and the **basic cosine cycle**, respectively. Each of these basic cycles is repeated over and over to get the final, complete graphs of $y = \sin x$ and $y = \cos x$. The important fact is that *as the argument takes on values from 0 to 2π , we get one basic cycle of the function*. Note that the basic sine cycle is 0 at its beginning, middle, and end points. The basic cosine function starts with $y = 1$, ends with $y = 1$, and $y = -1$ at the midpoint of the cycle.

Now consider what the graph of $y = \sin\left(x - \frac{\pi}{4}\right)$ should look like. We know that one basic cycle of the sine function is produced as the argument goes from 0 to 2π . In this case, the argument is the expression $x - \frac{\pi}{4}$. We

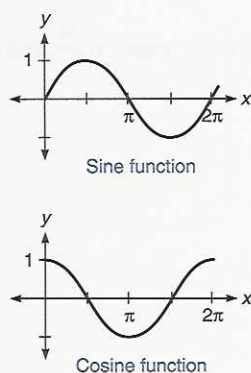


Figure 3-19

examine $x - \frac{\pi}{4}$ as it takes on all values from 0 to 2π to find out what values x takes on. We can do this algebraically, using

$$0 \leq x - \frac{\pi}{4} \leq 2\pi$$

This states that the argument runs (takes on values) from 0 to 2π . Now we can solve this statement for x by adding $\frac{\pi}{4}$ to each part.

$$\frac{\pi}{4} \leq x \leq 2\pi + \frac{\pi}{4}$$

$$\frac{\pi}{4} \leq x \leq \frac{9\pi}{4}$$

What we learn from this process is that for the expression $x - \frac{\pi}{4}$ to take on all values from 0 to 2π , x must take on all values from $\frac{\pi}{4}$ to $\frac{9\pi}{4}$. Now we can reason as follows:

1. We know that one basic cycle of the sine function is produced as the argument, in this case $x - \frac{\pi}{4}$, varies from 0 to 2π .
2. The expression $x - \frac{\pi}{4}$ varies from 0 to 2π as x varies from $\frac{\pi}{4}$ to $\frac{9\pi}{4}$.
3. Therefore, the expression $\sin\left(x - \frac{\pi}{4}\right)$ produces one basic cycle of the sine function as x varies from $\frac{\pi}{4}$ to $\frac{9\pi}{4}$.

Thus, our basic cycle does not start at 0 and end at 2π but starts at $\frac{\pi}{4}$ and ends at $\frac{9\pi}{4}$. See figure 3-20.

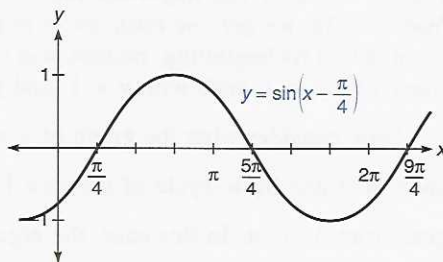


Figure 3-20

If we find the distance between $\frac{\pi}{4}$ and $\frac{9\pi}{4}$, it is $\frac{9\pi}{4} - \frac{\pi}{4} = \frac{8\pi}{4} = 2\pi$. This is the “length” of one basic cycle and is called the “period” of the function. We will define the period of the sine and cosine functions shortly. Right now, let us simply observe that a good rule for marking the x -axis is to divide it by using increments of one half of the period. To find this amount, divide the period by two (or multiply by one half). Thus, for convenience we marked the horizontal scale in increments of $\frac{2\pi}{2} = \pi$, starting at $\frac{\pi}{4}$. Note that the function crosses the x -axis halfway between the beginning and end of the cycle (at $\frac{5\pi}{4}$) and has high and low points halfway between this point and the beginning and end points of the cycle.

Now, since we know that the sine function is periodic, and we have graphed one cycle, we repeat this cycle to obtain the complete graph. See figure 3-21.

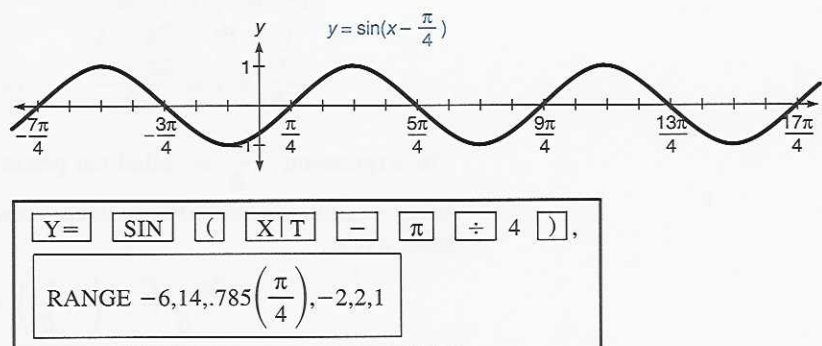


Figure 3-21

Although we looked at the previous function in some detail, we can state the process we used in just a few steps. To understand these steps, remember the underlying idea we used:

We know what the graphs of the sine and cosine functions look like as their arguments take on values from 0 to 2π . We therefore find out what values x has to take on for the argument to take on all values from 0 to 2π . As x takes on these values, we get one basic cycle of the sine or cosine function.

We now state the procedure to graph sine and cosine functions where the argument is of the form $Bx + C$, $B > 0$.

Graphing

$$y = A \sin(Bx + C) + D \text{ and}$$

$$y = A \cos(Bx + C) + D, B > 0$$

1. Solve $0 \leq Bx + C \leq 2\pi$ for x . This gives the left and right end points for one basic cycle.
2. Label the amplitude $|A|$. Use the left and right end points found in step 1, along with the amplitude, to draw one basic cycle. Reflect about the horizontal axis if $A < 0$.
3. Repeat this cycle to obtain as much of the graph as desired.
4. Apply a vertical shift D if necessary.

Note We will discuss the case where $B < 0$ shortly.

We need to define the term period, used above, and another term, phase shift, before proceeding with more examples. To do this we perform step 1 for the general case. Solving $0 \leq Bx + C \leq 2\pi$ for x :

$$0 \leq Bx + C \leq 2\pi$$

$$-C \leq Bx \leq 2\pi - C \quad \text{Subtract } C \text{ from each expression}$$

$$-\frac{C}{B} \leq x \leq \frac{2\pi - C}{B} \quad \text{Divide each expression by } B$$

The expression $-\frac{C}{B}$ is called the **phase shift** of the sine or cosine function being examined. The difference between the left and right end points of the basic cycle,

$$\begin{aligned} \frac{2\pi - C}{B} - \left(-\frac{C}{B}\right) &= \frac{2\pi - C}{B} + \frac{C}{B} \\ &= \frac{2\pi - C + C}{B} \\ &= \frac{2\pi}{B} \end{aligned}$$

is called the **period** of the sine or cosine function. B is also the number of complete cycles in 2π units.

It is not necessary to memorize these general expressions since the method we are using will produce these results anyway. With these terms, however, we can now state a precise *guideline for marking off the x -axis* when we graph. Mark the axis in increments of one half of the period, starting at the phase shift.

■ **Example 3-3 B**

1. Graph $y = 2 \cos 4x$. Show three cycles. State the amplitude, period, and phase shift.

Amplitude is 2, with no reflection about the x -axis.

Step 1: Solve $0 \leq 4x \leq 2\pi$ for x .

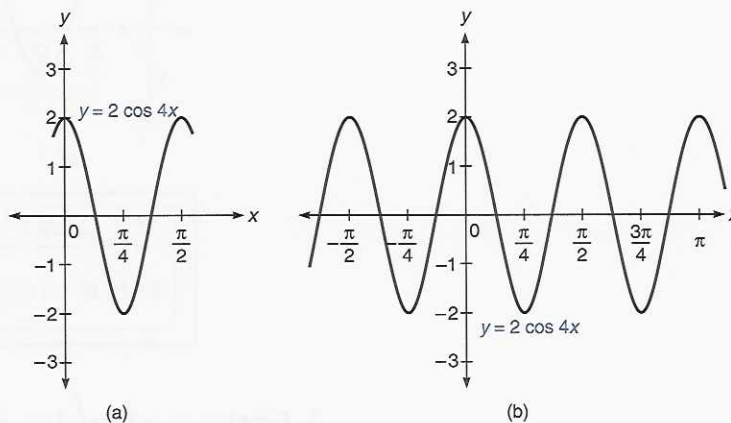
$$\frac{0}{4} \leq \frac{4x}{4} \leq \frac{2\pi}{4} \quad \text{Divide by 4}$$

$$0 \leq x \leq \frac{\pi}{2}$$

We know that one basic cycle of the cosine function starts at 0 and ends at $\frac{\pi}{2}$. The period is $\frac{\pi}{2}$, and the phase shift is 0. The x -axis will be marked off in increments of one half of the period: $\frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$.

Step 2: Draw one basic cycle with amplitude 2. See part (a) of the figure.

Step 3: We get two more basic cycles by marking off one more period to the right of $\frac{\pi}{2}$, and one more to the left of 0. We then draw in the basic cycles. See part (b) of the figure.



$Y = 2 \cos 4 X \quad T, \quad \text{RANGE } -2.4, .785, -3.3, 1$

2. Graph $y = \cos(3x - \pi)$. Show three cycles. State the amplitude, period, and phase shift.

Step 1: $0 \leq 3x - \pi \leq 2\pi$

$$\pi \leq 3x \leq 3\pi \quad \text{Add } \pi \text{ to each expression}$$

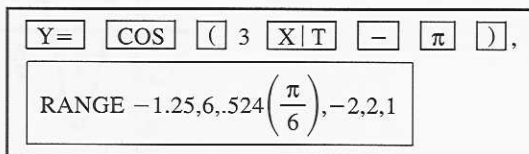
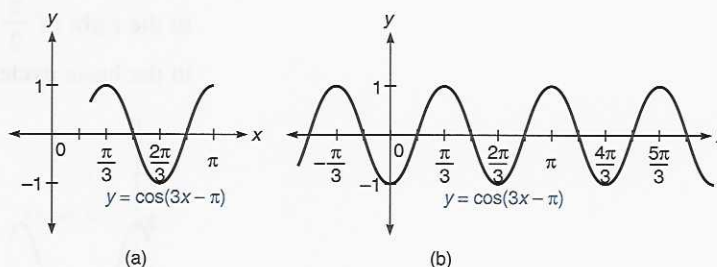
$$\frac{\pi}{3} \leq x \leq \pi \quad \text{Divide each expression by 3}$$

We know that we get one basic cycle of the cosine function as x varies from $\frac{\pi}{3}$ to π . The phase shift is $\frac{\pi}{3}$, and the period is

$\pi - \frac{\pi}{3} = \frac{2\pi}{3}$. We mark the x -axis in increments of $\frac{1}{2} \cdot \frac{2\pi}{3} = \frac{\pi}{3}$, starting at $\frac{\pi}{3}$, the phase shift, because this is where one basic cycle will start.

Step 2: We mark the amplitude, 1, and draw one basic cycle. See part (a) of the figure.

Step 3: In part (b) of the figure we show one more cycle on each “side” of the basic cycle.



3. Graph $y = -3 \sin\left(3x - \frac{\pi}{4}\right)$. Show three cycles. State the amplitude, period, and phase shift.

The amplitude will be $|-3| = 3$ for this function. Because $-3 < 0$, there will be a reflection of the graph about the horizontal axis.

Step 1: $0 \leq 3x - \frac{\pi}{4} \leq 2\pi$

$$0 \leq 12x - \pi \leq 8\pi \quad \text{Multiply each expression by 4}$$

$$\pi \leq 12x \leq 9\pi$$

$$\frac{\pi}{12} \leq x \leq \frac{9\pi}{12} \text{ or } \frac{\pi}{12} \leq x \leq \frac{3\pi}{4}$$

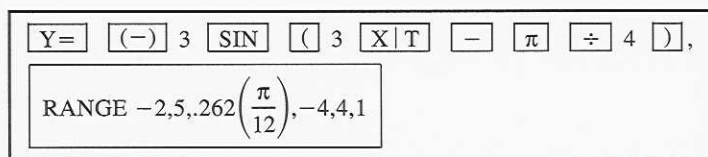
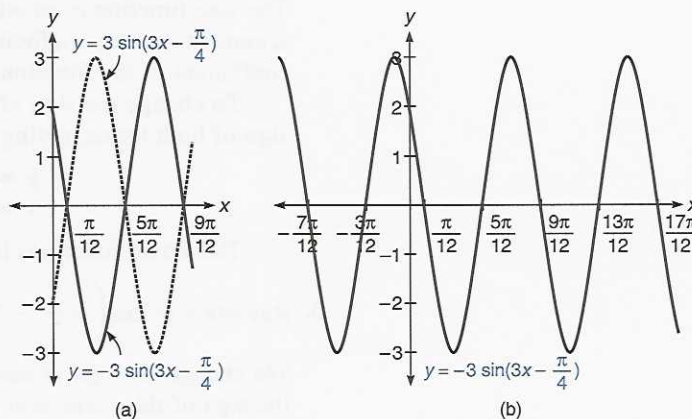
The phase shift is $\frac{\pi}{12}$, and the period is $\frac{9\pi}{12} - \frac{\pi}{12} = \frac{8\pi}{12} = \frac{2\pi}{3}$.

We mark the x -axis in increments of $\frac{1}{2} \cdot \frac{2\pi}{3} = \frac{\pi}{3}$, starting at $\frac{\pi}{12}$. Actually, we will use the value $\frac{4\pi}{12}$ instead of $\frac{\pi}{3}$ for convenience.

Step 2: We mark the x -axis in increments of $\frac{4\pi}{12}$ and draw a sine cycle

between $\frac{\pi}{12}$ and $\frac{9\pi}{12}$ with amplitude 3. This is shown by the dashed lines in part (a) of the figure. Its reflection about the horizontal axis is shown in solid lines.

Step 3: We mark off more increments of $\frac{4\pi}{12}$ and draw two more cycles. See part (b) of the figure.



If B is negative in the argument $Bx + C$, we use the odd and even identities to get an equivalent expression with B positive. Recall these identities (section 3-1):

$$\begin{aligned}\sin(-x) &= -(\sin x) \\ \cos(-x) &= \cos x\end{aligned}$$

Concept

We can change the sign of the argument of the sine function and still have an equivalent expression if we change the sign of the coefficient of the sine function itself. We can change the sign of the argument of the cosine function and still have an equivalent expression. We do *not* change the sign of the coefficient of the cosine function.

Example 3-3 C

1. Rewrite $y = 2 \cos(-3x)$ so that the coefficient of x is positive.

Changing the sign of the argument, $-3x$, we get $3x$. We do not change the sign of the coefficient of the cosine function since it is an even function. Thus,

$$\begin{aligned} y &= 2 \cos(-3x) \text{ becomes} \\ y &= 2 \cos 3x \end{aligned}$$

These are equivalent functions, so they have the same graph.

2. Rewrite $y = 3 \sin(-2x + \pi)$ so that the coefficient of x is positive.

The sine function is an odd function, so we change the sign of both the argument and the coefficient of the function. We change the sign of the coefficient of the function, 3, to -3 .

To change the sign of the argument, $-2x + \pi$, we must change the sign of both terms, giving $2x - \pi$. Thus,

$$\begin{aligned} y &= 3 \sin(-2x + \pi) \text{ becomes} \\ y &= -3 \sin(2x - \pi) \end{aligned}$$

These two functions have the same graph.

3. Rewrite $y = \cos\left(-\frac{x}{2} - 3\right)$ so that the coefficient of x is positive.

We change the sign of each term of the argument, but we do not change the sign of the coefficient of the function itself. Thus,

$$\begin{aligned} y &= \cos\left(-\frac{x}{2} - 3\right) \text{ becomes} \\ y &= \cos\left(\frac{x}{2} + 3\right) \end{aligned}$$

4. Rewrite $y = -\sin\left(-\frac{x}{3}\right)$ so that the coefficient of x is positive.

Change the sign of $-\frac{x}{3}$ to $\frac{x}{3}$, and of the coefficient of the sine function, -1 , to 1.

$$\begin{aligned} y &= -\sin\left(-\frac{x}{3}\right) \text{ becomes} \\ y &= \sin \frac{x}{3} \end{aligned}$$

Example 3-3 D illustrates how to use the odd/even properties to help graph a function.

■ Example 3-3 D

Graph $y = -2 \sin\left(-\frac{2x}{3} + 1\right)$. Show three cycles, and state the amplitude, period, and phase shift.

Since the x term of the argument is negative, we use the odd property of the sine function to change both the sign of the argument and the sign of the coefficient of the function. This gives us the equivalent function

$$y = 2 \sin\left(\frac{2x}{3} - 1\right)$$

Step 1: $0 \leq \frac{2x}{3} - 1 \leq 2\pi$

$$0 \leq 2x - 3 \leq 6\pi$$

$$3 \leq 2x \leq 6\pi + 3$$

$$\frac{3}{2} \leq x \leq \frac{6\pi + 3}{2}$$

Phase shift is $\frac{3}{2}$, and period is $\frac{6\pi + 3}{2} - \frac{3}{2} = \frac{6\pi}{2} = 3\pi$. We mark the x -axis by using $\frac{3}{2}$ as our starting point and using increments of $\frac{3\pi}{2}$, which is one half of the period. In this case, we also compute decimal approximations to the results to make it easier to plot our points. Several of the computations are

$$\frac{3}{2} + \frac{3\pi}{2} = \frac{3 + 3\pi}{2} \approx 6.2$$

$$\frac{3 + 3\pi}{2} + \frac{3\pi}{2} = \frac{3 + 6\pi}{2} \approx 10.9$$

$$\frac{3 + 6\pi}{2} + \frac{3\pi}{2} = \frac{3 + 9\pi}{2} \approx 15.6$$

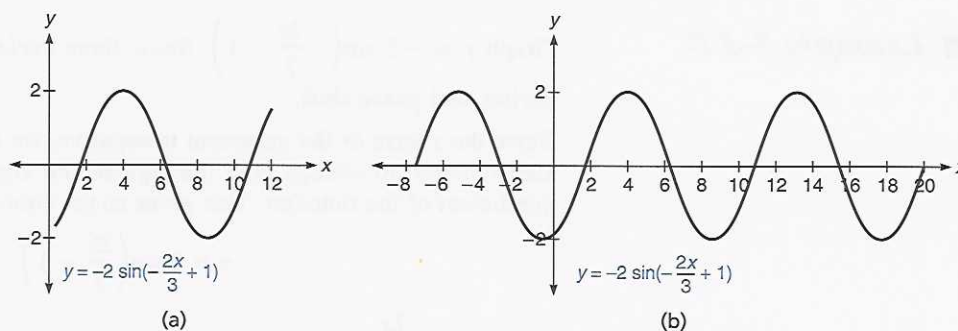
and

$$\frac{3}{2} - \frac{3\pi}{2} = \frac{3 - 3\pi}{2} \approx -3.2$$

$$\frac{3 - 3\pi}{2} - \frac{3\pi}{2} = \frac{3 - 6\pi}{2} \approx -7.9$$

Step 2: One basic cycle is shown in part (a) of the figure.

Step 3: Two more cycles are shown in part (b) of the figure.



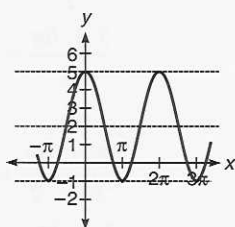
Y=	(-)	2	SIN	(	(-)	2	X	/	3	+	1	)	,
RANGE -8,20,2,-3,3,1													

We can observe at this point that the method we have been using has given us both horizontal translations and scaling factors. Phase shift is a horizontal translation, and if we divide the period, $\frac{2\pi}{B}$, by 2π (the period of the basic sine or cosine function), we get $\frac{1}{B}$, a horizontal scale factor. Normally we do not actually compute the horizontal scale factor.

There are times when we will want to find the equation of a function, given some of its properties.

■ Example 3-3 E

- Find the equation of the cosine function in the figure.

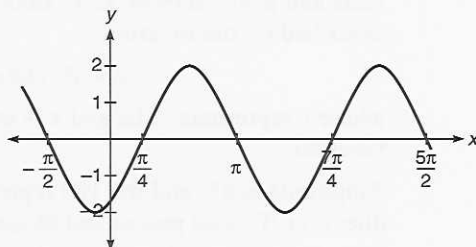


We know the equation is of the form

$$y = A \cos(Bx + C) + D$$

It is shifted up 2 units, so D is $+2$. The distance between the high and low points is 6. The amplitude is one half this value, or 3, so $|A| = 3$. Since a basic cycle starts at 0 and ends at 2π , we know the argument is simply x . Thus, the equation is $y = 3 \cos x + 2$.

2. Find an equation of the sine function in the figure.



We know that the equation is of the form

$$y = A \sin(Bx + C) + D$$

Since the distance between the high and low points of the graph is 4, we know that A is 2. Also, there is no vertical shift, so D is 0.

We need to find the argument of the function. Note that a cycle of this function starts at $\frac{\pi}{4}$ and ends at $\frac{7\pi}{4}$. We can work backward from this information.

We know that we get one basic cycle as x takes on values between these two points; that is, $\frac{\pi}{4} \leq x \leq \frac{7\pi}{4}$.

Our objective is to arrange the values so that the left value is 0 and the right value is 2π . (Remember, we are working back to the argument of the function.)

First, we want the left value to be 0.

$$\begin{array}{ll} \pi \leq 4x \leq 7\pi & \text{Multiply by 4} \\ 0 \leq 4x - \pi \leq 6\pi & \text{Subtract } \pi \end{array}$$

Now we want the right point to be 2π . Dividing by 3 will do this.

$$\begin{array}{l} \frac{0}{3} \leq \frac{4x - \pi}{3} \leq \frac{6\pi}{3} \\ 0 \leq \frac{4x}{3} - \frac{\pi}{3} \leq 2\pi \end{array}$$

We thus find the argument is $\frac{4x}{3} - \frac{\pi}{3}$. Thus, our final answer is

$$y = 2 \sin\left(\frac{4x}{3} - \frac{\pi}{3}\right).$$

In some applications we want to express values of x in degrees as opposed to radians. Our procedures are the same, except that our limits for the basic cycles are 0° and 360° instead of 0 and 2π .

■ Example 3-3 F

1. An AC (alternating current) signal with peak-to-peak voltage of 170 volts and phase shift of 120° , riding on a DC level of 100 volts, could be described by the function

$$y = 85 \sin(x + 120^\circ) + 100$$

where y represents volts and x is in degrees. Graph one cycle of this function.

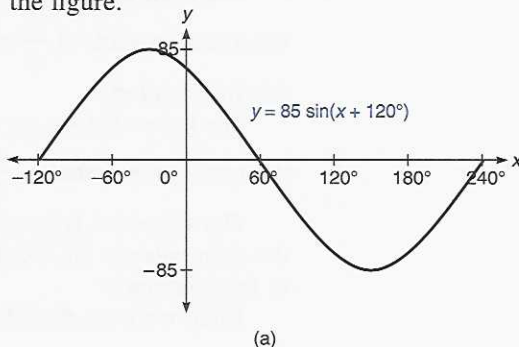
Amplitude is 85, and the 100 represents a vertical shift in the positive direction. To find period and phase shift we proceed as follows:

Step 1: $0^\circ \leq x + 120^\circ \leq 360^\circ$

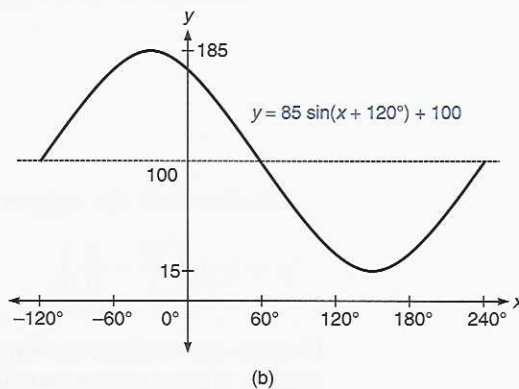
$$-120^\circ \leq x \leq 240^\circ$$

Phase shift is -120° and period is $240^\circ - (-120^\circ) = 360^\circ$. We mark the x -axis in increments of half the period, 180° , starting at -120° .

Step 2: We see that a basic cycle begins at -120° and ends at 240° . See the figure.



Step 3: We also label the amplitude, 85. We then shift the graph vertically by 100 units. See the figure.



Put the calculator in DEGREE mode (use the **MODE** key).

Y= 85 **SIN** (**X** **T** **+** 120 **)** **+** 100 **ENTER** 100,
RANGE -130,250,60,-10,190,20

2. An electronic signal is to be modeled with the sine function. The peak-to-peak voltage is 340 volts (amplitude is 170 volts). There is a phase shift of 30° , and the period is 120° . The signal is at 200 volts above ground potential. (There is a vertical shift of 200.) Find the sine function that will model this signal.

We know that the function is of the form

$$y = A \sin(Bx + C) + D$$

and that $A = 170$ and $D = 200$. To find B and C we can proceed “backward.” We know that we get one basic cycle as x varies between 30° , the phase shift, and $30^\circ + 120^\circ$, or phase shift + period.

$$30^\circ \leq x \leq 30^\circ + 120^\circ$$

Now adjust this so that phase shift is 0° and period is 360° .

$$30^\circ \leq x \leq 150^\circ$$

Subtract 30° from each expression to get 0° phase shift.

$$0^\circ \leq x - 30^\circ \leq 120^\circ$$

Multiply each term by 3, since this will make the end point 360° .

$$0^\circ \leq 3x - 90^\circ \leq 360^\circ$$

Thus, the argument of the function is $3x - 90^\circ$, and the function we want is

$$y = 170 \sin(3x - 90^\circ) + 200$$

Mastery points

Can you

- Graph an equation of the form

$$y = A \sin(Bx + C) + D \text{ or}$$

$$y = A \cos(Bx + C) + D?$$

- Find a sine or cosine equation that is appropriate, given values of A and D and the initial and terminal points of a basic cycle?

Exercise 3-3

Graph three cycles of the following functions.

1. $y = 5 \sin x$

2. $y = 5 \cos x$

3. $y = \frac{2}{3} \cos x$

4. $y = \frac{1}{5} \sin x$

5. $y = -4 \cos x$

6. $y = -2 \sin x$

7. $y = -\frac{1}{3} \sin x$

8. $y = -\frac{5}{2} \sin x$

9. $y = 2 \sin x + 1$

10. $y = 3 \cos x - 2$

11. $y = -\frac{3}{4} \cos x - 2$

12. $y = -\frac{1}{2} \sin x + 3$

Graph three cycles of the following functions. State the amplitude, period, and phase shift of each.

13. $y = 2 \sin 4x$

14. $y = 3 \cos \frac{x}{2}$

15. $y = \cos\left(x - \frac{\pi}{2}\right)$

16. $y = 3 \sin(2x + \pi)$

17. $y = \frac{2}{3} \sin(3x + \pi)$

18. $y = \frac{5}{8} \cos 5x$

19. $y = -\cos 3x$

20. $y = -\sin x$

21. $y = -\cos\left(2x + \frac{\pi}{2}\right)$

22. $y = -\sin\left(3x - \frac{\pi}{3}\right)$

23. $y = \sin(3x + 2\pi)$

24. $y = \cos(2x - 3\pi)$

25. $y = \cos 2\pi x$

26. $y = \sin \pi x$

27. $y = 2 \sin 3x + 2$

28. $y = 3 \cos 2x - 3$

29. $y = -3 \cos x + 1$

30. $y = -\sin 4x + 1$

31. $y = 2 \sin(2x - \pi) + 1$

32. $y = 3 \sin(3x + \pi) - 3$

33. $y = \sin \pi x + 1$

34. $y = 2 \cos \frac{\pi x}{2} - 2$

Use the odd/even properties of the sine and cosine functions to rewrite each of the following functions as an equivalent function in which the coefficient of x is positive.

35. $y = \sin(-2x)$

36. $y = \cos(-x)$

37. $y = -\cos(-3x)$

38. $y = -\sin(-5x)$

39. $y = \sin(-x - 3)$

40. $y = \cos(-2x + 4)$

41. $y = \sin(-x) - 3$

42. $y = \cos(-2x) + 4$

43. $y = -3 \cos\left(-2x + \frac{\pi}{2}\right)$

44. $y = 2 \sin\left(-\frac{x}{3} - \pi\right)$

Use the odd/even properties of the sine and cosine functions to rewrite each of the following functions as an equivalent function in which the coefficient of x is positive. Then graph three cycles of the function.

45. $y = \sin(-x)$

46. $y = \cos(-2x)$

47. $y = \cos\left(-x - \frac{\pi}{3}\right)$

48. $y = 2 \sin(-2x + \pi)$

49. $y = -\sin(-2\pi x + \pi)$

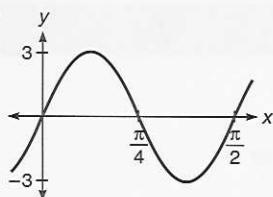
50. $y = -\cos(-\pi x)$

51. $y = \sin(-\pi x + 1)$

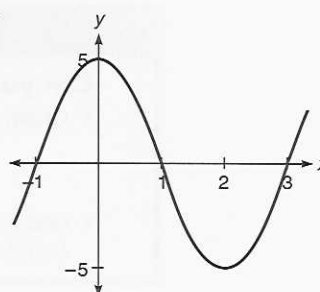
52. $y = 2 \cos(-3\pi x - 2)$

Assume that each of the following graphs is the graph of a sine function of the form $y = A \sin(Bx + C) + D$. Find values of A , B , C , and D that would produce each graph.

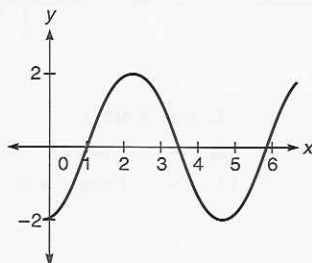
53.



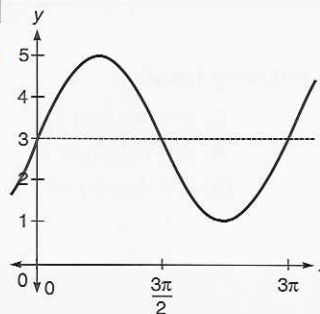
54.



55.



56.



57. Do problem 53, assuming that the graph is a cosine function of the form $y = A \cos(Bx + C) + D$.

58. Do problem 54, assuming that the graph is a cosine function of the form $y = A \cos(Bx + C) + D$.

59. Do problem 55, assuming that the graph is a cosine function of the form $y = A \cos(Bx + C) + D$.

60. Do problem 56, assuming that the graph is a cosine function of the form $y = A \cos(Bx + C) + D$.

Graph one cycle of each of the following functions. Mark the horizontal axis in degrees.

61. $y = 3 \sin(x + 60^\circ)$

62. $y = -50 \cos(x - 120^\circ)$

63. $y = 25 \cos 3x$

64. $y = 10 \sin(2x - 180^\circ)$

65. An electronic signal modeled with the sine function has a peak-to-peak voltage of 120 volts (amplitude is 60 volts), phase shift of 90° , and period of 54° . Find an equation of the sine function that will model this signal.

66. An ocean wave is being modeled with the sine function. Its amplitude is 6 feet and its phase shift (with respect to another wave) is -180° . If the period is 720° , find an equation of the sine function that will model this wave.

67. One of the components of a function that could describe the earth's ice ages for the last 500,000 years is described by a sine function with amplitude 0.5, period $\frac{360^\circ}{43}$, 0° phase shift, and vertical translation 23.5. Find an equation for this component.

68. The activity of sunspots seems to follow an 11-year cycle. Assuming that this activity can be roughly modeled with a sine wave, construct a sine function with period $\frac{360^\circ}{11}$, amplitude 1, phase shift 90° , and vertical translation 2.

69. Graph the following functions on the same set of axes:

$$y = \sin x; y = \sin 3x; \text{ and } y = \sin \frac{x}{3}.$$

70. Graph the following functions on the same set of axes:

$$y = \sin x; y = \sin\left(x + \frac{\pi}{2}\right); \text{ and } y = \sin x + \frac{\pi}{2}.$$

71. Graph the following functions on the same set of axes:

$$y = \cos\left(\frac{\pi}{2} - x\right) \text{ and } y = \sin x. \text{ Draw a conclusion from the graph. [Hint: Rewrite as } y = \cos\left(-x + \frac{\pi}{2}\right).]$$

3-4 Linear transformations of the tangent, cotangent, secant, and cosecant functions (optional)

The tangent and cotangent functions

The tangent and cotangent functions are π -periodic, so the basic cycle for each is π units long instead of the 2π units for the sine and cosine functions. Figure 3-22 shows a basic cycle for the tangent and cotangent functions.

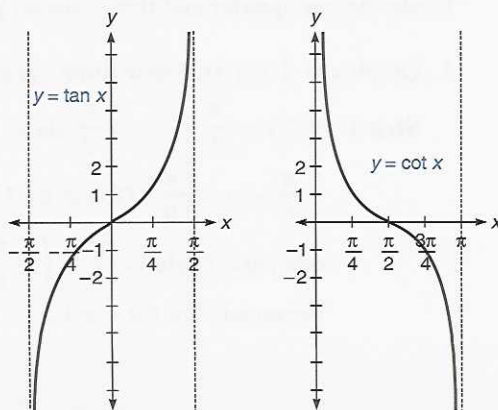


Figure 3-22

Note that the basic tangent cycle starts at $-\frac{\pi}{2}$ and ends at $\frac{\pi}{2}$. The basic cotangent cycle starts at 0 and ends at π . Also note that the functions are +1 or -1 at the points that are $\frac{1}{4}$ and $\frac{3}{4}$ of the distance between the cycle end points. We will call these the *one-quarter* and *three-quarter* points.

Graphing functions of the form $y = A \tan(Bx + C)$ and $y = A \cot(Bx + C)$ is done in a manner very similar to that for the sine and cosine functions. Although the concept of amplitude does not make sense for these functions, the vertical scaling factor A does affect the graph. In fact, these functions take on values of $\pm A$ at the one-quarter and three-quarter points (unless there is a vertical shift) instead of ± 1 .

To graph functions of the form

$$y = A \tan(Bx + C) \text{ and}$$

$$y = A \cot(Bx + C)$$

1. For the tangent function, solve

$$-\frac{\pi}{2} < Bx + C < \frac{\pi}{2}$$

for x .

For the cotangent function, solve

$$0 < Bx + C < \pi$$

for x .

2. Step 1 gives the left and right end points for one basic cycle. Draw this cycle. Label the "one-quarter" and "three-quarter" points with $y = A$ and $y = -A$, as appropriate. If $A < 0$, this cycle is reflected about the horizontal axis.
3. Repeat this cycle to obtain as much of the graph as desired.

We will not concern ourselves with defining the period and phase shift for the tangent and cotangent functions. A *guideline* for marking off the x -axis is to use increments of one fourth of the length of one basic cycle to locate the one-quarter and three-quarter points.

■ Example 3-4 A

1. Graph $y = \frac{1}{2} \tan 3x$. Show three cycles.

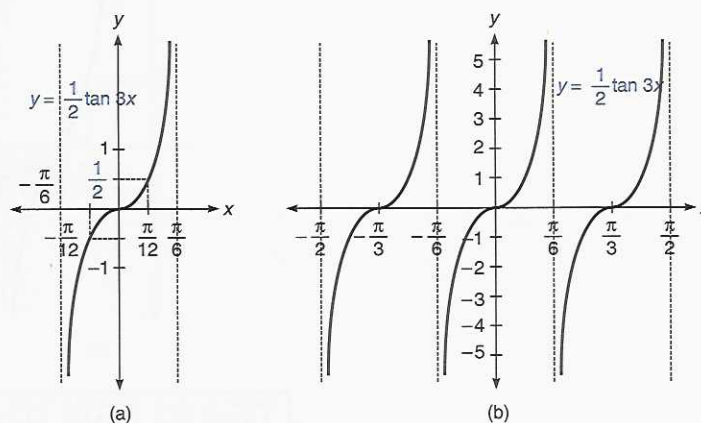
Step 1: Solve $-\frac{\pi}{2} < 3x < \frac{\pi}{2}$ for x .

$-\frac{\pi}{6} < x < \frac{\pi}{6}$. Divide by 3 (or multiply by $\frac{1}{3}$). The length of

one basic cycle is $\frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$. We use $\frac{1}{4} \cdot \frac{\pi}{3} = \frac{\pi}{12}$ increments on the x -axis.

Step 2: We now know that we get one basic tangent cycle starting at $-\frac{\pi}{6}$ and ending at $\frac{\pi}{6}$. We draw vertical asymptotes at these points and sketch one cycle of the tangent function. With the vertical scaling factor of $\frac{1}{2}$ we label the one-quarter and three-quarter points as shown in part (a) of the figure.

Step 3: Two more cycles are shown in part (b) of the figure.



Y=	.5	TAN	3	X T,
RANGE $-2,2,.524\left(\frac{\pi}{6}\right),-4,4,1$				

2. Graph $y = \cot\left(2x - \frac{\pi}{3}\right)$. Show three cycles.

Step 1: We put the argument between 0 and π and solve for x :

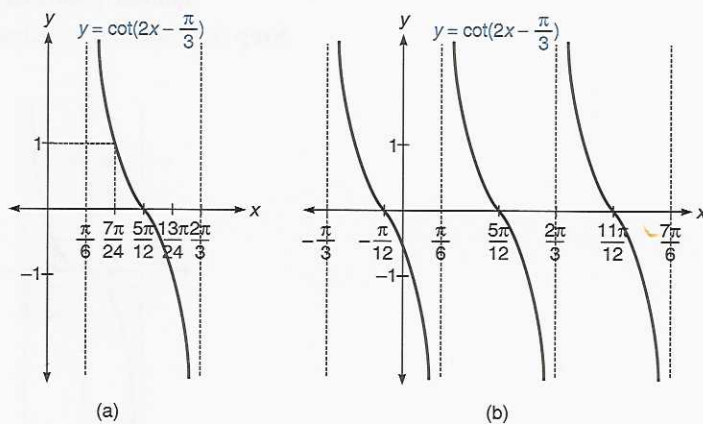
$$\begin{aligned}
 0 &< 2x - \frac{\pi}{3} < \pi \\
 0 &< 6x - \pi < 3\pi && \text{Multiply by 3} \\
 \pi &< 6x < 4\pi && \text{Add } \pi \\
 \frac{\pi}{6} &< x < \frac{4\pi}{6} && \text{Divide by 6} \\
 \text{or} \\
 \frac{\pi}{6} &< x < \frac{2\pi}{3}
 \end{aligned}$$

A basic cycle is $\frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{2}$ units long. We mark the x -axis

by adding or subtracting increments of $\frac{\pi}{8}$ units from $\frac{\pi}{6}$.

Step 2: We have one basic cycle between $\frac{\pi}{6}$ and $\frac{2\pi}{3}$; the basic graph is shown in part (a) of the figure.

Step 3: The finished graph, including three cycles, is shown in part (b) of the figure.



Y=	(	TAN	(	2	X T	-	pi	÷	3	)	)	x ⁻¹	,
RANGE -1.5,4,0.524,-3,3,1													

If the coefficient of x is negative, we use the fact that the tangent and cotangent functions are odd to rewrite the function with this coefficient positive.

■ Example 3-4 B

Graph $y = -\cot(-\pi x)$. Show three cycles.

Since the coefficient of x , $-\pi$, is negative, we rewrite this function as

$$y = \cot \pi x.$$

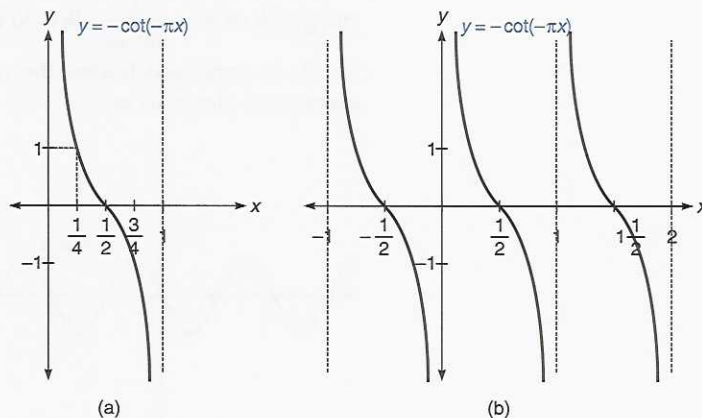
Step 1: $0 < \pi x < \pi$

$$0 < x < 1 \quad \text{Divide by } \pi$$

A basic cycle is 1 unit long, and we use $\frac{1}{4}$ for an increment on the x -axis.

Step 2: One basic cycle is shown in part (a) of the figure.

Step 3: Two more cycles are shown in part (b) of the figure.



Y= (-) (TAN (-) π X|T) x^{-1} ,
 RANGE -1.2,2.2,.5,-3.3,1

The secant and cosecant functions

To graph variations of the secant and cosecant functions, we use the fact that they are reciprocals of the cosine and sine functions, respectively. Figure 3-11 shows the graph of the cosecant function and figure 3-12 shows the graph of the secant function. Observe that the ranges are $|y| \geq 1$, and that they have vertical asymptotes where their reciprocal function, cosine or sine, is zero.

Consider the graph of a function of the form

$$y = A \csc(Bx + C)$$

We know it is a modification of the graph shown in figure 3-11. Since it is equivalent to the graph of

$$y = A \left(\frac{1}{\sin(Bx + C)} \right)$$

we can construct the graph of $y = \sin(Bx + C)$ and, graphically, form the reciprocal to get the graph we want.

For example, consider the graph of $y = 3 \csc 2x$. This will have the same graph as $y = 3 \left(\frac{1}{\sin 2x} \right)$. Thus, we can first graph $y = \sin 2x$ and graphically form the reciprocal, as we did in section 3-3.

The graph of $y = \sin 2x$ is shown in figure 3-23. In figure 3-24 we show the graph of $y = \frac{1}{\sin 2x}$. We do this by drawing vertical asymptotes wherever $\sin 2x$ is zero, and letting the reciprocal graph go to infinity as $\sin 2x$ gets closer and closer to zero.

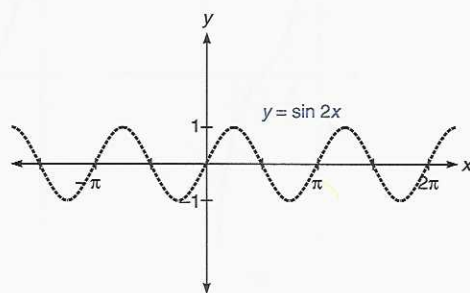


Figure 3-23

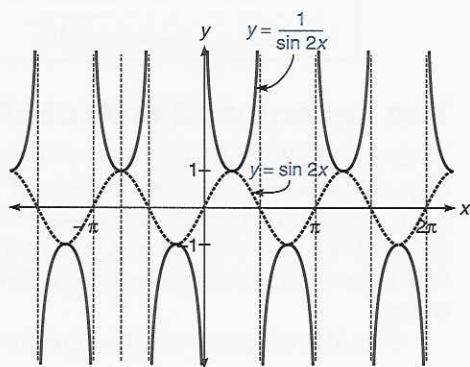
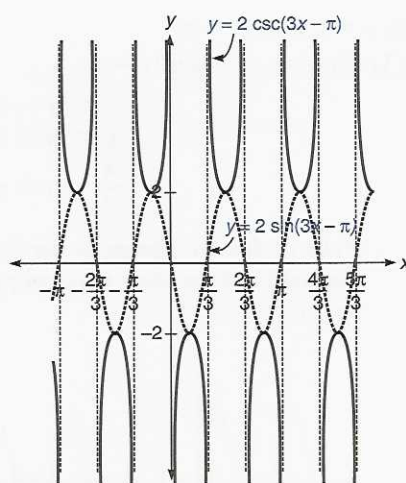


Figure 3-24

In figure 3-25 we show the graph of $y = 3\left(\frac{1}{\sin 2x}\right)$. Each point is three times higher or lower than each corresponding point on the graph in figure 3-25. This is also the graph we wanted originally.



Y= 3 (SIN 2 X | T) x^{-1} ,
 RANGE -4,7,1.57,-5,5,1

Figure 3-25

Observe that we could have originally graphed $y = 3 \sin 2x$ and used this as our guide, since in the graph of $y = 3 \sin 2x$, $|y| \leq 3$, while in the graph of $y = 3 \csc 2x$, $|y| \geq 3$.

Based on this example, and what we have done in the preceding sections of this chapter, we can state the following.

Procedure for graphing functions of the form

$$y = A \csc(Bx + C) \text{ and } y = A \sec(Bx + C)$$

1. Graph $y = A \sin(Bx + C)$ or $y = A \cos(Bx + C)$, whichever is the appropriate reciprocal function.
2. Graphically form the reciprocal by drawing vertical asymptotes wherever the graph in step 1 is 0, and draw the graph getting larger and larger in absolute value wherever the reciprocal function approaches 0.

■ **Example 3-4 C**

Graph $y = 2 \csc(3x - \pi)$.

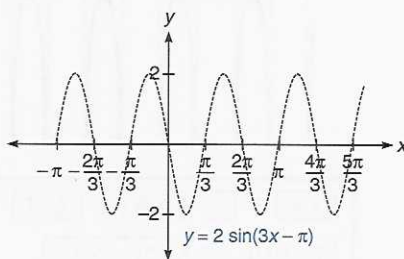
Step 1: Graph $y = 2 \sin(3x - \pi)$.

$$0 \leq 3x - \pi \leq 2\pi$$

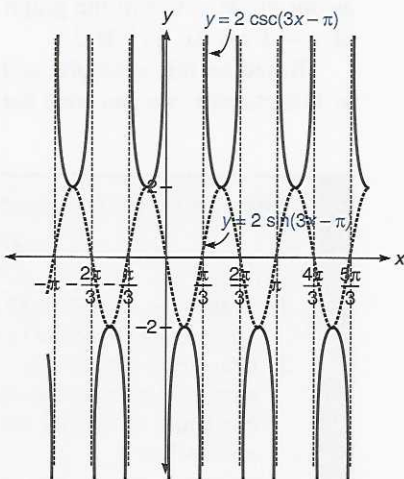
$$\pi \leq 3x \leq 3\pi$$

$$\frac{\pi}{3} \leq x \leq \pi$$

Four cycles are shown in the first figure. We draw this graph using dashed lines, because it is not part of the graph we have been asked to show.



Step 2: We draw vertical asymptotes wherever the graph of $y = 2 \sin(3x - \pi)$ is 0; that is, where it crosses the x -axis. We then form the reciprocal function, as shown in the next figure.



Y=	2	(	SIN	(	3	X/T	-	pi	)	)	x ⁻¹	,
RANGE -3.2,6,1.05($\frac{\pi}{3}$),-4,4,1												

If the coefficient of x is negative, we use the odd/even properties of the sine and cosine functions as appropriate.

■ **Example 3-4 D**

Graph $y = 2 \csc\left(-\frac{2\pi x}{3}\right)$.

Step 1: Graph $y = 2 \sin\left(-\frac{2\pi x}{3}\right)$. Since the argument is negative, we use the odd property of the sine function to rewrite this as $y = -2 \sin \frac{2\pi x}{3}$.

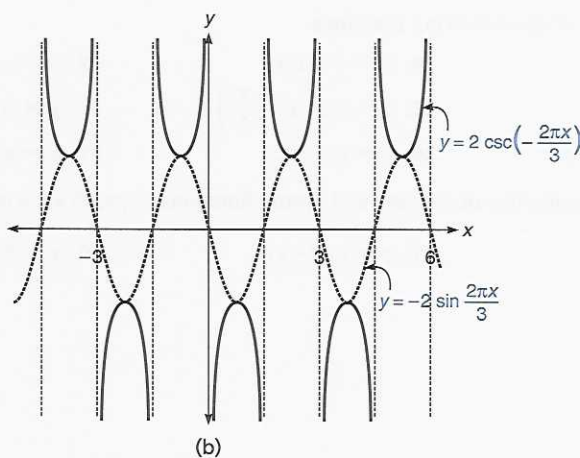
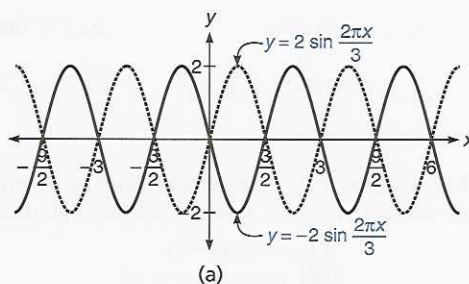
$$0 \leq \frac{2\pi x}{3} \leq 2\pi$$

$$0 \leq 2\pi x \leq 6\pi \quad \text{Multiply by 3}$$

$$0 \leq x \leq 3 \quad \text{Divide by } 2\pi$$

The graph of $y = -2 \sin \frac{2\pi x}{3}$ is shown in the figure, part (a).

Step 2: The reciprocal function is shown in part (b) of the figure.



Y= 2 (SIN ((-) 2 π X T ÷ 3)) x⁻¹,
 RANGE -4.5,6.5,1.5,-4.4,1

Mastery points

Can you

- Graph functions of the form

$$y = A \tan(Bx + C), \text{ and}$$

$$y = A \cot(Bx + C)?$$

- Graph functions of the form

$$y = A \sec(Bx + C) \text{ and}$$

$$y = A \csc(Bx + C)?$$

Exercise 3-4

Graph three cycles of the following functions.

1. $y = 5 \tan x$

2. $y = -4 \cot x$

3. $y = \tan 4x$

4. $y = \cot \frac{x}{2}$

5. $y = \cot\left(x - \frac{\pi}{2}\right)$

6. $y = 3 \tan(2x + \pi)$

7. $y = -\cot\left(2x + \frac{\pi}{2}\right)$

8. $y = -\tan\left(3x - \frac{\pi}{3}\right)$

9. $y = \cot 2\pi x$

10. $y = \tan \pi x$

Use the odd/even properties of the tangent and cotangent functions to rewrite each of the following functions as an equivalent function in which the coefficient of x is positive. Then graph three cycles of the function.

11. $y = \tan(-2x)$

12. $y = \cot(-x)$

13. $y = -\cot(-\pi x)$

14. $y = -\tan(-2\pi x)$

15. $y = \tan(-x - \pi)$

16. $y = \cot(-2x + 4\pi)$

Graph three cycles of the following functions.

17. $y = \frac{2}{3} \csc x$

18. $y = \frac{1}{5} \sec x$

19. $y = -4 \csc x$

20. $y = 2 \sec 4x$

21. $y = 3 \csc \frac{x}{2}$

22. $y = \csc\left(x - \frac{\pi}{2}\right)$

23. $y = 3 \sec(2x + \pi)$

24. $y = \frac{2}{3} \sec(3x + \pi)$

25. $y = \csc(2x - 3\pi)$

26. $y = \csc 2\pi x$

27. $y = \sec \pi x$

Use the odd/even properties of the sine and cosine functions to graph each of the following functions.

28. $y = \sec(-2x)$

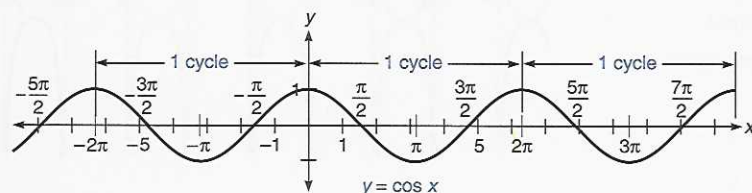
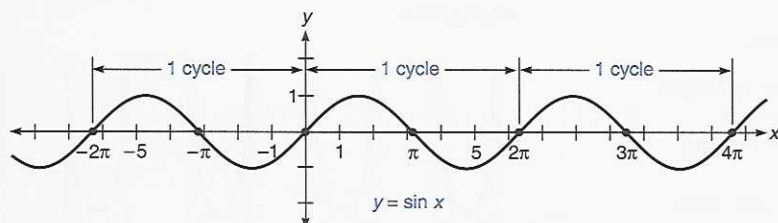
29. $y = \csc(-x)$

30. $y = 3 \csc\left(-2x + \frac{\pi}{2}\right)$

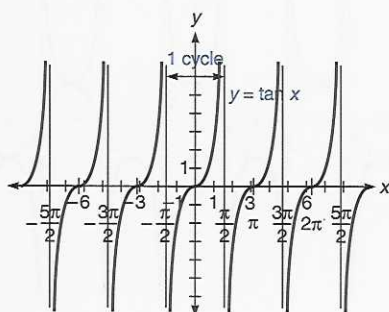
31. $y = 2 \sec\left(-\frac{x}{3} - \pi\right)$

Chapter 3 summary

- Basic graphs
- Graph of the sine and cosine functions.



- Graph of the tangent function.



- To graph sine and cosine functions of the form

$$y = A \sin(Bx + C) + D \text{ and}$$

$$y = A \cos(Bx + C) + D, \text{ where } B > 0$$

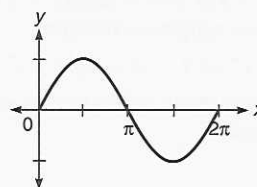
1. Solve $0 \leq Bx + C \leq 2\pi$ so x is the middle member.
 - This gives the left and right end points for one basic cycle.
 - The left end point is the phase shift.
 - The difference between the end points is the period.

2. The amplitude is $|A|$.

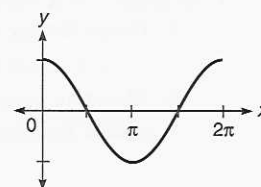
- This is the height of the basic graph above and below the x -axis.
- The graph is shifted about the horizontal axis if $A < 0$.

Draw one basic cycle with the information from steps 1 and 2.

3. Repeat the cycle obtained from steps 1 to 3 to obtain more of the graph.
4. Shift the graph vertically D units.



Basic sine cycle



Basic cosine cycle

- If the coefficient of x , B , is negative in the argument $Bx + C$ we first use the odd and even properties to get an equivalent expression with B positive.

- To graph functions of the form

$$y = A \tan(Bx + C) \text{ and } y = A \cot(Bx + C), B > 0$$

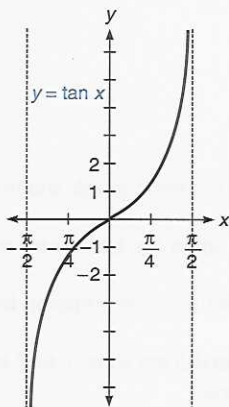
- For the tangent function, solve $-\frac{\pi}{2} < Bx + C$

$< \frac{\pi}{2}$ for x ; for the cotangent function solve

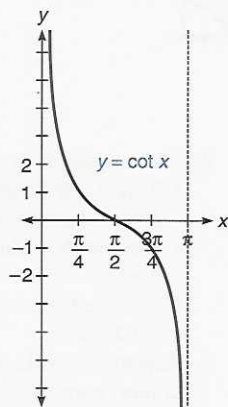
$0 < Bx + C < \pi$ for x . This gives the left and right end points for one basic cycle. The difference between the end points is the period. The left end point is the phase shift.

- Use the values from step 1 to draw one basic cycle. Label the one-quarter and three-quarter points with $y = A$ and $y = -A$ as appropriate. Repeat this cycle to obtain as much of the graph as desired.

Basic tangent cycle



Basic cotangent cycle



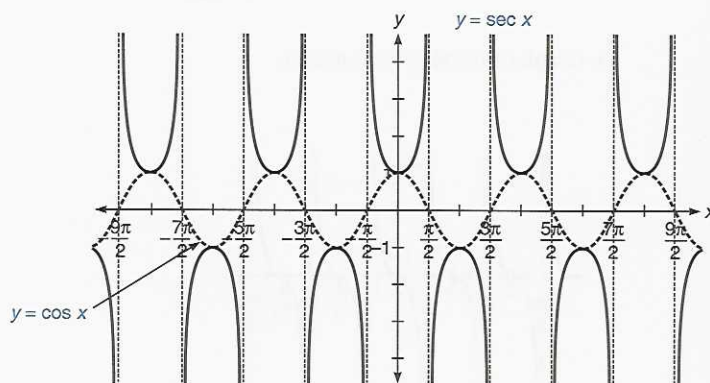
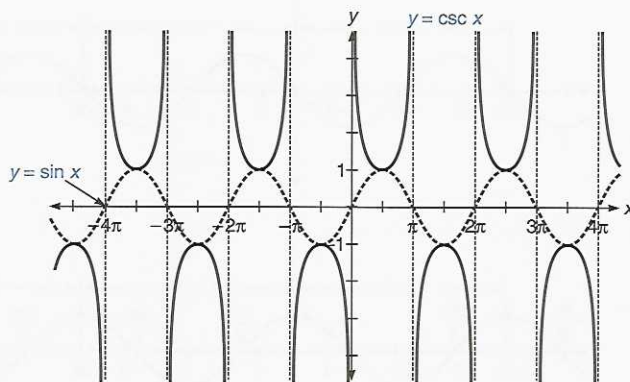
- To graph $y = A \csc(Bx + C)$ or $y = A \sec(Bx + C)$:

- Graph the appropriate reciprocal function,

$$y = A \sin(Bx + C) \text{ or } y = A \cos(Bx + C)$$

- Sketch in vertical asymptotes wherever the sine or cosine function is zero.

- Create the cosecant or secant graph by starting at the highest and lowest points of the sine or cosine graph and sketching values that increase in absolute value from that point as x approaches the vertical asymptotes. Note that these functions are not defined at the asymptotes.



- Summary of the properties of the sine, cosine, and tangent functions (k an integer).

Function	Domain	Range	Period
$y = \sin x$	R	$-1 \leq y \leq 1$	2π
$y = \cos x$	R	$-1 \leq y \leq 1$	2π
$y = \tan x$	$x \neq \frac{\pi}{2} + k\pi$	R	π
	$\sin(-x) = -\sin x$	(odd)	
	$\cos(-x) = \cos x$	(even)	
	$\tan(-x) = -\tan x$	(odd)	

- Summary of the properties of the cosecant, secant, and cotangent functions (k an integer).

Function	Domain	Range	Period
$y = \csc x$	$x \neq k\pi$	$ y \geq 1$	2π
$y = \sec x$	$x \neq \frac{\pi}{2} + k\pi$	$ y \geq 1$	2π
$y = \cot x$	$x \neq k\pi$	R	π
	$\csc(-x) = -\csc x$	(odd)	
	$\sec(-x) = \sec x$	(even)	
	$\cot(-x) = -\cot x$	(odd)	

Chapter 3 review

[3-1]

- Sketch the graph of the sine function; state the domain, range, and period of the sine function.
- Using the graph of $y = \cos x$ as a guide, describe all values of x for which $\cos x$ is 1.

Use the appropriate property, even or odd, to calculate the exact function value.

- $\cos\left(-\frac{\pi}{6}\right)$
- $\tan\left(-\frac{4\pi}{3}\right)$
- $\sin\left(-\frac{5\pi}{6}\right)$
- $\sec\left(-\frac{\pi}{4}\right)$

Test the function for the even/odd property.

- $f(x) = \frac{x^2 - 1}{x}$
- $f(x) = x \sin x$
- $f(x) = \tan x \cdot \cos x$

[3-2]

- Sketch the graph of the cosecant function.
- State the domain and range of the cotangent function.
- Show that $f(x) = \frac{x}{\sec x}$ is an odd function.
- Show that the function $f(x) = \sec x \cdot \sin^2 x + x^4$ is an even function.

[3-3] Graph three cycles of the following functions. State the amplitude, period, and phase shift of each.

- $y = 2 \sin x$
- $y = -\frac{2}{3} \cos x$
- $y = 3 \sin x - 2$
- $y = 2 \sin 3x$
- $y = \cos\left(x + \frac{\pi}{3}\right)$
- $y = 2 \sin\left(\frac{x}{2} + \frac{\pi}{3}\right)$
- $y = \cos 3x\pi$
- $y = 3 \cos 2x - 3$

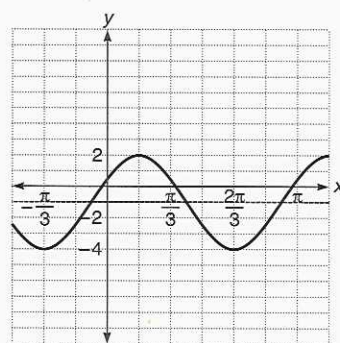
Use the odd/even properties of the sine and cosine functions to rewrite each of the following functions as an equivalent function in which the coefficient of x is positive. Then graph three cycles of the function.

22. $y = \cos\left(-2x + \frac{\pi}{2}\right)$ 23. $y = 3 \sin(-x + \pi)$

24. Graph three cycles of the function

$y = 2 \sin(3x + 60^\circ)$. Mark the horizontal axis in degrees.

25. Assume that the following graph is of a cosine function of the form $y = A \cos(Bx + C) + D$. Find the values of A , B , C , and D and rewrite the function using these values.



26. Most people have heard about the theory of biorhythms. This theory maintains that at birth three cycles are started—physical, emotional, and intellectual. The physical cycle has a period of 23 (days). Assuming an amplitude of 1, a phase shift of -10 (days), and no vertical translation, create an equation that describes the physical cycle in terms of the sine function.

[3–4] Graph three cycles of the following functions.

27. $y = \tan 4x$ 28. $y = \tan(3x + \pi)$
 29. $y = 2 \cot\left(x - \frac{\pi}{4}\right)$ 30. $y = -2 \sec 3x$
 31. $y = \csc \frac{x}{2}$ 32. $y = \sec(2x - \pi)$
 33. $y = \csc 3\pi x$

Chapter 3 test

- Using the graph of $y = \sin x$ as a guide, describe all values of x for which $\sin x$ is -1 .
- Use the appropriate property, even or odd, to calculate the exact function value of $\tan\left(-\frac{5\pi}{3}\right)$.
- Test the function $f(x) = x + \sin x$ for the even/odd property.

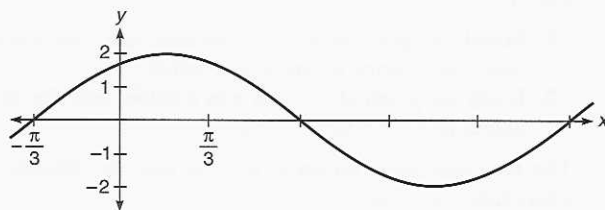
Graph three cycles of the following functions. State the amplitude, period, and phase shift of each.

4. $y = 2 \sin x + 2$ 5. $y = 3 \cos 2x$
 6. $y = 3 \sin\left(\frac{x}{3} + \frac{\pi}{2}\right)$

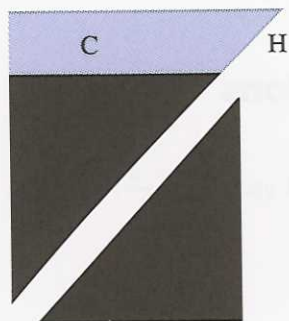
Graph three cycles of the following functions.

7. $y = 3 \tan \pi x$ 8. $y = \sec(3x - \pi)$
 9. $y = -\csc 4\pi x$

10. Assume that the graph is of the form $y = A \sin(Bx + C) + D$. Find values of A , B , C , and D that would produce the graph and write the corresponding equation.



- Sketch the graph of the secant function.
- Show that the function $f(x) = \sec x \cdot \sin x + x^3$ is an odd function.
- In the theory of biorhythms, the emotional cycle has a period of 28 (days). Assuming an amplitude of 1, a phase shift of 5 (days), and no vertical translation, create an equation that describes the emotional cycle in terms of the sine function.
- An electronic signal is to be modeled with the sine function. Amplitude is 25 volts, phase shift is -20° , period is 150° , and there is a vertical shift of 10 volts. Find a sine function that will model this signal.



The Inverse Trigonometric Functions

In previous chapters we have used what we called the inverse trigonometric functions. For example, we stated that if $\sin \theta = 0.3$, then one value of θ is $\theta = \sin^{-1} 0.3$. In this chapter we examine these inverse trigonometric functions in complete detail.

We use these functions both to solve trigonometric equations like that in the last paragraph, to solve triangles, and to describe angles in terms of given information. These functions are also useful in advanced mathematics (calculus in particular) where they provide a means to simplify certain algebraic expressions.

We begin the chapter by examining the general topic of inverse functions, and then apply this topic to trigonometry.

4-1 The inverse of a function

In section 2-1 we introduced the idea of the inverse of a function. We stated that

- A function is a set of ordered pairs in which no first element is repeated.
- A one-to-one function is a function in which no second element is repeated.
- Reversing the elements of the ordered pairs of a function produces a function if, and only if, the function is one to one.

For example,

$$H = \{(1,5), (2,7), (-5,-5)\}$$

is a one-to-one function, and its inverse function is

$$H^{-1} = \{(5,1), (7,2), (-5,-5)\}$$

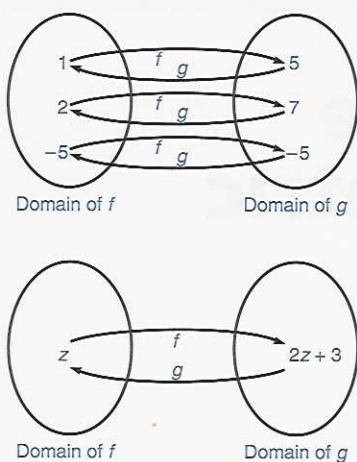


Figure 4-1

Showing that two given functions are inverses

Now consider the two functions $f(x) = 2x + 3$ and $g(x) = \frac{x-3}{2}$. By computation we could determine the following facts.

$$f(1) = 5 \text{ and } g(5) = 1$$

$$f(2) = 7 \text{ and } g(7) = 2$$

$$f(-5) = -7 \text{ and } g(-7) = -5$$

(Compare to H and H^{-1} on page 133 and figure 4-1.)

Whatever value z we try, f sends z to some value z' , and g sends z' back to z . In fact we can prove this; let z represent any real number. Then,

$$f(z) = 2z + 3 \text{ and}$$

$$g(2z + 3) = \frac{(2z + 3) - 3}{2} = z$$

Also

$$g(z) = \frac{z-3}{2} \text{ and}$$

$$f\left(\frac{z-3}{2}\right) = 2\left(\frac{z-3}{2}\right) + 3 = z$$

When two functions f and g act this way we say they are inverse functions. This is because, whenever an ordered pair (a, b) is in f , the ordered pair formed by reversing its elements (b, a) is in g . The functions H and H^{-1} , and f and g , above, are examples of this.

The notation for the inverse of a function is the superscript -1 , so using this notation, if $f(x) = 2x + 3$ (as above) we can say that $f^{-1}(x) = \frac{x-3}{2}$. Note

the superscript -1 , when applied to the name of a function, is *not* an exponent; it does not indicate division, as it does if applied as an exponent of a real valued expression.

$$f^{-1}(x) \text{ does not mean } \frac{1}{f(x)}$$

To show that two functions f and g are inverses of each other

Show that

- [1] If $f(x) = y$, then $g(y) = x$, and
- [2] If $g(x) = y$, then $f(y) = x$.

Note In practice “ y ” represents an expression in x .

■ Example 4-1 A

Show that f and g are the inverse functions of each other.

1. $f(x) = \frac{1}{3}x - 1$; $g(x) = 3x + 3$

[1] Assume $f(x) = y$, then $y = \frac{1}{3}x - 1$. Replace $f(x)$ by y

Show that $g(y) = x$:

$$\begin{aligned} g(y) &= 3y + 3 \\ &= 3\left(\frac{1}{3}x - 1\right) + 3 \\ &= x - 3 + 3 \\ &= x \end{aligned}$$

Replace x by y in $g(x) = 3x + 3$

Replace y by $\frac{1}{3}x - 1$

[2] Assume $g(x) = y$, then $y = 3x + 3$.

Show that $f(y) = x$:

$$\begin{aligned} f(y) &= \frac{1}{3}y - 1 \\ &= \frac{1}{3}(3x + 3) - 1 \\ &= x + 1 - 1 \\ &= x \end{aligned}$$

Replace x by y in $f(x) = \frac{1}{3}x - 1$

Replace y by $3x + 3$

Thus, we have shown conditions [1] and [2] above, so f and g are inverse functions.

2. $f(x) = \sqrt{x}$; $g(x) = x^2$, $x \geq 0$

[1] $y = \sqrt{x}$

Let y represent $f(x)$

$$\begin{aligned} g(y) &= y^2 \\ &= (\sqrt{x})^2 \\ &= x \end{aligned}$$

Determine $g(y)$

$$(\sqrt{a})^2 = a$$

[2] $y = x^2$, $x \geq 0$

Let y represent $g(x)$

$$\begin{aligned} f(y) &= \sqrt{y} \\ &= \sqrt{x^2} \\ &= x \end{aligned}$$

Determine $f(y)$

$$\sqrt{a^2} = a \text{ if } a \geq 0$$

Thus, we have shown conditions [1] and [2], so f and g are inverse functions. ■

Graphical analysis of relations for the function and one-to-one properties

The graph of a relation can be used to determine whether that relation is a function, and whether or not a function is one to one. This is done by the **vertical line test** and the **horizontal line test**.

Vertical line test for a function

If no vertical line crosses the graph of a relation in more than one place, the relation is a function.

The vertical line test works for the following reason. Assume a vertical line crosses a graph at more than one point. Since these two points are in a vertical line their first components (the x -values) are equal. Therefore, the function must have two points in which the first element repeats, and it is therefore not a function.

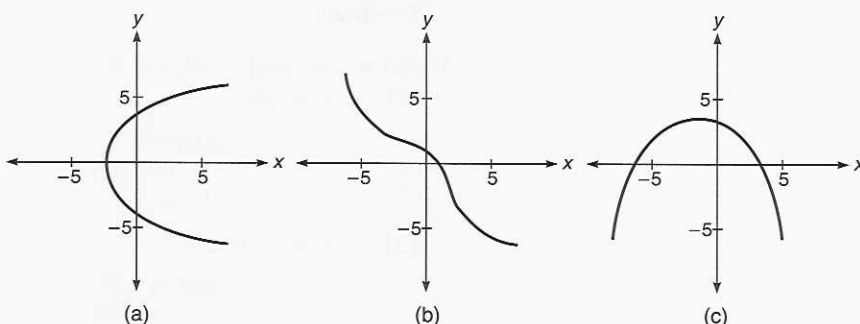
Horizontal line test for a one-to-one function

If no horizontal line crosses the graph of a function in more than one place, the function is one to one.

The horizontal line test works for reasons similar to those for the vertical line test. If a horizontal line crosses a function at two (or more) points, then these are different domain elements (first components) with the same range elements (second component). Therefore the function is not one to one.

Example 4-1 B

Tell which relations are functions, and which functions are one to one.



1. Relation (a) is not a function since there are clearly many vertical lines that would intersect the graph in at least two places.
2. Relation (b) is a function since no vertical line will intersect the graph in more than one place. It is also one to one since no horizontal line will intersect the graph in more than one place.
3. Relation (c) is a function by the vertical line test, but not a one-to-one function.

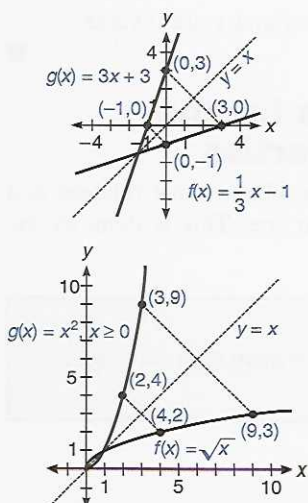


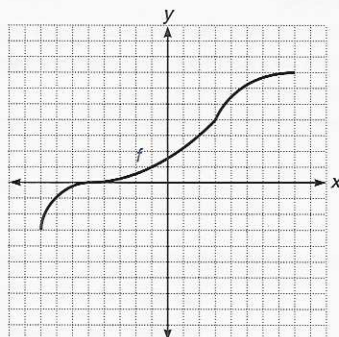
Figure 4-2

The graph of a function's inverse

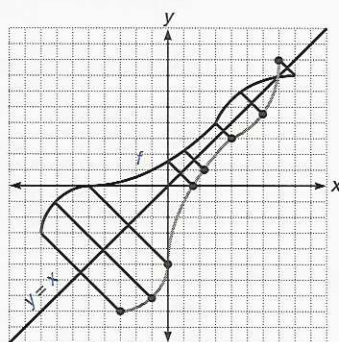
The fact that the ordered pairs reverse in a function's inverse function means that *the graph of f^{-1} is a reflection of the graph of f about the line $y = x$* . By way of example, observe the graphs of the functions in example 4-1 A. These are shown in figure 4-2. To draw a graph that is symmetric about the line $y = x$ to a given graph, we draw lines perpendicular to the line $y = x$, as shown, and plot points at equal distances from this line, but on the other side of this line. Since the ordered pairs of f all reverse in f^{-1} *the domain of f is the range of f^{-1} , and the range of f is the domain of f^{-1}* .

Example 4-1 C

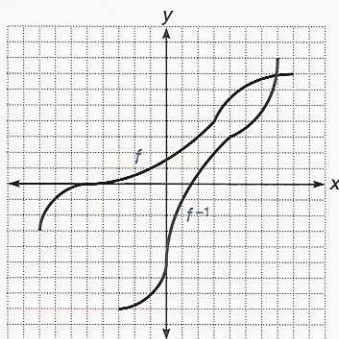
Given the graph of the one-to-one function f , sketch the graph of its inverse function f^{-1} .



To graph f^{-1} we construct the line $y = x$, which is a straight line passing through the origin with a slope of 1. We then construct various straight lines perpendicular to this line, starting on the graph of the function, and extending an equal distance to the other side of the line $y = x$.



Connecting the points that result from this process gives the graph of f^{-1} .



Mastery points

Can you

- Demonstrate that two functions are inverses of each other?
- Given a graph that represents a one-to-one function, graph its inverse function?

Exercise 4-1

Show that the following functions f and g are inverses of each other. Assume the domains as indicated are correct.

1. $f(x) = 2x - 7; g(x) = \frac{1}{2}x + 3\frac{1}{2}$

3. $f(x) = \frac{1}{3}x + \frac{8}{3}; g(x) = 3x - 8$

5. $f(x) = 2x - 5; g(x) = \frac{1}{2}(x + 5)$

7. $f(x) = \frac{2}{x-3}; g(x) = \frac{2}{x} + 3$

9. $f(x) = 7 - \frac{3}{x}; g(x) = \frac{3}{7-x}$

11. $f(x) = \frac{x}{x-1}; g(x) = \frac{x}{x-1}$

13. $f(x) = x^2 - 9, x \geq 0; g(x) = \sqrt{x+9}$

15. $f(x) = x^3; g(x) = \sqrt[3]{x}$

17. $f(x) = x^2 - 2x + 3, x \geq 1; g(x) = \sqrt{x-2} + 1$

19. $f(x) = \frac{2x}{x-3}; g(x) = \frac{3x}{x-2}$

2. $f(x) = -\frac{1}{3}x + \frac{1}{2}; g(x) = -3x + \frac{3}{2}$

4. $f(x) = x - 1; g(x) = x + 1$

6. $f(x) = \frac{x}{5} - 3; g(x) = 5(x + 3)$

8. $f(x) = \frac{3}{x+2}; g(x) = \frac{3}{x} - 2$

10. $f(x) = 5 + \frac{3}{x-1}; g(x) = \frac{3}{x-5} + 1$

12. $f(x) = \frac{x+1}{x-5}; g(x) = \frac{5x+1}{x-1}$

14. $f(x) = \sqrt{4-2x}; g(x) = 2 - \frac{1}{2}x^2, x \geq 0$

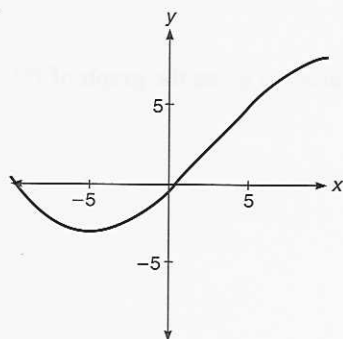
16. $f(x) = x^3 - 3; g(x) = \sqrt[3]{x+3}$

18. $f(x) = \sqrt{x+9} - 2; g(x) = x^2 + 4x - 5, x \geq -2$

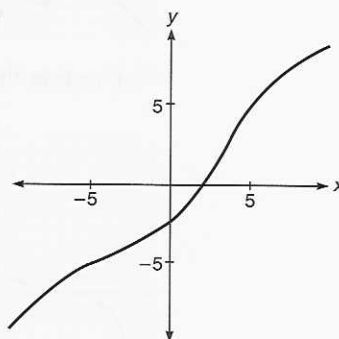
20. $f(x) = \frac{x-3}{x-2}; g(x) = 2 - \frac{1}{x-1}$

Tell which relation is a function, and which functions are one to one.

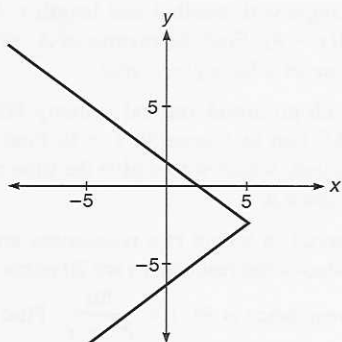
21.



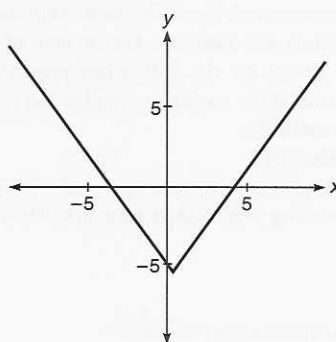
22.



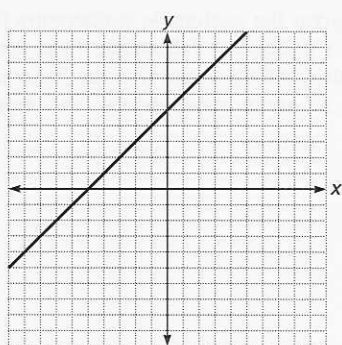
23.



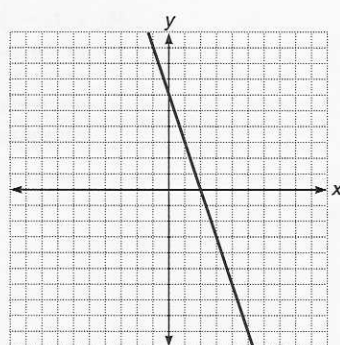
24.

Sketch the graph of f^{-1} , given the graph of the one-to-one function f .

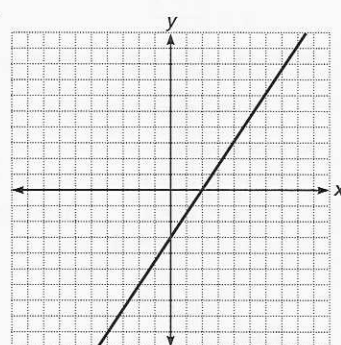
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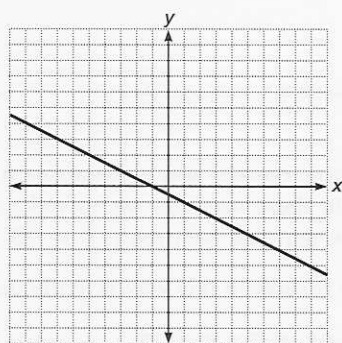
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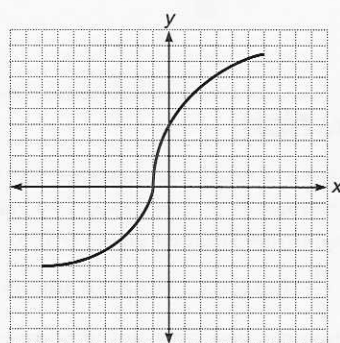
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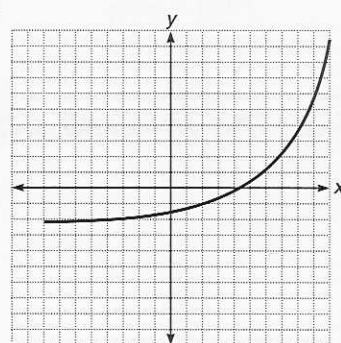
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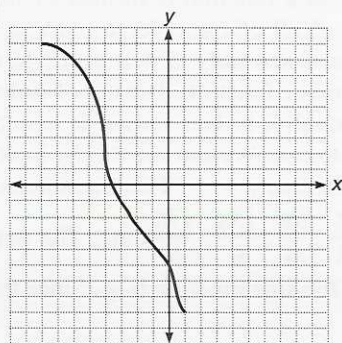
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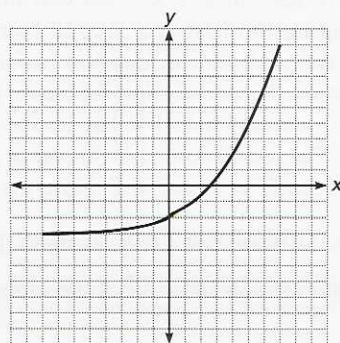
30.



31.



32.



 When a function is described by an algebraic expression involving polynomials and radicals, the inverse of the function can often be found by the following procedure, which is described in terms of the variables x and y , but could be in terms of any two variables:

1. Replace the $f(x)$ symbol by y .
2. Exchange the two variables; replace x by y and y by x .
3. Solve for y . The resulting expression in x describes the inverse function.

Example

Find the inverse of the function $f(x) = \frac{x+1}{x}$.

1. Replace the $f(x)$ symbol by y :

$$y = \frac{x+1}{x}$$

2. Exchange x and y variables:

$$x = \frac{y+1}{y}$$

3. Solve for y :

$$xy = y + 1$$

$$xy - y = 1$$

All y terms on one side

$$y(x - 1) = 1$$

$$y = \frac{1}{x-1}$$

This is f^{-1} :

$$f^{-1}(x) = \frac{1}{x-1}$$

33. The area of a rectangle with width 4 and length $x + 4$, $x \geq 0$, is $A(x) = 4(x + 4)$. Find the inverse of A , which would give the value of x for a given area.

34. A falling object with no initial vertical velocity falls a distance $d(t) = 16t^2$ feet in t seconds, $t > 0$. Find the inverse of this function, which would give the time necessary to fall a distance d .

35. In an electronic circuit in which two resistances are in parallel, and the value of the resistances are 20 ohms and x ohms, the total resistance is $R(x) = \frac{20x}{20+x}$. Find the inverse of this function, which would give the value of x required for a total resistance R .

36. $C(t) = \frac{5}{9}(t - 32)$ gives the centigrade temperature for a given temperature t in degrees Fahrenheit. Find the inverse of this function, which would find the Fahrenheit temperature for a given temperature in degrees centigrade.

4-2 The inverse sine function

We have already encountered the inverse sine, cosine, and tangent functions in sections 1-3, 1-4, 2-3, 2-4, and 2-6. In these sections we solved for θ in equations such as $\sin \theta = 0.5$. We know that one solution is $\theta = 30^\circ$, or $\theta = \frac{\pi}{6}$ (radians). Using the sine function in reverse in this way is really using the inverse sine function, which we will study in this section.

In section 2-1 we said that the inverse of a function is formed by interchanging the first and second components of all of the ordered pairs in the function and that a function has an inverse if and only if it is a one-to-one function. Since the sine function, like any other function, is a set of ordered pairs, we may ask if it has an inverse. Naturally, it does *only* if it is one to one. The sine function is, however, not one to one. We can see this by looking at its graph in figure 4-3. Note the horizontal line $y = \frac{1}{2}$. This line intersects with (crosses) the graph at many points: $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$, etc. The points of intersection show all of the places where the sine function is $\frac{1}{2}$. Since there

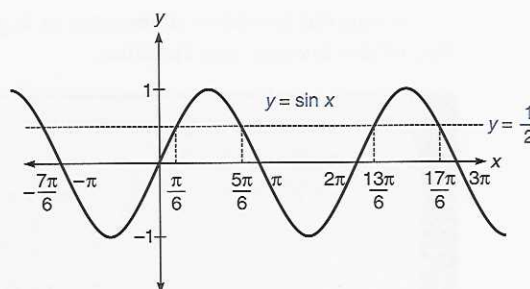


Figure 4-3

is more than one such point, there is more than one ordered pair in the sine function that has $\frac{1}{2}$ as its second element. Some of these ordered pairs are $(\frac{\pi}{6}, \frac{1}{2})$, $(\frac{5\pi}{6}, \frac{1}{2})$, $(\frac{13\pi}{6}, \frac{1}{2})$, etc. Thus, there is a repetition in the second element of these ordered pairs, and the sine function is, therefore, not one to one.

In general, if we can find a horizontal line that intersects the graph of a function in more than one point, then the function cannot be one to one. This graphic test is the horizontal line test, presented in section 4-1.

Since we can quickly see from our knowledge of the graphs of the six trigonometric functions (see chapter 3) that they would all fail the horizontal line test, we know that none of these functions is one to one.

Experience has shown, however, that there is a real need for inverses of these functions. By compromising a little, we can indeed define inverse functions for the six trigonometric functions. The compromise we must make is to form the inverse of only a small part of each function. This part is chosen so that it will include the entire range of the given function but will be one to one. We do this by limiting the domain.

For the sine function we select that portion whose ordered pairs are defined for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. See figure 4-4. Note that this new function is one to one, since there is no horizontal line that would intersect the graph in more than one point. Since it is one to one, it has an inverse function. This inverse function is denoted by $\sin^{-1}$, which is read "inverse sine function." The ordered pairs of this inverse sine function are the reversals of the ordered pairs of the one-to-one portion of the sine function we selected.

We know that if $y = \sin^{-1}x$, then the ordered pair (x, y) is in the inverse sine function. The ordered pair (y, x) is therefore in the selected one-to-one portion of the sine function. If (y, x) is in this part of the sine function, then

$$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \text{ and } x = \sin y.$$

Note Remember that if an ordered pair is in the sine function, then the second element of the pair is the sine of the first.

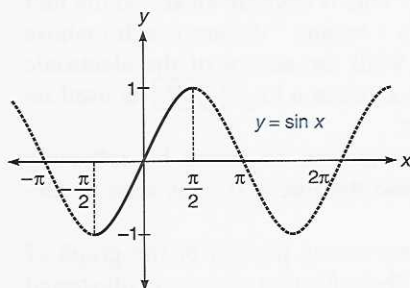


Figure 4-4

Using the previous statements as a guide, we make the following definition of the inverse sine function.

The inverse sine function

$y = \sin^{-1}x$ means

1. $\sin y = x$
2. $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
3. $|x| \leq 1$

Concept

If we think of $\sin^{-1}x$ as representing an angle, and since we know that angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ are in the first and fourth quadrants, we can interpret $y = \sin^{-1}x$ in the following way: “ $\sin^{-1}x$ is the angle in quadrant I or quadrant IV whose sine is x .” (We are referring to negative angles in quadrant IV.)

The domain of the inverse sine function is $-1 \leq x \leq 1$, or $|x| \leq 1$, which is the range of the sine function. The range of the inverse sine function is

$-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$, which is the domain of the one-to-one portion of the sine function that we selected. Note that these are parts 2 and 3 of the definition. More explicitly,

$$\text{Domain}_{\sin^{-1}}: |x| \leq 1$$

$$\text{Range}_{\sin^{-1}}: -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

Another notation for $\sin^{-1}x$ is **arcsin** x . This is because an arc on the unit circle can represent an angle, and so **arcsin** x means “the arc (angle) whose sine is x , in quadrant I or quadrant IV.” With the advent of the electronic calculator a third notation might be **invsin** x , since a key **INV** is used on some calculators to find inverse sine values.

The graph of the inverse of any function can be found by reflecting each point in the graph of the function across the line $y = x$, as seen in section 4-1.

To graph $y = \sin^{-1}x$, we reflect the one-to-one portion of the graph of $y = \sin x$ (figure 4-4) across the line $y = x$. The reflecting process is illustrated in figure 4-5, and the final graph of $y = \sin^{-1}x$ is shown in figure 4-6.

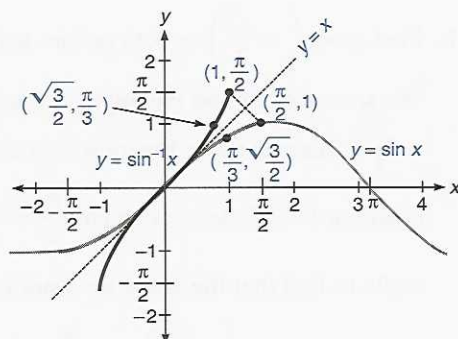
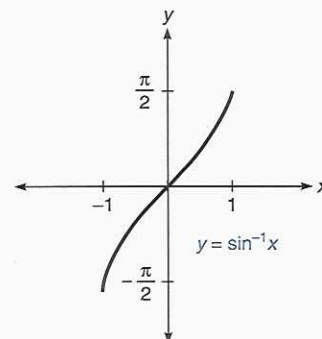


Figure 4-5



$Y_1 = \sin^{-1} X$ RANGE $-3.14159, 3.14159$
--

Figure 4-6

Although the inverse trigonometric functions are defined using radian measure, we often want the result in degrees. For the inverse sine function we therefore want angles between -90° and 90° , since these correspond to $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ radians.

Example 4-2 A

1. Find $\sin^{-1}\frac{1}{2}$ in both radians and degrees.

To use the definition, we write $y = \sin^{-1}\frac{1}{2}$. Then we know that

$$\sin y = \frac{1}{2}$$

and

$$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

We know that $\sin \frac{\pi}{6} = \frac{1}{2}$, and $\frac{\pi}{6}$ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, so

$$y = \frac{\pi}{6} = \sin^{-1}\frac{1}{2}.$$

It is often easier to solve these problems if we restate them verbally. Remember that $\sin^{-1}\frac{1}{2}$ can be interpreted to mean “the angle in quadrant I or quadrant IV whose sine is $\frac{1}{2}$.” We have memorized the fact that this is $\frac{\pi}{6}$.

Thus, $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$. Note that this corresponds to 30° .

2. Find $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$ in both radians and degrees.

We are asked to find the angle in quadrant I or quadrant IV whose sine is $-\frac{\sqrt{2}}{2}$. Since the sine function is positive in quadrant I, we must look in

quadrant IV. We know that $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, so we use $\frac{\pi}{4}$ as a reference

angle to find that the angle we want is $-\frac{\pi}{4}$.

Thus, $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$. This corresponds to -45° . ■

We memorized the sine values for certain angles, such as $\frac{\pi}{6}$ (30°), $\frac{\pi}{4}$ (45°), and $\frac{\pi}{3}$ (60°). In most cases, however, we can obtain only decimal approximations. For this we use calculators, as shown below and in sections 1-3, 1-4, 2-4, and 2-6. We will round radian answers to two decimal places and degree answers to one decimal place.

■ Example 4-2 B

1. Find $\sin^{-1}0.5312$ in both radians and degrees.

Put the calculator in radian mode. Most calculators use either a $\boxed{\text{SIN}^{-1}}$ key or the $\boxed{\text{INV}}$ (or $\boxed{\text{ARC}}$ or $\boxed{2\text{nd}}$) before the $\boxed{\text{SIN}}$ key.

.5312 $\boxed{\text{INV}}$ $\boxed{\text{SIN}}$ or Display: $\boxed{0.560016290}$

.5312 $\boxed{\text{SIN}^{-1}}$

$\boxed{\text{TI-81}}$ $\boxed{2\text{nd}}$ $\boxed{\text{SIN}}$.5312 $\boxed{\text{ENTER}}$

Thus, $\sin^{-1}0.5312 \approx 0.56$.

To obtain the result in degrees, put the calculator in degree mode before performing the calculations. When we round to one decimal place, we get 32.1° .

2. Find $\arcsin(-0.9249)$ in both radians and degrees.

We want the arc (angle) whose sine is -0.9249 and is in quadrant I or quadrant IV. The negative value tells us we want an angle in quadrant IV.

Make sure the calculator is in radian mode.

.9249 $\boxed{+/-}$ $\boxed{\text{INV}}$ $\boxed{\text{SIN}}$ (or $\boxed{\text{SIN}^{-1}}$) Display: $\boxed{-1.180772498}$

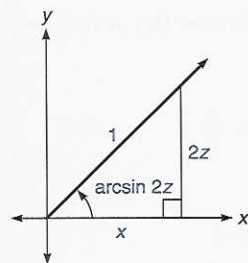
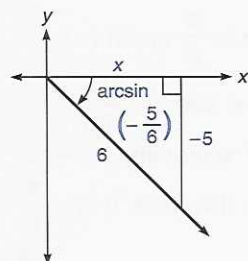
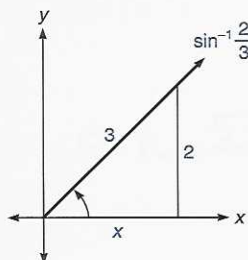
$\boxed{\text{TI-81}}$ $\boxed{2\text{nd}}$ $\boxed{\text{SIN}}$ $\boxed{(-)}$.9249 $\boxed{\text{ENTER}}$

and round to two decimal places, or -1.18 . Thus, $\arcsin(-0.9249) \approx -1.18$.

Performing the same calculator steps in degree mode gives -67.7° . ■

It is often important to be able to simplify expressions that involve combinations of the trigonometric and inverse trigonometric functions. This can often be done with the aid of the reference triangle we studied in section 2-4.

■ Example 4-2 C



1. Simplify $\tan(\sin^{-1}\frac{2}{3})$.

Since $\frac{2}{3}$ is positive, the angle represented by $\sin^{-1}\frac{2}{3}$ is in quadrant I. (Remember, the range of the $\sin^{-1}$ function is quadrant I and quadrant IV.) Since $\sin^{-1}\frac{2}{3}$ means the angle in quadrant I whose sine is $\frac{2}{3}$, we can draw a reference triangle (section 2-4) in quadrant I for this angle. See the figure. Using the Pythagorean theorem, we calculate the third side of the triangle to be $\sqrt{5}$. From this we can see that the tangent of this angle is $\frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$. Thus, $\tan(\sin^{-1}\frac{2}{3}) = \frac{2\sqrt{5}}{5}$.

2. Simplify $\sec[\arcsin(-\frac{5}{6})]$.

$\arcsin(-\frac{5}{6})$ represents an angle in quadrant IV whose sine is $-\frac{5}{6}$. A reference triangle for such an angle is shown in the figure.

We compute the length of side x to be $\sqrt{11}$. Now we can compute the cosine of this angle to be $\frac{\sqrt{11}}{6}$, and the secant of this angle is then the reciprocal of the cosine. That is, $\frac{6}{\sqrt{11}} = \frac{6}{\sqrt{11}} \cdot \frac{\sqrt{11}}{\sqrt{11}} = \frac{6\sqrt{11}}{11}$. Thus, $\sec[\arcsin(-\frac{5}{6})] = \frac{6\sqrt{11}}{11}$.

3. Simplify $\tan(\arcsin 2z)$, if $z > 0$.

Since $z > 0$, then $2z$ is also positive, so $\arcsin 2z$ represents an angle in quadrant I whose sine is $2z$. A reference triangle for such an angle is shown in the figure.

We find the horizontal side x by the Pythagorean theorem.

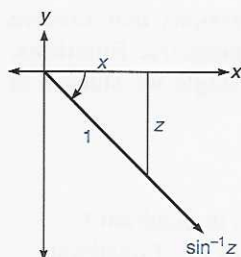
$$\begin{aligned} 1^2 &= x^2 + (2z)^2 \\ x^2 &= 1 - 4z^2 \\ x &= \sqrt{1 - 4z^2} \end{aligned}$$

Since x is positive, we chose the positive solution when we took the square root.

We can now find the tangent of this angle. We form the ratio of the side opposite the angle to the side adjacent to the angle to get $\frac{2z}{\sqrt{1 - 4z^2}}$.

We will not rationalize this denominator.

$$\text{Thus, } \tan(\arcsin 2z) = \frac{2z}{\sqrt{1 - 4z^2}}, \text{ if } z > 0.$$

4. Compute $\cos(\sin^{-1}z)$, $z < 0$.

Since $z < 0$, $\sin^{-1}z$ represents an angle in quadrant IV. A reference triangle for such an angle is shown in the figure. Note that we do not write $-z$, but simply z , for the directed vertical distance, since z already represents a negative quantity.

We find the length x .

$$\begin{aligned}x^2 + z^2 &= 1^2 \\x^2 &= 1 - z^2 \\x &= \sqrt{1 - z^2}\end{aligned}$$

Now we compute the cosine of this angle. It is $\frac{\sqrt{1 - z^2}}{1}$ or $\sqrt{1 - z^2}$.

Thus, $\cos(\sin^{-1}z) = \sqrt{1 - z^2}$ if $z < 0$.

5. Compute $\sin^{-1}\left(\sin \frac{7\pi}{6}\right)$.

We know that $\sin \frac{7\pi}{6} = -\frac{1}{2}$ since its reference angle is $\frac{\pi}{6}$ and it is in

quadrant III. (See section 2-3.) Thus, $\sin^{-1}\left(\sin \frac{7\pi}{6}\right) = \sin^{-1}\left(-\frac{1}{2}\right)$.

Thus, we need the angle in quadrant I or quadrant IV whose sine is $-\frac{1}{2}$.

The angle is in quadrant IV since its sine is negative; therefore, it is $-\frac{\pi}{6}$ in radians or -30° . ■

Part 5 of example 4-2 C shows that $\sin^{-1}(\sin x)$ is not necessarily x . If, however, x is an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, then this is true (try a few examples). Thus, we note:

$$\sin^{-1}(\sin x) = x \text{ if and only if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

Also,

$$\sin(\sin^{-1}x) = x \text{ if and only if } -1 \leq x \leq 1$$

This last restriction applies simply because the domain of the inverse sine function is $-1 \leq x \leq 1$.

Mastery points

Can you

- Compute exact and approximate values for the inverse of the sine function?
- Simplify expressions that combine the trigonometric functions and the inverse sine function?
- State the domain and range of the inverse sine function?

Exercise 4-2

1. Sketch the graph of the inverse sine function.
2. State the domain and range of the inverse sine function.

Find exact values for each of the following expressions. State the results in both radians and degrees.

3. $\sin^{-1}(-\frac{1}{2})$
4. $\arcsin \frac{\sqrt{3}}{2}$
5. $\sin^{-1}0$
6. $\arcsin(-\frac{\sqrt{2}}{2})$
7. $\arcsin(-\frac{\sqrt{3}}{2})$
8. $\sin^{-1}1$

Find approximate values for the following expressions in both radians and degrees. Round the values for radians to two decimal places and for degrees to one decimal place.

9. $\sin^{-1}0.8823$
10. $\arcsin 0.8253$
11. $\sin^{-1}0.9323$
12. $\sin^{-1}0.6442$
13. $\sin^{-1}(-0.9976)$
14. $\sin^{-1}(-0.2955)$
15. $\arcsin(-0.2571)$
16. $\arcsin(-0.9888)$

Simplify each of the following expressions.

17. $\tan(\arcsin \frac{5}{8})$
18. $\cos(\arcsin \frac{3}{5})$
19. $\sec[\sin^{-1}(-\frac{2}{3})]$
20. $\tan[\arcsin(-0.8)]$
21. $\cot(\sin^{-1} \frac{\sqrt{3}}{5})$
22. $\csc(\sin^{-1} \frac{\sqrt{2}}{6})$
23. $\cos(\arcsin 0.3)$
24. $\tan(\arcsin 0.4)$
25. $\cos(\sin^{-1}z), z > 0$
26. $\cos(\sin^{-1}3z), z < 0$
27. $\tan[\sin^{-1}(1+z)], 1+z < 0$
28. $\cos(\arcsin \sqrt{z})$
29. $\sec(\arcsin \sqrt{2z})$
30. $\cot(\sin^{-1} \sqrt{z-1})$
31. $\sin^{-1}(\sin \frac{\pi}{6})$
32. $\arcsin(\tan \frac{\pi}{4})$
33. $\arcsin(\cos \frac{2\pi}{3})$
34. $\sin^{-1}(\sin \frac{11\pi}{6})$
35. $\sin^{-1}(\cos 0)$
36. $\arcsin(\sin \frac{5\pi}{6})$

37.  Recall that a function f is periodic if there is a number $p, p > 0$, such that $f(x) = f(x + p)$ for all x in the domain of the function. Can a periodic function be one to one? Give a reason for your answer.

4-3 The inverse cosine and inverse tangent functions

The inverse of the cosine function is formed in the same way as the inverse of the sine function. We first select a one-to-one portion. This is chosen to be between 0 and π . See figure 4-7.

The graph of $y = \cos^{-1}x$ is the reflection of this portion of the cosine function across the line $y = x$ (shown in figure 4-7 by a dashed line). The graph of the inverse cosine function is shown separately in figure 4-8.

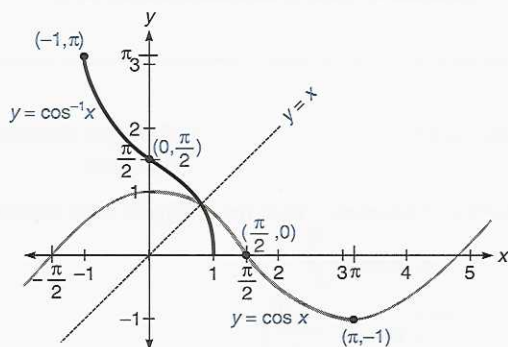
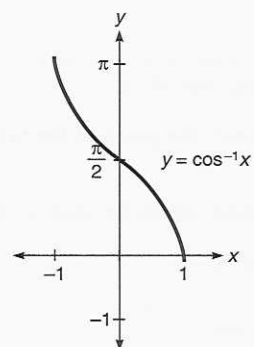


Figure 4-7



$$Y_1 = \cos^{-1} X$$

RANGE $-3.3, .5, -.5, 3.5, .785$

Figure 4-8

We can see from the graph that

$$\text{Domain}_{\cos^{-1}}: |x| \leq 1$$

$$\text{Range}_{\cos^{-1}}: 0 \leq y \leq \pi$$

The inverse cosine function is defined as follows:

The inverse cosine function

$y = \cos^{-1}x$ means

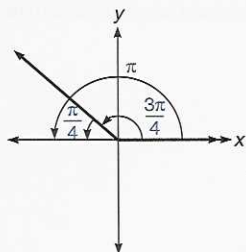
1. $\cos y = x$
2. $0 \leq y \leq \pi$
3. $|x| \leq 1$

Concept

$\cos^{-1}x$ is the angle in quadrant I or quadrant II whose cosine is x .

Arccos x and $\text{invcos } x$ mean the same thing as $\cos^{-1}x$. Note that in degrees the inverse cosine varies from 0° to 180° .

Example 4-3 A



1. Find $\cos^{-1}\frac{1}{2}$ in both radians and degrees.

We want the angle in quadrant I or quadrant II whose cosine is $\frac{1}{2}$. Since this is a positive value, the angle is in quadrant I. We know $\cos$

$$\frac{\pi}{3} = \frac{1}{2}, \text{ so } \cos^{-1}\frac{1}{2} = \frac{\pi}{3}. \text{ Note that this is equivalent to } 60^\circ.$$

2. Find $\arccos\left(-\frac{\sqrt{2}}{2}\right)$ in both radians and degrees.

We want the angle in quadrant I or quadrant II whose cosine is $-\frac{\sqrt{2}}{2}$.

Since this is a negative value, the angle is in quadrant II. We know $\cos$

$$\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \text{ so we use } \frac{\pi}{4} \text{ as a reference angle in quadrant II. This gives us}$$

$$\pi - \frac{\pi}{4} = \frac{3\pi}{4} \text{ for our angle. See the figure.}$$

$$\text{Thus, } \arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}, \text{ which is equivalent to } 135^\circ. \quad \blacksquare$$

As with the inverse sine function, we sometimes need approximate answers. This is illustrated in example 4-3 B.

Example 4-3 B

1. Find $\cos^{-1}0.9638$ in both radians and degrees.

We are looking for the angle in quadrant I or quadrant II whose cosine is 0.9638. Since this is a positive value, the angle is in quadrant I. Putting the calculator in radian mode,

$$.9638 \quad \boxed{\text{INV}} \quad \boxed{\text{COS}} \quad (\text{or } \boxed{\text{COS}^{-1}}) \quad \text{Display: } \boxed{0.269890866}$$

$$\boxed{\text{TI-81}} \quad \boxed{2\text{nd}} \quad \boxed{\text{COS}} \quad .9638 \quad \boxed{\text{ENTER}}$$

Thus, $\cos^{-1}0.9638 \approx 0.27$.

If the calculation is done with the calculator in degree mode the result is approximately 15.5° .

2. Find $\arccos(-0.5141)$ in both radians and degrees.

We are looking for the angle in quadrant I or quadrant II whose cosine is -0.5141 . Since this value is negative, the angle is in quadrant II.

In radian mode,

$$.5141 \quad \boxed{+/-} \quad \boxed{\text{INV}} \quad \boxed{\text{COS}} \quad \text{Display: } \boxed{2.110754365}$$

$$\boxed{\text{TI-81}} \quad \boxed{2\text{nd}} \quad \boxed{\text{COS}} \quad \boxed{(-)} \quad .5141 \quad \boxed{\text{ENTER}}$$

Thus, $\arccos(-0.5141) \approx 2.11$. When calculated in degree mode the result is approximately 120.9° . $\blacksquare$

The inverse of the tangent function is formed in the same way as the inverses of the sine and cosine functions. We first select a one-to-one portion.

This is chosen to be between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. (In fact, this is the same as our basic tangent cycle from section 3-1.) The graph of $y = \tan^{-1}x$ is the reflection of this basic cycle across the line $y = x$. The graph of the basic cycle and its reflection are shown in figure 4-9 and the graph of the inverse tangent function is shown in figure 4-10.

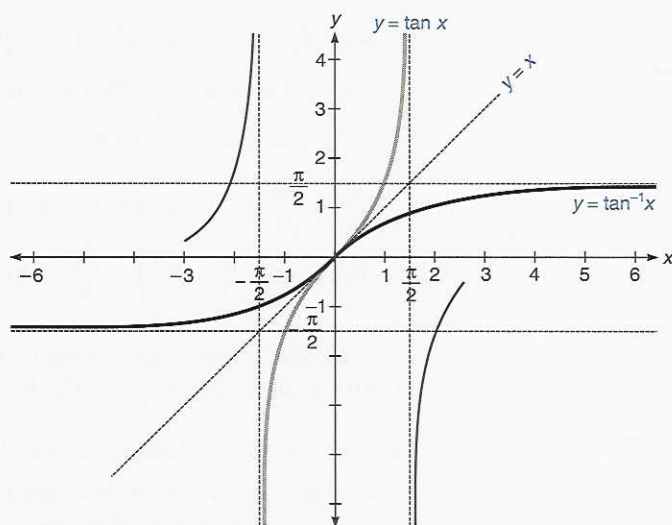


Figure 4-9

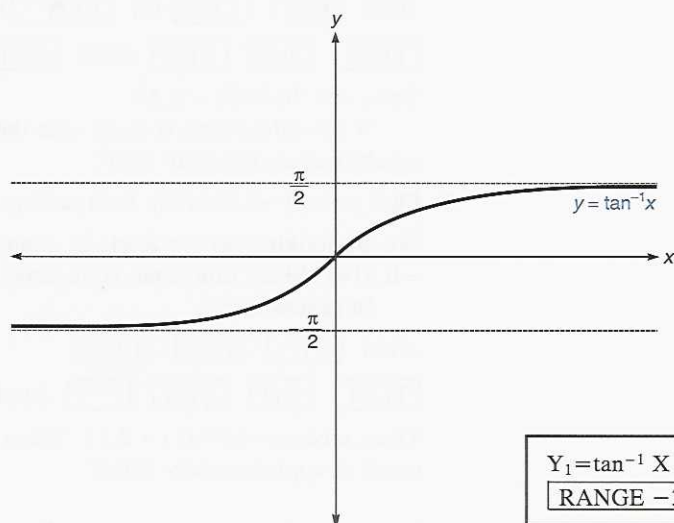


Figure 4-10

The domain of the inverse tangent function is all the reals, and the range is all values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Observe that the range is in quadrants I and IV, as is the range of the inverse sine function, except that the range of the inverse tangent function does not include the points $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ themselves.

Thus,

$$\text{Domain}_{\tan^{-1}}: R$$

$$\text{Range}_{\tan^{-1}}: -\frac{\pi}{2} < y < \frac{\pi}{2}$$

The inverse tangent function

$y = \tan^{-1}x$ means

1. $\tan y = x$ and
2. $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Concept

$\tan^{-1}x$ means the angle in quadrant I or quadrant IV whose tangent is x .

Note Since x can take on any value, we do not restrict it in the definition as we did for the inverse sine and inverse cosine functions.

Arctan x and **invtan** x are other notations for $\tan^{-1}x$.

■ Example 4-3 C

1. Find $\tan^{-1}\sqrt{3}$ in both radians and degrees.

We want the angle in quadrant I or quadrant IV whose tangent is $\sqrt{3}$. Since this is a positive value, the angle is in quadrant I. We know $\tan \frac{\pi}{3} = \sqrt{3}$, so $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$. This is 60° , also.

2. Find $\arctan(-1)$ in both radians and degrees.

We want the angle in quadrant I or quadrant IV whose tangent is -1 . Since this is a negative value, the angle is in quadrant IV. We know $\tan \frac{\pi}{4} = 1$, so we use $\frac{\pi}{4}$ as a reference angle in quadrant IV. This gives us $-\frac{\pi}{4}$, or -45° , for our angle.

$$\text{Thus, } \arctan(-1) = -\frac{\pi}{4}, \text{ or } -45^\circ.$$

Example 4-3 D illustrates obtaining approximate values with the calculator.

Example 4-3 D

Find $\arctan(-1.9208)$ in both radians and degrees.

We are looking for the angle in quadrant I or quadrant IV whose tangent is -1.9208 . Since this value is negative the angle is in quadrant IV.

1.9208 $\boxed{+/-}$ $\boxed{INV}$ $\boxed{TAN}$ (or $\boxed{TAN^{-1}}$) Display: $\boxed{-1.090791948}$
 $\boxed{TI-81}$ $\boxed{2nd}$ $\boxed{TAN}$ $\boxed{(-)}$ 1.9208 $\boxed{ENTER}$

Thus, the result is about -1.09 radians and -62.5° . ■

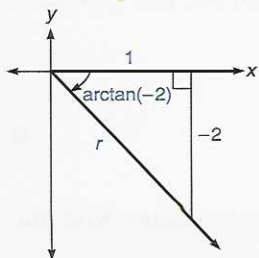
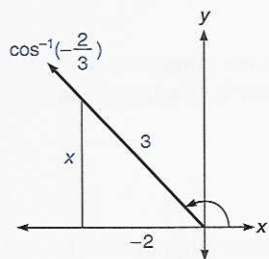
The domains and ranges of the inverses of the sine, cosine, and tangent functions are summarized in table 4-1. Also indicated are the quadrants to which the ranges correspond.

Function	Domain	Range	Quadrants
$y = \sin^{-1}x$	$ x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$	I, IV
$y = \cos^{-1}x$	$ x \leq 1$	$0 \leq y \leq \pi$	I, II
$y = \tan^{-1}x$	R	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	I, IV

Table 4-1

We can simplify expressions that involve combinations of the trigonometric and inverse trigonometric functions using the reference triangle, as we did in the previous section.

Example 4-3 E



1. Simplify $\tan[\cos^{-1}(-\frac{2}{3})]$.

Since $-\frac{2}{3}$ is negative, the angle represented by $\cos^{-1}(-\frac{2}{3})$ is in quadrant II. (Remember, the range of this function is quadrants I and II.) Since $\cos^{-1}(-\frac{2}{3})$ means the angle in quadrant II whose cosine is $-\frac{2}{3}$, we draw a reference triangle in quadrant II for this angle. See the figure.

Using the Pythagorean theorem, we calculate the third side of the triangle to be $\sqrt{5}$. From this we can see that the tangent of this angle is $\frac{\sqrt{5}}{-2} = -\frac{\sqrt{5}}{2}$.

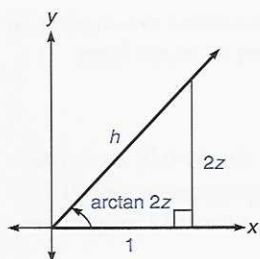
$$\text{Thus, } \tan\left[\cos^{-1}\left(-\frac{2}{3}\right)\right] = -\frac{\sqrt{5}}{2}.$$

2. Simplify $\sec[\arctan(-2)]$.

$\arctan(-2)$ represents an angle in quadrant IV whose tangent is -2 . A reference triangle for such an angle is shown in the figure.

Using the Pythagorean theorem, we compute the length of the hypotenuse to be $\sqrt{5}$. From this triangle we can find the secant of this angle. We see that the cosine is $\frac{1}{\sqrt{5}}$, so the secant is the reciprocal, or $\sqrt{5}$.

$$\text{Thus, } \sec[\arctan(-2)] = \sqrt{5}.$$



3. Simplify $\sin(\arctan 2z)$, if $z > 0$.

Since $z > 0$, then $2z$ is also positive, so $\arctan 2z$ represents an angle in quadrant I whose tangent is $2z$. A reference triangle for such an angle is shown in the figure.

We find the hypotenuse h using the Pythagorean theorem.

$$\begin{aligned} 1^2 + (2z)^2 &= h^2 \\ 1 + 4z^2 &= h^2 \\ \sqrt{1 + 4z^2} &= h \end{aligned}$$

We can now find the sine of this angle. We form the ratio of the side opposite the angle to the hypotenuse to get $\frac{2z}{\sqrt{1 + 4z^2}}$. We will not rationalize this denominator.

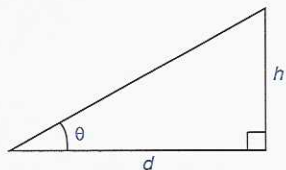
$$\text{Thus, } \sin(\arctan 2z) = \frac{2z}{\sqrt{1 + 4z^2}} \text{ if } z > 0.$$

4. Find $\tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right]$.

$$\begin{aligned} \tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right] &= \tan^{-1}(-1) \\ &= -\frac{\pi}{4} \text{ or } -45^\circ \end{aligned}$$

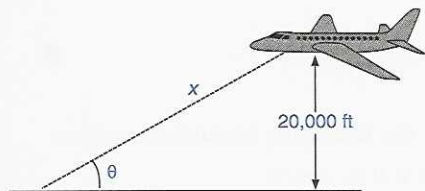
One application of the inverse trigonometric functions is to describe an angle in a given situation with a mathematical expression.

■ Example 4-3 F



1. Describe angle θ in the figure using an inverse trigonometric function.

We can see that the tangent of θ is $\frac{h}{d}$, so “ θ is an angle whose tangent is $\frac{h}{d}$.” This is what $\tan^{-1}\frac{h}{d}$ means, so $\theta = \tan^{-1}\frac{h}{d}$.



2. A jet aircraft is flying at 20,000 feet. If x represents the distance from a ground observer to the aircraft (also in feet), describe the angle of elevation from the observer to the aircraft in terms of an inverse trigonometric function.

If we represent the situation as shown in the figure, where θ indicates the angle of elevation, we see that θ is an angle whose sine is $\frac{20,000}{x}$. This is what the expression $\sin^{-1}\frac{20,000}{x}$ means.

$$\text{Thus, since } \sin \theta = \frac{20,000}{x}, \theta = \sin^{-1}\frac{20,000}{x}.$$

Another way in which the inverse trigonometric functions are used is to describe one of the solutions to a trigonometric equation in exact form.

■ Example 4-3 G

1. $\sin \theta = 0.7$. Describe one value of θ in exact form.

Since the equation tells us that θ is an angle whose sine is 0.7, we write $\theta = \sin^{-1}0.7$. This expression is exact. We could approximate $\sin^{-1}0.7$ with a calculator, but the decimal number we obtained would only be an approximation.

Thus, if $\sin \theta = 0.7$, one exact value of θ is $\sin^{-1}0.7$.

2. $\tan \theta = z$. Describe one value of θ in exact form.

We see that $\theta = \tan^{-1}z$, so one value would be $\tan^{-1}z$.

3. $\cos 2\theta = 0.55$. Describe one value of θ in exact form.

$2\theta = \cos^{-1}0.55$, so $\theta = \frac{\cos^{-1}0.55}{2}$ is one value of θ .

4. $3 \sin \theta = 0.69$. Describe one value of θ in exact form.

If we divide both sides by 3, we see that $\sin \theta = 0.23$. Thus, $\theta = \sin^{-1}0.23$ gives one exact value of θ .

5. $A \sin Bx = C$. Describe one value of x in exact form.

Dividing both sides by A :

$$\sin Bx = \frac{C}{A}$$

Then,

$$Bx = \sin^{-1}\frac{C}{A}$$

Dividing both sides by B :

$$x = \frac{\sin^{-1}\frac{C}{A}}{B}$$

Thus, one value of x that satisfies the equation $A \sin Bx = C$ is

$$\frac{\sin^{-1}\frac{C}{A}}{B}.$$

Finally, it is not too hard to verify that the following identities are true:

$$\cos^{-1}(\cos x) = x \text{ if and only if } 0 \leq x \leq \pi$$

$$\cos(\cos^{-1}x) = x \text{ if and only if } -1 \leq x \leq 1$$

$$\tan^{-1}(\tan x) = x \text{ if and only if } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\tan(\tan^{-1}x) = x \text{ for all } x$$

Mastery points**Can you**

- Compute exact and approximate values for the inverses of the cosine and tangent functions?
- Simplify expressions that combine the trigonometric functions and their inverses?
- State the domains and ranges of these inverse trigonometric functions?
- Apply these inverse trigonometric functions to describe angles in physical situations?
- Apply these inverse trigonometric functions to give one exact solution to trigonometric equations?

Exercise 4-3

1. Sketch the graph of each function.

- inverse sine
- inverse cosine
- inverse tangent

2. State the domain and range for each function.

- inverse sine
- inverse cosine
- inverse tangent

Find exact values for each of the following expressions in both radians and degrees.

- | | | | | |
|------------------------------|---------------------------------|-----------------------------------|--------------------------------------|--------------------------------------|
| 3. $\cos^{-1}(-\frac{1}{2})$ | 4. $\arccos \frac{\sqrt{3}}{2}$ | 5. $\cos^{-1}0$ | 6. $\arccos(-\frac{\sqrt{2}}{2})$ | 7. $\arcsin \frac{\sqrt{3}}{2}$ |
| 8. $\tan^{-1}1$ | 9. $\arctan(-\sqrt{3})$ | 10. $\cos^{-1}\frac{\sqrt{2}}{2}$ | 11. $\tan^{-1}(-\frac{\sqrt{3}}{3})$ | 12. $\cos^{-1}(-\frac{\sqrt{3}}{2})$ |

Find approximate values for the following expressions, in both degrees and radians. Round degrees to tenths and radians to hundredths.

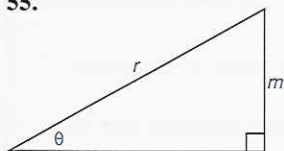
- | | | | |
|--------------------------|--------------------------|------------------------|------------------------|
| 13. $\sin^{-1}0.8823$ | 14. $\arctan 11.08$ | 15. $\arccos 0.8253$ | 16. $\arcsin 0.6131$ |
| 17. $\tan^{-1}0.9316$ | 18. $\cos^{-1}0.6442$ | 19. $\arcsin 0.7961$ | 20. $\sin^{-1}0.8776$ |
| 21. $\sin^{-1}(-0.9976)$ | 22. $\cos^{-1}(-0.2955)$ | 23. $\arctan(-0.2553)$ | 24. $\arccos(-0.9888)$ |
| 25. $\arccos(-0.9902)$ | 26. $\tan^{-1}(-3.4776)$ | | |

Simplify each of the following expressions.

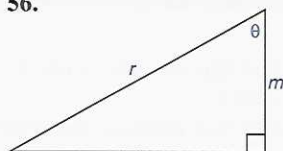
- | | | | |
|---------------------------------------|----------------------------------|-------------------------------------|---|
| 27. $\cos(\tan^{-1}\frac{3}{5})$ | 28. $\sin(\cos^{-1}\frac{1}{4})$ | 29. $\sin(\arccos \frac{5}{8})$ | 30. $\cos(\arctan 2)$ |
| 31. $\tan[\cos^{-1}(-\frac{2}{3})]$ | 32. $\sin[\arccos(-0.8)]$ | 33. $\sin(\tan^{-1}\sqrt{5})$ | 34. $\csc(\cos^{-1}\frac{\sqrt{2}}{6})$ |
| 35. $\cos(\arctan 0.3)$ | 36. $\sin(\arctan 0.4)$ | 37. $\sin(\cos^{-1}z), z > 0$ | 38. $\cos(\arccos z), z > 0$ |
| 39. $\tan(\arccos z), z > 0$ | 40. $\sin(\tan^{-1}z), z > 0$ | 41. $\cos(\arctan z), z < 0$ | 42. $\tan(\cos^{-1}z), z < 0$ |
| 43. $\sin(\cos^{-1}3z), z < 0$ | 44. $\cos(\arctan 2z), z > 0$ | 45. $\sec[\tan^{-1}(1+z)], z > 0$ | 46. $\sin(\arccos \sqrt{z})$ |
| 47. $\cos(\arctan \sqrt{2z})$ | 48. $\tan(\cos^{-1}\sqrt{z-1})$ | 49. $\tan^{-1}(\sin \frac{\pi}{2})$ | 50. $\cos^{-1}(\sin \frac{7\pi}{6})$ |
| 51. $\cos^{-1}(\cos \frac{11\pi}{6})$ | 52. $\tan^{-1}(\cos 0)$ | 53. $\arccos(\tan \frac{5\pi}{4})$ | 54. $\cos^{-1}(\cos \frac{3\pi}{2})$ |

In the following problems state the angle θ shown in each diagram in terms of an inverse trigonometric function.

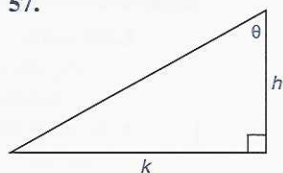
55.



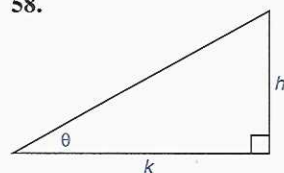
56.



57.



58.



59. A picture hangs on a wall so that the bottom of the picture is 5 feet above the floor. The picture is 2 feet high. Describe the angle subtended by the picture at the eye of an observer, in terms of an inverse trigonometric function, if the observer's eye is also 5 feet above the floor and the observer is x feet away from the wall.

60. A radar antenna will track the launch of a rocket from a point 12,500 feet from the rocket. Both are at the same ground elevation at launch. If a represents the altitude of the rocket in feet, describe the angle of elevation of the rocket at the radar site in terms of an inverse trigonometric function.

61. An aircraft is flying toward an airport at an elevation of 3,500 feet above the airport. Describe the angle of depression of the airport at the aircraft in terms of the distance z from the aircraft directly to the airport, using an inverse trigonometric function.

In the following problems describe one value of θ , or x , in exact form, in terms of an inverse trigonometric function.

62. $\sin \theta = 0.75$

63. $\cos \theta = -0.8$

64. $\tan \theta = 3$

65. $\tan \theta = 4.1$

66. $2 \sin \theta = 1.6$

67. $3 \tan \theta = 5$

68. $\frac{1}{2} \sin \theta = -0.1$

69. $\frac{\tan \theta}{5} = 10$

70. $\sin 2\theta = 0.76$

71. $\tan 3\theta = 9$

72. $\cos \frac{\theta}{3} = -0.42$

73. $\sin \frac{3\theta}{2} = -0.56$

74. $4 \cos 3\theta = 3$

75. $2 \sin 4\theta = 1.5$

76. $\frac{3 \cos 2\theta}{8} = \frac{7}{40}$

77. $\frac{6 \sin 5\theta}{5} = \frac{10}{13}$

78. $\frac{A \cos Bx}{C} = D$

79. $A \tan(Bx + C) = D$

80. $\cos(x - 2) = 0.2$

81. $\sin(2x + 3) = 0.6$

82. Many computer languages only provide an arctangent function. In these situations we must program our own arcsine and arccosine functions. Use appropriate reference triangles to show that the following are identities.

$$\text{a. } \arcsin(x) = \begin{cases} -\frac{\pi}{2} & \text{if } x = -1 \\ \arctan\left(\frac{x}{\sqrt{1-x^2}}\right) & \text{if } |x| < 1 \\ \frac{\pi}{2} & \text{if } x = 1 \end{cases}$$

$$\text{b. } \arccos(x) = \begin{cases} \arctan\left(\frac{\sqrt{1-x^2}}{x}\right) & \text{if } 0 < x \leq 1 \\ \frac{\pi}{2} & \text{if } x = 0 \\ \arctan\left(\frac{\sqrt{1-x^2}}{x}\right) + \pi & \text{if } -1 \leq x < 0 \end{cases}$$

4-4 The inverse cotangent, secant, and cosecant functions

The reciprocal trigonometric functions (cotangent, secant, cosecant) and their inverses were useful for computations before the advent of electronic calculating devices like calculators and computers; today they have no practical use in computation. This is why calculators do not have keys for the reciprocal functions, and why programming languages, such as FORTRAN, BASIC, or Pascal, do not support these functions. These functions still have value in expressing certain expressions in higher mathematics, however, and that is why we study them here.

The inverse cotangent function

We define the inverse cotangent function by reversing the ordered pairs in the basic cotangent cycle; that is, we restrict the domain to $0 < x < \pi$. Figure 4-11 shows the inverse cotangent function, $\cot^{-1}$. The domain is R , and the range is $0 < y < \pi$. As we might suspect **arccot** x also means $\cot^{-1}x$.

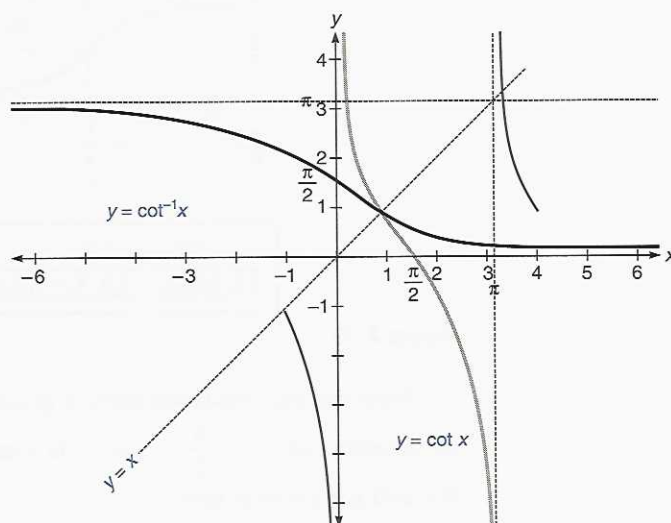


Figure 4-11

The inverse cotangent function

$y = \cot^{-1}x$ means

1. $\cot y = x$
2. $0 < y < \pi$

There are several ways to compute values of the inverse cotangent function. One way is to use the identity

$$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$$

This identity can be seen by considering the graphs of $y = \cot^{-1}x$ (figure 4-11) and $y = \tan^{-1}x$ (figure 4-12). Figure 4-12 shows the sequence of operations that will transform one graph into the other. Part (a) shows $y = \tan^{-1}x$. Part (b) shows the transformation caused by the scaling factor -1 . Part (c) shows the vertical shift caused by adding $\frac{\pi}{2}$. This result is the same as the graph of the inverse cotangent function.

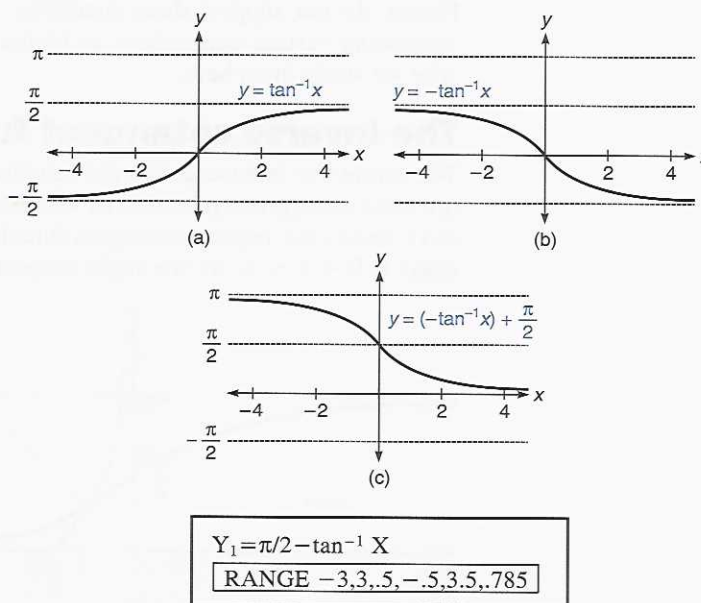


Figure 4-12

Note that this transformation of graphs does not absolutely guarantee that the identity $\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$ is true. It should be proven algebraically. We will not prove it here.

The inverse secant function

Recall the identity $\sec \theta = \frac{1}{\cos \theta}$. We use this to define the inverse of the secant function. Suppose we stated that $y = \sec^{-1}x$. Then $\sec y = x$. We proceed as shown.

$$y = \sec^{-1}x$$

$$\sec y = x$$

$$\frac{1}{\cos y} = x$$

$$\cos y = \frac{1}{x}$$

$$y = \cos^{-1}\frac{1}{x}$$

$$\sec^{-1}x = \cos^{-1}\frac{1}{x}$$

An expression we want to define

We expect the expression to have this property

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\frac{1}{\cos y} = x; 1 = x \cos y; \frac{1}{x} = \cos y$$

Since y is the angle whose cosine is $\frac{1}{x}$

We use this sequence of steps to motivate our definition. As expected, $\operatorname{arcsec} x$ also means $\sec^{-1}x$.

The inverse secant function

$$\sec^{-1}x = \cos^{-1}\frac{1}{x} \text{ if } |x| \geq 1$$

We require $|x| \geq 1$ so that $\left|\frac{1}{x}\right| \leq 1$, as required by the inverse cosine function. The range of the secant function is the range of the inverse cosine function since the latter defines the former, except that $\frac{\pi}{2}$ is not in the range.

This is because $\frac{\pi}{2} = \cos^{-1}0$, and there is no value of x such that $\frac{1}{x} = 0$.

Inverse cosecant function

The identity $\csc \theta = \frac{1}{\sin \theta}$, and reasoning similar to that above, leads to the following definition for the inverse cosecant (**arcsecant**) function.

The inverse cosecant function

$$\csc^{-1}x = \sin^{-1}\frac{1}{x} \text{ if } |x| \geq 1$$

For reasons similar to those stated earlier, the domain of the inverse cosecant function is $|x| \geq 1$, and the range is the same as that of the inverse sine function, except for 0, which is $\sin^{-1}0$, and $\frac{1}{x}$ can not take on the value 0.

Note It is not worth the effort to study the graphs of the functions of this section. We presented the graph of the inverse cotangent function only to justify the identity presented earlier.

Summary of properties

Table 4-2 summarizes the domains and ranges of the functions introduced previously.

Function	Domain	Range	Quadrants
$y = \cot^{-1}x$	R	$0 < y < \pi$	I, II
$y = \sec^{-1}x$	$ x \geq 1$	$0 \leq y \leq \pi, y \neq \frac{\pi}{2}$	I, II
$y = \csc^{-1}x$	$ x \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$	I, IV

Table 4-2

Note in this table that quadrant I always corresponds to a nonnegative domain element (a nonnegative value of x), and quadrant II or IV corresponds to negative domain elements.

Example 4-4 A

Find the given values in both radians and degrees. Round radians to two decimal places and degrees to one decimal place where necessary.

1. $\cot^{-1}(-4)$

$$\begin{aligned}
 &= \frac{\pi}{2} - \tan^{-1}(-4) && \cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x \\
 &\approx 2.90 \text{ (radians)} && \text{Calculator in radian mode} \\
 &= 90^\circ - \tan^{-1}(-4) \\
 &\approx 166.0^\circ && \text{Calculator in degree mode}
 \end{aligned}$$

Thus, $\cot^{-1}(-4) \approx 166.0^\circ$ or 2.90 (radians).

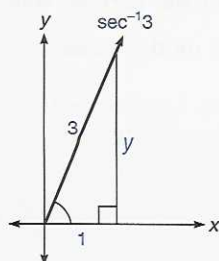
2. $\sec^{-1} 2$

$$\begin{aligned}
 &= \cos^{-1} \frac{1}{2} && \text{Definition of } \sec^{-1}\theta \\
 &= 60^\circ \text{ or } \frac{\pi}{3}
 \end{aligned}$$

Thus, $\sec^{-1} 2 = 60^\circ$ or $\frac{\pi}{3}$.

As we saw in section 4-3 some expressions that involve both the trigonometric and inverse trigonometric functions can be simplified by using a reference triangle.

Example 4-4 B



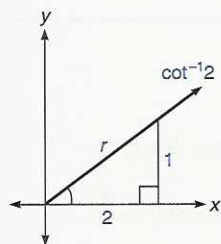
Simplify the expression.

1. $\sin(\sec^{-1}3)$

$\sec^{-1}3 = \cos^{-1}\frac{1}{3}$, by definition. This is a first quadrant angle. The figure shows a reference triangle in this quadrant in which the cosine of the angle is $\frac{1}{3}$.

$$y = \sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$$

The sine of this angle is $\frac{\text{opposite}}{\text{hypotenuse}} = \frac{2\sqrt{2}}{3}$.

2. $\sec(\cot^{-1}2)$

$\cot^{-1}2$ is an angle in quadrant I. Since its cotangent is 2, its tangent is $\frac{1}{2}$. A reference triangle for a quadrant I angle with tangent $\frac{1}{2}$ is shown in the figure.

$r = \sqrt{5}$ (Pythagorean theorem), so the cosine of the angle is $\frac{2}{\sqrt{5}}$,

and therefore the secant is $\frac{\sqrt{5}}{2}$. Thus, $\sec(\cot^{-1}2) = \frac{\sqrt{5}}{2}$.

Note The identity $\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$ would not be helpful in simplifying this expression. It is used for computing values of $\cot^{-1}x$.

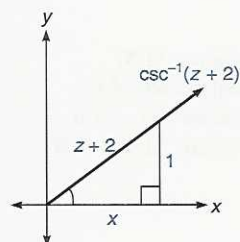
3. $\cos(\csc^{-1}(z+2))$, $z > 0$

$\csc^{-1}(z+2) = \sin^{-1}\frac{1}{z+2}$ by definition. Since $\frac{1}{z+2}$ is positive,

$\sin^{-1}\frac{1}{z+2}$ is an angle in quadrant I (see the figure). We find y by the Pythagorean theorem.

$$\begin{aligned}(z+2)^2 &= 1^2 + y^2 \\ z^2 + 4z + 4 - 1 &= y^2 \\ \sqrt{z^2 + 4z + 3} &= y\end{aligned}$$

The cosine of the angle is $\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{y}{z+2} = \frac{\sqrt{z^2 + 4z + 3}}{z+2}$.



Mastery points

Can you

- State the domain and range of the inverse cotangent, cosecant, and secant functions?
- Find exact values, in both radians and degrees, for expressions of the form $\cot^{-1}x$, $\csc^{-1}x$, and $\sec^{-1}x$, for appropriate values of x , using the definitions of these functions?
- Find approximate values, in both radians and degrees, for expressions of the form $\sin^{-1}x$, $\cos^{-1}x$, and $\tan^{-1}x$, using a calculator and the definitions of these functions?
- Simplify expressions that involve combinations of the trigonometric and inverse trigonometric functions, using a reference triangle where appropriate?

Exercise 4-4

Compute the exact value of the following expressions.

- | | | | | |
|---------------------------|---|------------------------------|--------------------------------|--|
| 1. $\csc^{-1}2$ | 2. $\operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right)$ | 3. $\operatorname{arccot} 1$ | 4. $\operatorname{arccsc}(-1)$ | 5. $\sec^{-1}(-2)$ |
| 6. $\cot^{-1}(-\sqrt{3})$ | 7. $\operatorname{arccsc}\frac{2\sqrt{3}}{3}$ | 8. $\csc^{-1}\sqrt{2}$ | 9. $\operatorname{arccot} 0$ | 10. $\operatorname{arcsec}(-\sqrt{2})$ |

Compute approximate values of the following expressions in both radians (to hundredths) and degrees (to tenths).

- | | | | |
|------------------------------------|--------------------------------------|--------------------------------------|--------------------------------------|
| 11. $\csc^{-1}3.3534$ | 12. $\cot^{-1}0.5080$ | 13. $\operatorname{arcsec}(-2.9986)$ | 14. $\operatorname{arccsc}(-2.5087)$ |
| 15. $\operatorname{arccot} 5.1997$ | 16. $\sec^{-1}(-2.0126)$ | 17. $\sec^{-1}(-11.1261)$ | 18. $\csc^{-1}(-3.8898)$ |
| 19. $\operatorname{arccsc} 3.1790$ | 20. $\operatorname{arccot}(-8.3534)$ | | |

Simplify the following expressions.

- | | | | |
|--|--|--|---|
| 21. $\sin(\csc^{-1}3)$ | 22. $\cos(\cot^{-1}2)$ | 23. $\cot(\operatorname{arcsec} 4)$ | 24. $\sin(\sec^{-1}1.5)$ |
| 25. $\csc(\operatorname{arccot} 5)$ | 26. $\sec(\operatorname{arccsc}\frac{7}{4})$ | 27. $\cos[\operatorname{arccsc}(-\frac{5}{4})]$ | 28. $\tan[\operatorname{arccsc}(-\frac{5}{3})]$ |
| 29. $\tan[\sec^{-1}(-\frac{6}{5})]$ | 30. $\csc[\sec^{-1}(-\frac{8}{7})]$ | 31. $\sin(\csc^{-1}z), z > 0$ | 32. $\cot(\sec^{-1}z), z > 0$ |
| 33. $\sec(\cot^{-1}z), z < 0$ | 34. $\tan(\operatorname{arcsec} z), z < 0$ | 35. $\cos(\operatorname{arcsec} 2z), z > 0$ | 36. $\sin(\csc^{-1}3z), z < 0$ |
| 37. $\tan[\sec^{-1}(z+1)], z+1 > 0$ | | 38. $\csc[\sec^{-1}(1-z)], 1-z > 0$ | |
| 39. $\csc\left(\operatorname{arccsc}\frac{3}{z}\right), z > 0$ | | 40. $\sec\left(\cot^{-1}\frac{2}{z+1}\right), z+1 > 0$ | |

Chapter 4 summary

- To show that two functions f and g are inverses of each other

Show that [1] If $f(x) = y$, then $g(y) = x$, and
[2] If $g(x) = y$, then $f(y) = x$.

- Vertical line test for a function** If no vertical line crosses the graph of a relation in more than one place, the relation is a function.
- Horizontal line test for a one-to-one function** If no horizontal line crosses the graph of a function in more than one place, the function is one to one.
- Inverse sine function**
 $y = \sin^{-1}x$ means
 - $\sin y = x$
 - $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
 - $-1 \leq x \leq 1$
- Inverse cosine function**
 $y = \cos^{-1}x$ means
 - $\cos y = x$
 - $0 \leq y \leq \pi$
 - $-1 \leq x \leq 1$

- Inverse tangent function**

$y = \tan^{-1}x$ means

- $\tan y = x$
- $-\frac{\pi}{2} < y < \frac{\pi}{2}$

- Inverse cotangent function**

$y = \cot^{-1}x$ means

- $\cot y = x$
- $0 < y < \pi$

$$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$$

- Inverse secant function** $\sec^{-1}x = \cos^{-1}\frac{1}{x}$ if $|x| \geq 1$

- Inverse cosecant function** $\csc^{-1}x = \sin^{-1}\frac{1}{x}$ if $|x| \geq 1$

- Summary of the properties of the inverse sine, cosine, and tangent functions.

Function	Domain	Range	Quadrants
$y = \sin^{-1}x$	$ x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$	I, IV
$y = \cos^{-1}x$	$ x \leq 1$	$0 \leq y \leq \pi$	I, II
$y = \tan^{-1}x$	R	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	I, IV

- Summary of the properties of the inverse cotangent, secant, and cosecant functions.

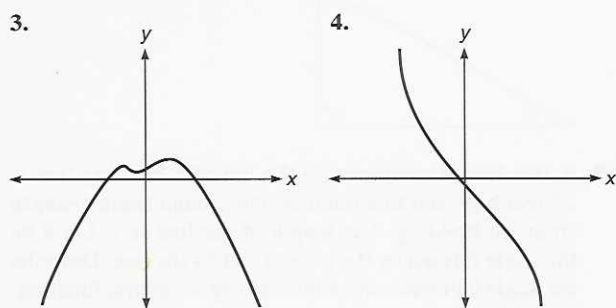
Function	Domain	Range	Quadrants
$y = \cot^{-1}x$	R	$0 < y < \pi$	I, II
$y = \sec^{-1}x$	$ x \geq 1$	$0 \leq y \leq \pi, y \neq \frac{\pi}{2}$	I, II
$y = \csc^{-1}x$	$ x \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$	I, IV

Chapter 4 review

[4-1] Show that the following functions f and g are inverses of each other.

- $f(x) = 3x - 5; g(x) = \frac{x+5}{3}$
- $f(x) = \frac{3x}{x+1}; g(x) = \frac{-x}{x-3}$

Each of the following diagrams shows the graph of a function. Use the horizontal line test to determine if the function is or is not one to one.



[4-2]

- Sketch the graph of the inverse sine function.
- State the domain and range of the inverse sine function.

Find exact values for each of the following expressions in both radians and degrees.

- $\sin^{-1}(-\frac{1}{2})$
- $\arcsin \frac{\sqrt{3}}{2}$
- $\sin^{-1}\frac{1}{2}$
- $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$
- $\arcsin 0$
- $\sin^{-1}(-1)$

Find approximate values for the following expressions in both radians and degrees.

- $\sin^{-1}0.9737$
- $\arcsin(-0.4882)$

Simplify the following expressions.

- $\cos(\sin^{-1}\frac{3}{5})$
- $\tan(\sin^{-1}\frac{1}{4})$
- $\sec[\sin^{-1}(-\frac{2}{3})]$
- $\tan(\sin^{-1}z), z > 0$
- $\cos[\arcsin(1+z)], 1+z > 0$

[4-3]

- Sketch the graph of the inverse cosine function.
- State the domain and range of the inverse tangent function.

Find exact values for each of the following expressions in both radians and degrees.

- $\cos^{-1}(-\frac{1}{2})$
- $\arccos \frac{\sqrt{3}}{2}$
- $\cos^{-1}\frac{1}{2}$
- $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$
- $\arctan \sqrt{3}$
- $\tan^{-1}(-1)$

Find approximate values for the following expressions in both radians and degrees.

28. $\tan^{-1}1.5601$ 29. $\arccos 0.4882$
 30. $\arccos(-0.3051)$

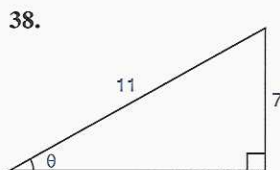
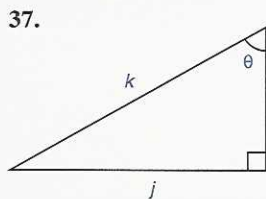
Simplify the following expressions.

31. $\cos(\tan^{-1}\frac{3}{5})$ 32. $\sec(\cos^{-1}\frac{1}{4})$
 33. $\tan[\cos^{-1}(-\frac{2}{3})]$ 34. $\cos^{-1}(\sin\frac{7\pi}{6})$

35. $\tan(\cos^{-1}z), z > 0$

36. $\cos[\arctan(1+z)], 1+z > 0$

For the following diagrams state the angle θ in terms of an inverse trigonometric function.



39. A camera is to be placed on the ground x feet from a flagpole that is 35 feet high. Describe the angle determined by the flagpole at the camera in terms of an inverse trigonometric function.

In the following problems describe one value of θ in exact form, in terms of an inverse trigonometric function.

40. $\cos \theta = 0.89$ 41. $\sin \theta = -0.88$
 42. $\frac{1}{2} \sin \theta = -0.1$ 43. $2 \tan \theta = 10$
 44. $\sin 2\theta = 0.76$ 45. $2 \cos 3\theta = 1.4$
 46. $\cos(2\theta + 3) = 0.6$ 47. $\sin 2\theta + 3 = 3.6$

[4–4] Compute the exact value of the following expressions in both radians and degrees.

48. $\sec^{-1}2$ 49. $\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)$
 50. $\operatorname{arccsc}(-2)$ 51. $\cot^{-1}(-\sqrt{3})$

Compute approximate values of the following expressions in both radians and degrees.

52. $\csc^{-1}4.3864$ 53. $\cot^{-1}1.5601$
 54. $\operatorname{arcsec}(-4.0420)$ 55. $\operatorname{arccsc}(-6.6917)$

Simplify the following expressions.

56. $\tan(\csc^{-1}3)$ 57. $\cot(\operatorname{arcsec} 4)$
 58. $\sin(\operatorname{arccot} 5)$ 59. $\csc[\operatorname{arcsec}(-\frac{7}{4})]$
 60. $\sin(\cot^{-1}z), z > 0$ 61. $\cot(\csc^{-1}z), z > 0$
 62. $\tan[\sec^{-1}(z+1)], z+1 > 0$
 63. $\csc[\sec^{-1}(1-z)], 1-z > 0$

Chapter 4 test

- Sketch the graph of the inverse tangent function.
- State the domain and range of the inverse cosecant function.

Find exact values for each of the following expressions in both radians and degrees.

3. $\sec^{-1}(-2)$ 4. $\arcsin\frac{\sqrt{2}}{2}$
 5. $\csc^{-1}2$ 6. $\arctan\sqrt{3}$

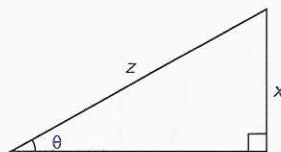
Find approximate values for the following expressions in both radians and degrees.

7. $\tan^{-1}1.4617$ 8. $\arccos(-0.4169)$
 9. $\operatorname{arcsec}(-1.4830)$ 10. $\operatorname{arccsc} 3.4971$

Simplify the following expressions.

11. $\cot(\sin^{-1}\frac{3}{5})$ 12. $\sin[\tan^{-1}(-4)]$
 13. $\tan[\sec^{-1}(-3)]$ 14. $\sec(\cos^{-1}z), z > 0$
 15. $\sin[\operatorname{arccot}(1+z)], 1+z > 0$
 16. $\tan(\sec^{-1}2z), z > 0$
 17. $\sec[\csc^{-1}(1-z)], 1-z > 0$

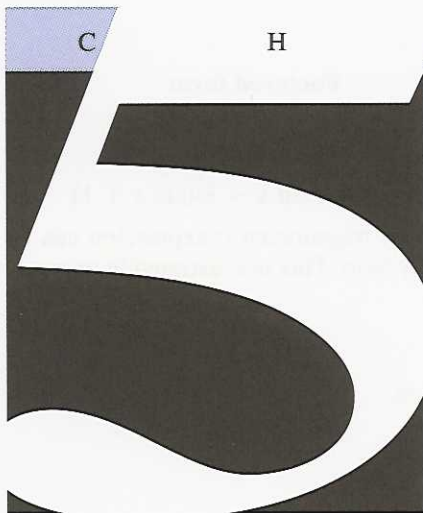
18. State the angle θ in terms of an inverse trigonometric function.



19. A taut line is connected to the top of a building that is 52 feet high and to a point on the ground some distance from the building. The length of the line is x . Let θ be the angle formed by the ground and by the line. Describe the angle θ in terms of an inverse trigonometric function.

In the following problems describe one value of θ in exact form in terms of an inverse trigonometric function.

20. $\sec \theta = 2.25$ 21. $2 \sin \theta = -1.88$
 22. $\sin 3\theta = 0.75$ 23. $4 \cos(2\theta - 3) = 2.4$



Trigonometric Equations

Recall that trigonometric equations were introduced in section 1–4, and revisited in 2–6. In this chapter, we first study trigonometric identities; these are very important in the study of the calculus and certain engineering applications. We then examine conditional trigonometric equations in more depth than we did previously.

Section 5–0 reviews important facts about equations and equation solving. Some of this material has been covered in this text, and the rest should be known from previous mathematics courses.

5–0 Equation solving—review

Factoring

To factor means to write as a product. Several common types of factoring that we will need in this chapter to deal with equations are categorized as common factor, difference of two squares, and quadratic trinomial. It is assumed that the student is familiar with each topic. They are presented here both with familiar algebraic forms and with similar trigonometric forms.

	Expression	Factored form
Common factor		
Algebraic	$a^2b - ab^2$	$ab(a - b)$
Trigonometric	$\sin^2\theta \cos \theta - \sin \theta \cos^2\theta$	$\sin \theta \cos \theta (\sin \theta - \cos \theta)$
Difference of two squares		
Algebraic	$a^4 - b^4$	$(a^2 - b^2)(a^2 + b^2)$ or $(a - b)(a + b)(a^2 + b^2)$
Trigonometric	$\sin^4\theta - \cos^4\theta$	$(\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)$ which becomes $(\sin \theta - \cos \theta)(\sin \theta + \cos \theta)(1)$

	Expression	Factored form
Quadratic trinomial		
Algebraic	$2x^2 - x - 3$	$(2x - 3)(x + 1)$
Trigonometric	$2 \sin^2 x - \sin x - 3$	$(2 \sin x - 3)(\sin x + 1)$

When it is difficult to see how a particular trigonometric expression can be factored, a method called substitution may help. This is illustrated in example 5-0 A.

■ Example 5-0 A

Factor $6 \tan^2 \theta - 11 \tan \theta + 3$

If this is difficult to factor, try substitution.

Let $u = \tan \theta$. Then $u^2 = \tan^2 \theta$.

Thus, we can rewrite the equation as

$$6u^2 - 11u + 3$$

which factors into

$$(2u - 3)(3u - 1) \quad \text{Quadratic trinomial}$$

Now replace u by $\tan \theta$:

$$(2 \tan \theta - 3)(3 \tan \theta - 1)$$

Equations

An equation is a statement of equality of two expressions.

Examples of equations are

$$3x = 27$$

$$3x^2 = 27$$

$$\tan \theta = 1$$

$$x + 3x = 4x$$

An equation involving only one variable is called an equation in one variable.

A **solution** to an equation in one variable is a real number that makes the statement of equality true when it replaces the variable.

For example, 9 is a solution to the equation $3x = 27$,

3 and -3 are both solutions to the equation $3x^2 = 27$

$\frac{\pi}{4}$ is a solution to the equation $\tan \theta = 1$

any number is a solution to the equation $x + 3x = 4x$

Identities and conditional equations

An **identity** is an equation that is true for all valid replacement values of the variable.

A **conditional equation** is an equation that is not true for all valid replacement values of the variable.

The equation

$$3x - 2(x + 5) = x - 10$$

is an identity, which can be seen by combining the terms in the left member, obtaining

$$x - 10 = x - 10$$

The left side will clearly equal the right side regardless of the value of x . We have seen that

$$\sin^2\theta + \cos^2\theta = 1$$

is an identity.

Most of this chapter is concerned with trigonometric identities.

Conditional linear equations

We have solved many linear trigonometric equations in previous sections, like the following.

1. $6 \sin x = 3$

$$\sin x = \frac{1}{2}$$

$$x = \sin^{-1}\frac{1}{2}$$

2. $3 \tan \theta - 1 = 1$

$$3 \tan \theta = 2$$

$$\tan \theta = \frac{2}{3}$$

$$\theta = \tan^{-1}\frac{2}{3}$$

Conditional quadratic equations

We have not previously discussed this type of equation. A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. There are two common methods for solving these equations: factoring and the quadratic formula.

Factoring uses the zero product property introduced in section 2-6:

$$\text{If } ab = 0, \text{ then } a = 0 \text{ or } b = 0.$$

The quadratic formula is as follows.

Quadratic formulaIf $ax^2 + bx + c = 0$, and $a \neq 0$, then

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

are both solutions to the equation.

The formula is usually abbreviated as $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Both methods for solving quadratic equations are illustrated in example 5-0 B.

Example 5-0 BSolve each quadratic equation for $\sin x$, $\cos x$, or $\tan x$, as appropriate.

1. $6 \tan^2 x - 11 \tan x = -3$

$$6 \tan^2 x - 11 \tan x + 3 = 0$$

$$(2 \tan x - 3)(3 \tan x - 1) = 0$$

$$2 \tan x - 3 = 0 \text{ or } 3 \tan x - 1 = 0$$

$$2 \tan x = 3 \text{ or } 3 \tan x = 1$$

$$\tan x = \frac{3}{2} \text{ or } \tan x = \frac{1}{3}$$

Add -3 to both members

Factor the left member

Zero product property

2. $2 \cos^2 x - \cos x - 2 = 0$

This expression does not factor, so we use the quadratic formula.

$$a = 2, b = -1, c = -2.$$

$$\cos x = \frac{-(-1) + \sqrt{(-1)^2 - 4(2)(-2)}}{2(2)} \quad \text{or} \quad \frac{-(-1) - \sqrt{(-1)^2 - 4(2)(-2)}}{2(2)}$$

$$\cos x = \frac{1 + \sqrt{17}}{4} \quad \text{or} \quad \frac{1 - \sqrt{17}}{4}$$

Equivalent equations using substitution

Consider the equation

$$(2x - 3)^2 - 3(2x - 3) - 10 = 0$$

If we replace $2x - 3$ by, say u , then we obtain the equation

$$u^2 - 3u - 10 = 0$$

which we solve as

$$(u - 5)(u + 2) = 0$$

$$u = -2 \text{ or } 5$$

We now replace u by $2x - 3$ to obtain

$$2x - 3 = -2 \text{ or } 2x - 3 = 5$$

$$2x = 1 \text{ or } 2x = 8$$

$$x = \frac{1}{2} \text{ or } x = 4$$

Anytime we can replace an expression by a variable like u we obtain an equivalent equation that may be easier to solve or otherwise manipulate.

■ Example 5-0 C

Use substitution to help solve each problem.

1. Solve the equation $2(5x + 3)^3 - (5x + 3)^2 = 0$.

Let $u = 5x + 3$. Then,

$$2u^3 - u^2 = 0$$

$$u^2(2u - 1) = 0$$

$$u^2 = 0 \text{ or } 2u - 1 = 0$$

$$u = 0 \text{ or } u = \frac{1}{2}$$

$$5x + 3 = 0 \text{ or } 5x + 3 = \frac{1}{2}$$

$$5x = -3 \text{ or } 5x = -\frac{5}{2}$$

$$x = -\frac{3}{5} \text{ or } x = \frac{1}{5}\left(-\frac{5}{2}\right) = -\frac{1}{2}$$

$$\frac{1}{2} - 3 = \frac{1}{2} - \frac{6}{2}$$

2. Simplify $\frac{\left(3\theta - \frac{\pi}{2}\right)^2 - 1}{1 - \left(3\theta - \frac{\pi}{2}\right)}$.

Let $u = 3\theta - \frac{\pi}{2}$. Then the expression is

$$\frac{u^2 - 1}{1 - u} = \frac{(u - 1)(u + 1)}{-(u - 1)} = \frac{u + 1}{-1} = -(u + 1) = -u - 1$$

Replace u by $3\theta - \frac{\pi}{2}$:

$$-u - 1 = -\left(3\theta - \frac{\pi}{2}\right) - 1 = -3\theta + \frac{\pi}{2} - 1$$

Mastery points

Can you

- Factor trigonometric expressions?
- Solve quadratic equations for x , $\sin x$, $\cos x$, or $\tan x$?
- Use substitution to help solve equations and simplify expressions?

Exercise 5-0

Note: The solutions to all of these exercises are given in appendix E. This section is not reviewed explicitly in the chapter review or the chapter test. It is designed to prepare for the rest of the chapter.

Factor each trigonometric expression.

1. $\sin^2\theta - \sin\theta$

4. $\sin^5\theta - \sin^3\theta$

7. $2\sin^2x - 7\sin x + 3$

2. $\cos^3\theta + 3\cos\theta$

5. $\cos^2x + \cos x - 20$

8. $9\cos^3\theta - 15\cos^2\theta - 6\cos\theta$

3. $\cos^4\theta - \cos^2\theta$

6. $\tan^2x + 2\tan x - 24$

9. $6\csc^2\theta - 5\csc\theta + 1$

Solve each quadratic equation for $\sin \theta$, $\cos \theta$, $\tan \theta$, etc., as appropriate.

10. $\tan^2 \theta - \tan \theta = 0$

11. $\tan^2 \theta - \tan \theta = 2$

12. $6 \sin^2 \theta + 5 \sin \theta + 1 = 0$

13. $\sec^4 \theta - 5 \sec^2 \theta + 4 = 0$

14. $36 \sin^4 \theta - 13 \sin^2 \theta + 1 = 0$

15. $\frac{2}{3} \sin \theta + \frac{1}{3 \sin \theta} = 1$

16. $\sin^2 \theta - 3 \sin \theta - 5 = 0$

17. $2 \sec^2 \theta + 3 \sec \theta - 7 = 0$

Use substitution to help solve each equation.

18. $(2x - 6)^3 - (2x - 6)^2 = 0$

19. $12\left(\frac{x}{3} - 1\right)^2 - 5\left(\frac{x}{3} - 1\right) - 2 = 0$

Use substitution to help simplify each expression.

20. $5\left(\frac{\pi}{2} - 3\right) + 3\left[\left(\frac{\pi}{2} - 3\right) - 7\right] - \frac{\pi}{2} + 3$

21. $\frac{2(3x - \frac{1}{2})^2 - 3(3x - \frac{1}{2}) + 1}{(3x - \frac{1}{2})^2 - 1}$

5-1 Basic trigonometric identities

Review of some identities

Recall from section 1-4 and 5-0 that an identity is an equation that is true for every allowed value of its variable (or variables). We have seen the following identities in previous sections.

Reciprocal identities

$$\csc \theta = \frac{1}{\sin \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad \cot \theta = \frac{1}{\tan \theta}$$

$$\sin \theta = \frac{1}{\csc \theta}, \quad \cos \theta = \frac{1}{\sec \theta}, \quad \tan \theta = \frac{1}{\cot \theta}$$

Tangent/cotangent identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

Fundamental identity of trigonometry

$$\sin^2 \theta + \cos^2 \theta = 1$$

Two other forms of the fundamental identity are $\sin^2 \theta = 1 - \cos^2 \theta$ and $\cos^2 \theta = 1 - \sin^2 \theta$.

If each term in the fundamental identity is divided by $\cos^2 \theta$ we obtain

$$\begin{aligned} \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} &= \frac{1}{\cos^2 \theta} \\ \left(\frac{\sin \theta}{\cos \theta}\right)^2 + 1 &= \left(\frac{1}{\cos \theta}\right)^2 \\ \tan^2 \theta + 1 &= \sec^2 \theta \end{aligned}$$

Similarly, if each term of the fundamental identity is divided by $\sin^2\theta$ we obtain the identity $\cot^2\theta + 1 = \csc^2\theta$. These two identities, along with the fundamental identity, are called the Pythagorean identities. They are summarized here.

Pythagorean identities

Useful forms

$\sin^2\theta + \cos^2\theta = 1$	$\sin^2\theta = 1 - \cos^2\theta$	$\cos^2\theta = 1 - \sin^2\theta$
$\tan^2\theta + 1 = \sec^2\theta$	$\tan^2\theta = \sec^2\theta - 1$	$\sec^2\theta - \tan^2\theta = 1$
$\cot^2\theta + 1 = \csc^2\theta$	$\cot^2\theta = \csc^2\theta - 1$	$\csc^2\theta - \cot^2\theta = 1$

Preliminary notes on algebra

When working with trigonometric equations there are certain algebraic principles that are used over and over. It is a good idea to get used to these principles, and the associated notation, now—it will make the rest of this chapter much easier.

These principles are illustrated in example 5-1 A. We also use the principle that we *do not leave fractions in a final answer when possible*. For example, $\frac{1}{\cot \theta}$ can be rewritten as $\tan \theta$. Also we note that a binomial of the form $x + y$ is called the **conjugate** of the binomial $x - y$, and vice versa.

■ Example 5-1 A

Perform the algebra indicated, and note the category of algebraic manipulation for later reference. Do not leave a fraction for an answer when possible.

1. *Separating fractions:* $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$

Rewrite $\frac{1 - \sin \theta}{\sin \theta}$ as two fractions.

$$\begin{aligned}\frac{1 - \sin \theta}{\sin \theta} &= \frac{1}{\sin \theta} - \frac{\sin \theta}{\sin \theta} \\ &= \csc \theta - 1\end{aligned}$$

2. *Multiplying binomial conjugates:* $(a + b)(a - b) = a^2 - b^2$
Multiply $(1 - \sin \theta)(1 + \sin \theta)$.

$$\begin{aligned}(1 - \sin \theta)(1 + \sin \theta) &= 1^2 - (\sin \theta)^2 \\ &= 1 - \sin^2\theta \\ &= \cos^2\theta \quad \text{Pythagorean identity}\end{aligned}$$

3. *Factoring quadratic binomials into conjugates:* $a^2 - b^2 = (a + b)(a - b)$

Factor the numerator of $\frac{\csc^2\theta - \cot^2\theta}{\csc \theta - \cot \theta}$.

$$\begin{aligned}\frac{\csc^2\theta - \cot^2\theta}{\csc \theta - \cot \theta} &= \frac{(\csc \theta - \cot \theta)(\csc \theta + \cot \theta)}{\csc \theta - \cot \theta} \\ &= \csc \theta + \cot \theta\end{aligned}$$

4. *Multiplying numerator and denominator of a fraction by a conjugate of the numerator or denominator:*

Multiply the numerator and denominator of $\frac{1 - \cos \theta}{\sin \theta}$ by the conjugate of the numerator.

$$\begin{aligned}\frac{1 - \cos \theta}{\sin \theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} &= \frac{1 - \cos^2 \theta}{\sin \theta(1 + \cos \theta)} \\ &= \frac{\sin^2 \theta}{\sin \theta(1 + \cos \theta)} \\ &= \frac{\sin \theta}{1 + \cos \theta}\end{aligned}$$

5. *Factoring the sign from a binomial: $-(a - b) = b - a$*

Simplify $\frac{\cos^2 \theta - 1}{\sin \theta}$.

$$\begin{aligned}\frac{\cos^2 \theta - 1}{\sin \theta} &= \frac{-(1 - \cos^2 \theta)}{\sin \theta} \\ &= \frac{-\sin^2 \theta}{\sin \theta} \\ &= -\sin \theta\end{aligned}$$

Transforming expressions

Identities and the principles illustrated above aid in simplifying and transforming trigonometric expressions. We proceed by replacing given parts of an expression by equivalent parts from the identities summarized above, as well as any other identities we have studied. *There is no single correct sequence of steps!* We proceed by trial and error, guided by past experience.

Although there are many ways to proceed in transforming an expression, we will note some guidelines for this process.

1. When functions appear raised to the second power, such as $\sin^2 \theta$, $\tan^2 \theta$, etc., look for expressions that appear in the Pythagorean identities.

- a. It may be possible to combine two terms into one.

example: $\frac{\sec^2 \theta - \tan^2 \theta}{\sin \theta}$ becomes $\frac{1}{\sin \theta}$, which becomes $\csc \theta$.

- b. It is always possible to rewrite one second degree term as two.

example: $\frac{\sec^2 \theta}{\tan^2 \theta + 1}$ becomes $\frac{\tan^2 \theta + 1}{\tan^2 \theta + 1}$, which becomes 1.

2. Sometimes it pays to rewrite all functions in terms of the sine and cosine functions.

Rewrite $\sec \theta$ as $\frac{1}{\cos \theta}$, $\csc \theta$ as $\frac{1}{\sin \theta}$, $\tan \theta$ as $\frac{\sin \theta}{\cos \theta}$, $\cot \theta$ as $\frac{\cos \theta}{\sin \theta}$.

3. Look for factors of the Pythagorean identities. These are things like $1 - \cos \theta$, which is a factor of $1 - \cos^2 \theta$, and $\sec \theta + \tan \theta$, which is a factor of $\sec^2 \theta - \tan^2 \theta$. These can often be transformed as in part 4 of example 5-1 A.

Example 5-1 B illustrates these guidelines.

■ Example 5-1 B

Simplify each expression into one term.

1. $1 - \cos^2 4\alpha$
 $\sin^2 4\alpha$

$$\sin^2 4\alpha + \cos^2 4\alpha = 1, \text{ so } 1 - \cos^2 4\alpha = \sin^2 4\alpha$$

2. $(1 - \sec x)(1 + \sec x)$
 $1 - \sec^2 x$
 $-(\sec^2 x - 1)$
 $-\tan^2 x$

$$(a - b)(a + b) = a^2 - b^2$$

Simplify the expression into as few factors as possible.

3. $\cot \theta \sec \theta \sin \theta$

$$\begin{aligned} & \cot \theta \sec \theta \sin \theta \\ & \frac{\cos \theta}{\sin \theta} \cdot \frac{1}{\cos \theta} \cdot \sin \theta \\ & \frac{\cancel{\cos \theta}}{\cancel{\sin \theta}} \cdot \frac{1}{\cancel{\cos \theta}} \cdot \cancel{\sin \theta} \\ & 1 \end{aligned}$$

Rewrite everything in terms of $\sin \theta$, $\cos \theta$

Divide out the common factors

Verifying identities

If we can show that one member of an equation can be transformed into the other member by replacing expressions using identities and performing algebraic transformations, then we say we have *verified* the equation to be an identity.

Although there are many ways to proceed in verifying an identity, there are some guidelines for this process. The guidelines 1 through 3 apply to verifying identities as well as simplifying individual expressions. Example 5-1 C illustrates another guideline, which applies to verifying identities.

4. Begin with the more complicated member of an equation, and try to simplify it.

A fraction is almost always considered more complicated than a nonfraction.

■ Example 5-1 C

Verify that each equation is an identity by showing that one member of the identity can be transformed into the other member.

1. $\tan \theta \csc \theta = \sec \theta$

$$\frac{\tan \theta \csc \theta}{\frac{\sin \theta}{\cos \theta} \cdot \frac{1}{\sin \theta}}$$

Begin with the left member

Rewrite everything in terms of $\sin \theta$ and $\cos \theta$

$$\frac{1}{\cos \theta}$$

Reduce by a factor of $\sin \theta$

$$\sec \theta$$

$$\sec \theta = \frac{1}{\cos \theta}$$

2. $\frac{1}{\sin \beta - \csc \beta} = -\tan \beta \sec \beta$

$$\frac{1}{\sin \beta - \csc \beta}$$

Begin with the left member; it is more complicated

$$\frac{1}{\sin \beta - \frac{1}{\sin \beta}}$$

Rewrite everything in terms of $\sin \theta$ and $\cos \theta$; we arrive at a complex fraction

$$\frac{1}{\sin \beta - \frac{1}{\sin \beta}} \cdot \frac{\sin \beta}{\sin \beta}$$

Multiply numerator and denominator by $\sin \beta$; this is to simplify the complex fraction

$$\frac{\sin \beta}{\sin^2 \beta - 1}$$

$$\sin \beta \left(\sin \beta - \frac{1}{\sin \beta} \right) = \sin^2 \beta - 1$$

$$\frac{\sin \beta}{-(1 - \sin^2 \beta)}$$

Factor the sign from the binomial denominator

$$\frac{\sin \beta}{-\cos^2 \beta}$$

$$-\frac{\sin \beta}{\cos \beta} \cdot \frac{1}{\cos \beta}$$

$$-\tan \beta \sec \beta$$

3. $\frac{\cos^2 \alpha}{1 + \sin \alpha} = 1 - \sin \alpha$

$$\frac{\cos^2 \alpha}{1 + \sin \alpha}$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

$$\frac{1 - \sin^2 \alpha}{1 + \sin \alpha}$$

$$\frac{(1 - \sin \alpha)(1 + \sin \alpha)}{1 + \sin \alpha}$$

$$m^2 - n^2 = (m - n)(m + n)$$

$$1 - \sin \alpha$$



It is not absolutely necessary to transform just one side or the other of an identity. When both members of an identity are very complicated it is easier to transform both members, as in example 5-1 D.

■ Example 5-1 D

Verify the identity.

$$\sin^2\theta \tan^2\theta + 1 = \sec^2\theta - \cos^2\theta \sec^2\theta + \cos^2\theta$$

Left member

$$\begin{aligned} &\sin^2\theta \tan^2\theta + 1 \\ &\sin^2\theta \tan^2\theta + \sin^2\theta + \cos^2\theta \\ &\sin^2\theta(\tan^2\theta + 1) + \cos^2\theta \\ &\sin^2\theta \sec^2\theta + \cos^2\theta \\ &\sin^2\theta \frac{1}{\cos^2\theta} + \cos^2\theta \\ &\tan^2\theta + \cos^2\theta \end{aligned}$$

Right member

$$\begin{aligned} &\sec^2\theta - \cos^2\theta \sec^2\theta + \cos^2\theta \\ &\sec^2\theta(1 - \cos^2\theta) + \cos^2\theta \\ &\sec^2\theta \sin^2\theta + \cos^2\theta \\ &\frac{1}{\cos^2\theta} \sin^2\theta + \cos^2\theta \\ &\tan^2\theta + \cos^2\theta \end{aligned}$$

Since the left side and right side can be transformed into the same expression, they are equivalent. This is true because we could actually take the steps on one side and add them to the other side in reverse order, arriving at the required result.

Of course most equations are not identities. To show that an equation is not an identity we need to find a value for the variable for which each member of the equation is defined, but that produces different results in the two members. This value is called a **counter example**; it shows that the equation is *not* an identity.

■ Example 5-1 E

Show by counter example that $\cos x + \sin x \cot x = \sin x$ is not an identity.

Choose a value for which both the left and right members are defined;

$x = \frac{\pi}{4}$ is such a value (most values would serve the purpose).

Left member

$$\begin{aligned} &\cos x + \sin x \cot x \\ &\cos \frac{\pi}{4} + \sin \frac{\pi}{4} \cot \frac{\pi}{4} \\ &\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot 1 \\ &\frac{2\sqrt{2}}{2} \\ &\sqrt{2} \end{aligned}$$

Right member

$$\begin{aligned} &\sin x \\ &\sin \frac{\pi}{4} \\ &\frac{\sqrt{2}}{2} \end{aligned}$$

Replace x by $\frac{\pi}{4}$

$$\cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}; \cot \frac{\pi}{4} = 1$$

Since $\sqrt{2} \neq \frac{\sqrt{2}}{2}$ we have shown that the given equation is not an identity. ■

As noted in example 5-1 E, most values will serve as a counter example for an equation that is not an identity. However, *avoid using zero*, since there are many equations that are not identities for which both sides evaluate to the same value when zero is used.

An example is $\sin \theta = 1 - \cos \theta$. This equation is *not* an identity, but observe that replacing θ by 0 produces $0 = 0$, which might lead one to believe that this equation is an identity.

Mastery points

Can you

- State the reciprocal identities, fundamental identity, and the remaining Pythagorean identities from memory?
- Recognize useful forms of the Pythagorean identities?
- Transform forms of the Pythagorean identities into simpler forms?
- Transform one side of an identity into the other side?
- Show that an equation is not an identity by a counter example?

Exercise 5-1

Each of the following expressions can be simplified into the form 1, -1 , $\sin \theta$, $\cos \theta$, $\tan \theta$, $\cot \theta$, $\sec \theta$, $\csc \theta$, $\sin^2 \theta$, $\cos^2 \theta$, $\tan^2 \theta$, $\cot^2 \theta$, $\sec^2 \theta$, or $\csc^2 \theta$. Show the transformation of each expression into one of these forms.

- $\frac{\sin \theta}{\tan \theta}$
- $\frac{\cos \theta}{\cot \theta}$
- $\tan \theta \csc \theta$
- $\sec \theta \cot \theta$
- $\cot^2 \theta \sin^2 \theta$
- $\sin^2 \theta \sec^2 \theta$
- $(\tan^2 \theta + 1)(1 - \sin^2 \theta)$
- $(1 - \cos^2 \theta)(1 + \cot^2 \theta)$
- $\frac{(\sec \theta + \tan \theta)(\sec \theta - \tan \theta)}{\cos^2 \theta}$
- $\frac{\csc \theta \sin \theta}{\cot \theta}$
- $\frac{\tan \theta \cot \theta}{\sin \theta}$
- $\cos \theta (\sec \theta - \cos \theta)$
- $\cos^2 \theta (1 + \cot^2 \theta)$
- $\csc^2 \theta (1 - \cos^2 \theta)$
- $\sin^2 \theta (\csc^2 \theta - 1)$
- $\cot^2 \theta - \csc^2 \theta$
- $\tan^2 \theta - \sec^2 \theta$
- $\tan^2 \theta (\cot^2 \theta + 1)$
- $\frac{\cot x}{\sec x}$
- $\sec^2 \theta (\csc^2 \theta - 1)$
- $\frac{\sec \theta}{\tan \theta \csc \theta}$
- $\frac{\cot \theta \sec \theta}{\csc \theta}$
- $\frac{\csc^2 \theta - 1}{\csc^2 \theta}$
- $\sin x + \cos x \cot x$
- $\cos x + \sin x \tan x$
- $\tan x \csc x \cos x$
- $\frac{\csc x + \sec x}{\tan x + 1}$
- $\sec x - \tan x \sin x$
- $(\csc x + 1)(\sec x - \tan x)$
- $(\csc x + \cot x)(1 - \cos x)$
- $\frac{\sec^4 y - \tan^4 y}{\sec^2 y + \tan^2 y}$

Verify the following identities.

$$33. \csc \theta + \cot \theta = \frac{1 + \cos \theta}{\sin \theta}$$

$$36. \frac{\sec \theta}{\csc \theta + \cot \theta} = \frac{\tan \theta}{1 + \cos \theta}$$

$$39. \frac{\tan^2 \theta + \sec^2 \theta}{\sec^2 \theta} = \sin^2 \theta + 1$$

$$42. \cot x \cos x = \csc x - \sin x$$

$$45. \frac{1}{\sec \theta - \cos \theta} = \cot \theta \csc \theta$$

$$48. \frac{\cot \theta}{\sec \theta} = \csc \theta - \sin \theta$$

$$51. \frac{\cot x + 1}{\cot x - 1} = \frac{\sin x + \cos x}{\cos x - \sin x}$$

$$54. \frac{1 + \sin y}{\cos y} = \frac{\cos y}{1 - \sin y}$$

$$57. \frac{1}{1 - \sin x} + \frac{1}{1 + \sin x} = 2 \sec^2 x$$

$$59. \sin^4 x - \cos^4 x = 1 - 2 \cos^2 x$$

$$61. \cot^2 x - \cos^2 x = \cot^2 x \cos^2 x$$

$$63. \frac{\tan y - \cot y}{\tan y + \cot y} = \frac{\tan^2 y - 1}{\sec^2 y}$$

$$65. \frac{1 - \sin x}{1 + \sin x} = (\tan x - \sec x)^2$$

$$67. \sec^4 x - \sec^2 x = \tan^4 x + \tan^2 x$$

$$69. 2 \cos^2 y - 1 = \cos^2 y - \sin^2 y$$

$$34. \tan \theta + \sec \theta = \frac{1 + \sin \theta}{\cos \theta}$$

$$37. \frac{1 + \csc \theta}{1 + \sec \theta} = \cot \theta \left(\frac{1 + \sin \theta}{1 + \cos \theta} \right)$$

$$40. \frac{\cot^2 \theta + \csc^2 \theta}{\csc^2 \theta} = 1 + \cos^2 \theta$$

$$43. \frac{1 + \cot^2 \theta}{\tan^2 \theta} = \cot^2 \theta \csc^2 \theta$$

$$46. \frac{1}{\cot \theta + \tan \theta} = \sin \theta \cos \theta$$

$$49. \frac{\tan^2 \theta}{\sec \theta - 1} = 1 + \sec \theta$$

$$52. \frac{1 + \sin y}{1 - \sin y} = \frac{\csc y + 1}{\csc y - 1}$$

$$55. \frac{\cos x}{\sec x - \tan x} = \frac{\cos^2 x}{1 - \sin x}$$

$$35. \frac{\csc \theta}{\sec \theta + \tan \theta} = \frac{\cos \theta}{\sin \theta + \sin^2 \theta}$$

$$38. \frac{\tan \theta + 1}{\tan \theta - 1} = \frac{\sin \theta + \cos \theta}{\sin \theta - \cos \theta}$$

$$41. \tan y \sin y = \sec y - \cos y$$

$$44. \frac{\cot^2 \theta}{\csc \theta + 1} = 1 - \csc \theta$$

$$47. \frac{\tan \theta}{\csc \theta} = \sin \theta \tan \theta$$

$$50. \frac{1}{\sec x - \tan x} = \sec x + \tan x$$

$$53. \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$$

$$56. \frac{\cos x}{\cos x + \sin x} = \frac{\cot x}{1 + \cot x}$$

$$58. \frac{\cos y}{\csc y + 1} + \frac{\cos y}{\csc y - 1} = 2 \tan y$$

$$60. (\tan^2 y + 1)(\cot^2 y + 1) = \sec^2 y + \csc^2 y$$

$$62. \csc^2 y + \sec^2 y = \sec^2 y \csc^2 y$$

$$64. \frac{\cot^2 x - 1}{\cot^2 x + 1} = 1 - 2 \sin^2 x$$

$$66. \sec^4 x - 1 = \tan^2 x \sec^2 x + \tan^2 x$$

$$68. \tan x - \cot x = \frac{\sin^2 x - \cos^2 x}{\sin x \cos x}$$

$$70. \cos^4 x - \sin^4 x = 1 - 2 \sin^2 x$$

In problems 71–80 show by counter example that each equation is not an identity.

$$71. \sin \theta = 1 - \cos \theta$$

$$72. \tan^2 \theta - \cot^2 \theta = 1$$

$$73. \sec \theta = \frac{1}{\csc \theta}$$

$$74. \sin \theta = \frac{1}{\cos \theta}$$

$$75. \sin^2 \theta - 2 \cos \theta \sin \theta + \cos^2 \theta = 2$$

$$77. \csc \theta + \sec \theta \cot \theta = 2$$

$$76. \tan^2 \theta - \tan \theta = 0$$

$$78. \sin \theta + 2 \sin \theta \cos \theta = 0$$

$$79. \frac{1 - \cos \theta}{1 + \cos \theta} = \sin^2 \theta$$

$$80. \frac{1}{\tan \theta + \csc \theta} = \sec \theta$$

81. Verify by calculation that $(1 - \csc^2 \theta)(1 - \sec^2 \theta) = 1$ for the values

a. $\theta = \frac{\pi}{6}$ b. $\theta = \frac{\pi}{4}$ c. Is this equation an identity?

82. Verify by calculation that $\frac{\sin \theta - \cos \theta}{\cos \theta} = \tan \theta - 1$ for the values

a. $\theta = \frac{\pi}{3}$ b. $\theta = \frac{3\pi}{4}$ c. Is this equation an identity?

83. Verify by calculation that $2 \sin^2 \theta + \sin \theta = 1$ for the values

a. $\theta = \frac{\pi}{6}$ b. $\theta = \frac{3\pi}{2}$ c. Is this equation an identity?

84. Verify by calculation that $\tan^4 \theta - \tan^2 \theta = 6$ for the values

a. $\theta = \frac{\pi}{3}$ b. $\theta = \frac{4\pi}{3}$ c. Is this equation an identity?

5-2 The sum and difference identities

Four important identities are called the sum and difference identities.

Sum and difference identities for sine and cosine

$$\begin{array}{ll} [1] & \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ [2] & \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \\ [3] & \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ [4] & \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \end{array}$$

The last three of these four identities can be developed using the first. Their verification is left as exercises. A demonstration that identity [1] is true is given in appendix B.

The sum and difference identities have several applications, illustrated in example 5-2 A.

■ Example 5-2 A

1. Use the fact that $\frac{7\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$ to find the exact value of $\cos \frac{7\pi}{12}$.

$$\begin{aligned} \cos \frac{7\pi}{12} &= \cos \left(\frac{\pi}{3} + \frac{\pi}{4} \right) \\ &= \cos \frac{\pi}{3} \cos \frac{\pi}{4} - \sin \frac{\pi}{3} \sin \frac{\pi}{4} && \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ &= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2} - \sqrt{6}}{4} \end{aligned}$$

2. Show that $\cos(\pi - \theta) = -\cos \theta$ for any angle θ .

$$\begin{aligned} \cos(\pi - \theta) &= \cos \pi \cos \theta + \sin \pi \sin \theta \\ &= (-1) \cos \theta + 0 \sin \theta \\ &= -\cos \theta \end{aligned}$$

Identity [1] can be used to prove the following identities (the proofs are left for the exercises). These identities are called the cofunction identities.

Cofunction identities

$$\begin{array}{ll} [5] & \sin \left(\frac{\pi}{2} - \theta \right) = \cos \theta & [6] & \cos \left(\frac{\pi}{2} - \theta \right) = \sin \theta \\ [7] & \tan \left(\frac{\pi}{2} - \theta \right) = \cot \theta & [8] & \cot \left(\frac{\pi}{2} - \theta \right) = \tan \theta \\ [9] & \sec \left(\frac{\pi}{2} - \theta \right) = \csc \theta & [10] & \csc \left(\frac{\pi}{2} - \theta \right) = \sec \theta \end{array}$$

The reason for the name of these identities is as follows.

When the sum of two angles is 90° , or $\frac{\pi}{2}$ radians, the angles are said to be **complementary**. The angles $\frac{\pi}{2} - \theta$ and θ add up to $\frac{\pi}{2}$, so they are complementary angles. Each is said to be the complement of the other. The cofunction identities say, in effect,

trig function (angle) = “co” trig function (complement of angle)

Thus, the sine and “co” sine appear in one identity, the tangent and “co” tangent appear in another, and the secant and “co” secant in the third. Whenever the sum of two angles is $\frac{\pi}{2}$ (or 90°), a trigonometric function of one equals the “co” trigonometric function of the other. Thus for example, the following statements are true:

$$\begin{array}{ll} \sin 50^\circ = \cos 40^\circ & 50^\circ + 40^\circ = 90^\circ \\ \sec \frac{\pi}{6} = \csc \frac{\pi}{3} & \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2} \\ \cot 130^\circ = \tan(-40^\circ) & 130^\circ + (-40^\circ) = 90^\circ \end{array}$$

■ Example 5-2 B

Rewrite each function value in terms of its cofunction.

1. $\sin 34^\circ$

$$\sin 34^\circ = \cos(90^\circ - 34^\circ) = \cos 56^\circ$$

2. $\csc \frac{2\pi}{5}$

$$\csc \frac{2\pi}{5} = \sec\left(\frac{\pi}{2} - \frac{2\pi}{5}\right) = \sec \frac{\pi}{10}$$

Simplify each expression.

3. $\frac{\sin 10^\circ}{\cos 80^\circ}$

$$\frac{\sin 10^\circ}{\cos 80^\circ} = \frac{\cos 80^\circ}{\cos 80^\circ} = 1$$

4. $\sin^2 \frac{\pi}{6} + \sin^2 \frac{\pi}{3}$

$$\sin^2 \frac{\pi}{6} + \sin^2 \frac{\pi}{3} = \sin^2 \frac{\pi}{6} + \cos^2\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{6} = 1 \quad \blacksquare$$

Two more important identities are the sum and difference formulas for the tangent function.

Sum and difference identities for tangent

$$[11] \quad \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$[12] \quad \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

The derivation of the first identity is as follows.

$$\begin{aligned} \tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta} \\ &= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta} \quad \begin{array}{l} \text{Divide numerator and} \\ \text{denominator by} \\ \cos \alpha \cos \beta \end{array} \\ &= \frac{\frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} \\ &= \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}}{1 - \frac{\sin \alpha}{\cos \alpha} \cdot \frac{\sin \beta}{\cos \beta}} = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \end{aligned}$$

Example 5-2 C illustrates using this identity.

■ **Example 5-2 C**

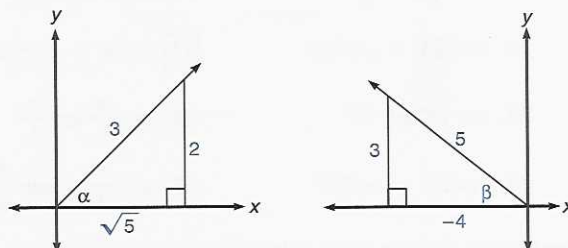
Use the fact that $15^\circ = 45^\circ - 30^\circ$ to find the exact value of $\tan 15^\circ$.

$$\begin{aligned} \tan 15^\circ &= \tan(45^\circ - 30^\circ) \\ &= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} = \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1\left(\frac{\sqrt{3}}{3}\right)} \\ &= \frac{1 - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}} \cdot \frac{3}{3} = \frac{3 - \sqrt{3}}{3 + \sqrt{3}} \cdot \frac{3 - \sqrt{3}}{3 - \sqrt{3}} \\ &= \frac{9 - 6\sqrt{3} + 3}{9 - 3} = \frac{12 - 6\sqrt{3}}{6} = 2 - \sqrt{3} \end{aligned}$$

Some problems can be solved by using the identities above and reference triangles (section 2-4).

■ **Example 5-2 D**

$\sin \alpha = \frac{2}{3}$, α in quadrant I; $\cos \beta = -\frac{4}{5}$, β in quadrant II. Find the exact value of $\cos(\alpha - \beta)$.



$$\cos \alpha = \frac{\sqrt{5}}{3}, \sin \beta = \frac{3}{5} \quad \text{We find the necessary values from the reference triangles}$$

$$\begin{aligned} \cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\ &= \frac{\sqrt{5}}{3} \cdot \left(-\frac{4}{5}\right) + \frac{2}{3} \cdot \frac{3}{5} \\ &= -\frac{4\sqrt{5}}{15} + \frac{6}{15} \\ &= \frac{-4\sqrt{5} + 6}{15} \end{aligned}$$

Mastery points

Can you

- State the sum and difference identities?
- State and apply the cofunction identities?
- Apply the sum and difference identities to find exact values of sine, cosine, and tangent for certain angles?
- Apply the sum and difference identities to find exact values of sine, cosine, and tangent for $\alpha + \beta$ and $\alpha - \beta$ given information about α and β ?
- Verify identities using the sum and difference identities?

Exercise 5-2

Rewrite each function in terms of its cofunction.

1. $\sin 18^\circ$

2. $\cos 42^\circ$

3. $\tan 8^\circ$

4. $\csc 100^\circ$

5. $\sec \frac{\pi}{3}$

6. $\cot \frac{\pi}{6}$

7. $\cos \frac{5\pi}{6}$

8. $\sin\left(-\frac{\pi}{3}\right)$

9. $\sec\left(-\frac{3\pi}{4}\right)$

10. $\csc\left(-\frac{\pi}{4}\right)$

Simplify each expression.

11. $\frac{\cos 65^\circ}{\sin 25^\circ}$

12. $\tan \frac{\pi}{3} \tan \frac{\pi}{6}$

13. $\cos 20^\circ \csc 70^\circ$

14. $\frac{\sin^2 5^\circ}{\cos^2 85^\circ}$

15. $\sin \frac{\pi}{5} \sec \frac{3\pi}{10}$

16. $\cos^2 25^\circ + \cos^2 65^\circ$

17. $\tan^2 8^\circ - \csc^2 82^\circ$

18. $\frac{\cos^2 30^\circ}{1 - \cos^2 60^\circ}$

19. $\tan 40^\circ \tan 50^\circ$

20. $\tan 19^\circ \tan 71^\circ$

21. $\sec \frac{\pi}{6} \sin \frac{\pi}{3}$

22. $\cot \frac{\pi}{5} \cot \frac{3\pi}{10}$

23. $\sin^2 10^\circ + \sin^2 80^\circ$

24. $\tan^2 25^\circ - \csc^2 65^\circ$

25. $\sec^2 \frac{\pi}{3} - \cot^2 \frac{\pi}{6}$

26. $\cos^2 \frac{3\pi}{8} + \cos^2 \frac{\pi}{8}$

Use the sum and difference identities to find the exact value of each of the following. Observe that each value is the sum or difference of values chosen from $\frac{\pi}{6}$ (30°), $\frac{\pi}{4}$ (45°), and $\frac{\pi}{3}$ (60°).

27. $\cos \frac{\pi}{12}$

28. $\tan \frac{\pi}{12}$

29. $\sin \frac{5\pi}{12}$

30. $\cos \frac{5\pi}{12}$

31. $\sin \frac{7\pi}{12}$

32. $\tan \frac{7\pi}{12}$

33. $\sin 15^\circ$

34. $\tan 15^\circ$

35. $\cos 105^\circ$

36. $\sin 105^\circ$

37. $\tan 75^\circ$

38. $\cos 15^\circ$

Each of the following problems presents information about two angles, α and β , including the quadrant in which the angle terminates. Use the information to find the required value.

39. $\cos \alpha = \frac{1}{3}$, quadrant I; $\sin \beta = \frac{3}{4}$, quadrant I. Find $\sin(\alpha + \beta)$.

41. $\sin \alpha = \frac{5}{13}$, quadrant II; $\cos \beta = -\frac{3}{4}$, quadrant III. Find $\tan(\alpha - \beta)$.

43. $\sin \alpha = -\frac{4}{5}$, quadrant IV; $\cos \beta = \frac{15}{17}$, quadrant IV. Find $\cos(\alpha + \beta)$.

45. $\sin \alpha = \frac{2}{3}$, quadrant I; $\cos \beta = -\frac{1}{3}$, quadrant III. Find $\cos(\alpha - \beta)$.

47. $\sin \alpha = \frac{2}{3}$, quadrant I; $\tan \beta = \frac{1}{4}$, quadrant I. Find $\sin(\alpha + \beta)$.

49. $\cos \alpha = \frac{5}{13}$, quadrant IV; $\tan \beta = -\frac{5}{12}$, quadrant IV. Find $\tan(\alpha - \beta)$.

51. $\tan \alpha = 2$, quadrant III; $\cos \beta = -\frac{3}{5}$, quadrant II. Find $\cos(\alpha - \beta)$.

53. $\sin \alpha = \frac{2}{3}$, quadrant I; $\sin \beta = -\frac{1}{5}$, quadrant III. Find $\sin(\alpha - \beta)$.

55. $\cos \alpha = -\frac{\sqrt{2}}{2}$, quadrant II; $\sin \beta = \frac{\sqrt{3}}{2}$, quadrant II. Find $\tan(\alpha + \beta)$.

40. $\cos \alpha = -\frac{12}{13}$, quadrant II; $\sin \beta = \frac{1}{2}$, quadrant II. Find $\cos(\alpha - \beta)$.

42. $\sin \alpha = -\frac{4}{5}$, quadrant IV; $\sin \beta = -\frac{1}{5}$, quadrant IV. Find $\sin(\alpha + \beta)$.

44. $\cos \alpha = -\frac{3}{5}$, quadrant II; $\sin \beta = -\frac{8}{17}$, quadrant III. Find $\sin(\alpha - \beta)$.

46. $\cos \alpha = \frac{\sqrt{2}}{2}$, quadrant IV; $\sin \beta = -\frac{\sqrt{3}}{2}$, quadrant III. Find $\tan(\alpha + \beta)$.

48. $\tan \alpha = \frac{3}{4}$, quadrant III; $\sin \beta = -\frac{4}{5}$, quadrant III. Find $\cos(\alpha - \beta)$.

50. $\cos \alpha = \frac{1}{2}$, quadrant I; $\cos \beta = \frac{\sqrt{2}}{2}$, quadrant IV. Find $\sin(\alpha + \beta)$.

52. $\sin \alpha = -\frac{15}{17}$, quadrant III; $\tan \beta = -\frac{3}{4}$, quadrant IV. Find $\tan(\alpha + \beta)$.

54. $\cos \alpha = -\frac{5}{13}$, quadrant III; $\cos \beta = -\frac{8}{17}$, quadrant III. Find $\cos(\alpha + \beta)$.

56. $\cos \alpha = -\frac{3}{5}$, quadrant III; $\sin \beta = \frac{1}{3}$, quadrant II. Find $\tan(\alpha - \beta)$.

Use the sum and difference identities to verify the following identities.

57. $\sin(\pi - \theta) = \sin \theta$

58. $\sin(\pi + \theta) = -\sin \theta$

59. $\cos(\pi - \theta) = -\cos \theta$

60. $\cos(\pi + \theta) = -\cos \theta$

61. $\tan(\pi - \theta) = -\tan \theta$

62. $\tan(\pi + \theta) = \tan \theta$

63. Use the sum formula to show that the sine function is 2π -periodic; that is, that $\sin(\theta + 2\pi) = \sin \theta$.

65. Use the sum formula to show that the tangent function is π -periodic; that is, that $\tan(\theta + \pi) = \tan \theta$.

64. Use the sum formula to show that the cosine function is 2π -periodic; that is, that $\cos(\theta + 2\pi) = \cos \theta$.

The following identities are important because they express a product of factors as a sum of terms. Verify each identity.

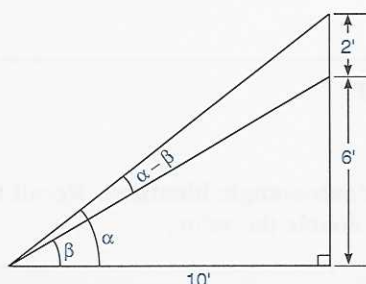
66. $\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$

67. $\cos \alpha \sin \beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$

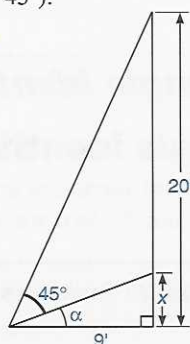
68. $\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$

69. $\sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

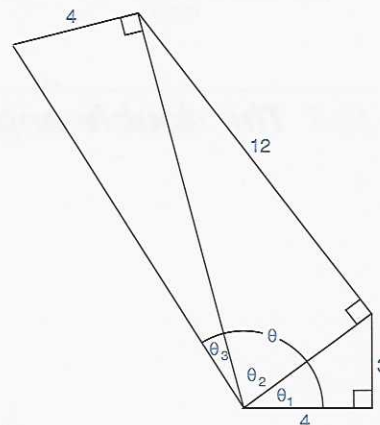
70.  A picture on a wall is 2 feet tall and 6 feet above eye level; see the diagram. Compute the exact value of $\sin(\alpha - \beta)$.



71.  Referring to the figure, find (a) the exact value of $\tan \alpha$, and (b) use this to find the exact value of x . *Hint:* Compute $\tan(\alpha + 45^\circ)$.



72.  Use the identities for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ to find $\sin \theta$ in the diagram.



 The following problems are designed to show that the sum and difference identities for sine and cosine [2], [3], and [4], and the cofunction identities, [5] through [10], are true, using the fact that identity [1] is true. The problems are in the necessary logical order.

73. Use identity [1], $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$, to verify identity [2], $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$. Do this by replacing β by $(-\beta)$ in the identity for $\cos(\alpha + \beta)$ and simplifying, using the even and odd properties for the sine and cosine functions.

74. Verify the identity [5], $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$, by using identity [2], letting $\alpha = \frac{\pi}{2}$ and $\beta = \theta$.

75. The identity [6], $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$, is really the same as identity [5]. Show this as follows. Let $\alpha = \frac{\pi}{2} - \theta$, so that $\theta = \frac{\pi}{2} - \alpha$. Replace $\frac{\pi}{2} - \theta$ by α in the left member of identity [5], then replace θ in the right member by $\frac{\pi}{2} - \alpha$. Then observe that α and θ are arbitrary values, so the result can be rewritten in terms of θ .

76. Verify identity [3], $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$, as follows.

$$\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

This is identity [5], which we know is true

$$\sin(\alpha + \beta) = \cos\left[\frac{\pi}{2} - (\alpha + \beta)\right]$$

Replace θ by $\alpha + \beta$

$$\sin(\alpha + \beta) = \cos\left[\left(\frac{\pi}{2} - \alpha\right) - \beta\right]$$

Regroup $\frac{\pi}{2} - \alpha - \beta$

Now use identity [2] to expand the right member of this equation, then apply identities [5] and [6] to simplify the result and obtain identity [3].

77. Use the identity [3], $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$, to verify the identity [4], $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$. Do this by replacing β by $(-\beta)$ in identity [3].

78. Verify identity [7], $\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$ using the fact that $\tan x = \frac{\sin x}{\cos x}$.

79. Verify identity [8], $\cot\left(\frac{\pi}{2} - \theta\right) = \tan \theta$. See problem 78 for guidance.

80. Verify identity [9], $\sec\left(\frac{\pi}{2} - \theta\right) = \csc \theta$ using the fact that $\sec x = \frac{1}{\cos x}$.

81. Verify identity [10], $\csc\left(\frac{\pi}{2} - \theta\right) = \sec \theta$. See problem 80 for guidance.

5-3 The double-angle and half-angle identities

Double-angle identities

Some more important identities are the **double-angle identities**. Recall that if we multiply a value by two we say we double the value.

Double-angle identities

[1]	$\sin 2\alpha = 2 \sin \alpha \cos \alpha$	[2-a]	$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$
		[2-b]	$\cos 2\alpha = 1 - 2 \sin^2 \alpha$
[3]	$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$	[2-c]	$\cos 2\alpha = 2 \cos^2 \alpha - 1$

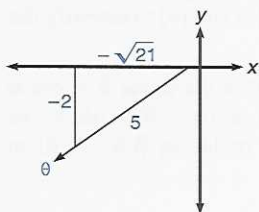
Observe that we present three identities for $\cos 2\alpha$. This is because identities [2-b] and [2-c] get so much use in the development of other identities.

The proof of [1] is as follows.

$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$	Sum identity from section 5-2
$\sin(\alpha + \alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha$	Let $\beta = \alpha$
$\sin 2\alpha = 2 \sin \alpha \cos \alpha$	$\alpha + \alpha = 2\alpha$

The verification of the remaining identities is left for the exercises. They are done in a similar way, starting with the identities for $\cos(\alpha + \beta)$ and $\tan(\alpha + \beta)$.

Example 5-3 A



1. If $\sin \theta = -\frac{2}{5}$ and θ lies in quadrant III, find exact values of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

First construct a reference triangle for θ to obtain any required trigonometric function values for that angle.

$$\begin{aligned}
 \sin 2\theta &= 2 \sin \theta \cos \theta \\
 &= 2 \left(-\frac{2}{5} \right) \left(-\frac{\sqrt{21}}{5} \right) & \sin \theta = -\frac{2}{5}; \cos \theta = -\frac{\sqrt{21}}{5} \\
 &= \frac{4\sqrt{21}}{25} \\
 \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\
 &= \left(-\frac{\sqrt{21}}{5} \right)^2 - \left(-\frac{2}{5} \right)^2 = \frac{21}{25} - \frac{4}{25} = \frac{17}{25}
 \end{aligned}$$

$$\begin{aligned}
 \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \\
 &= \frac{2\left(\frac{2}{\sqrt{21}}\right)}{1 - \left(\frac{2}{\sqrt{21}}\right)^2} = \frac{\frac{4}{\sqrt{21}}}{1 - \frac{4}{21}} \\
 &= \frac{\frac{4}{\sqrt{21}}}{\frac{17}{21}} = \frac{4}{\sqrt{21}} \cdot \frac{21}{17} = \frac{4\sqrt{21}}{17}
 \end{aligned}$$

$$\tan 2\theta \text{ could also be obtained from } \tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta}.$$

2. Find an identity for $\tan 3\theta$ in terms of $\tan \theta$.

$$\begin{aligned}
 \tan 3\theta &= \tan(2\theta + \theta) \\
 &= \frac{\tan 2\theta + \tan \theta}{1 - \tan 2\theta \tan \theta} \\
 &= \frac{\frac{2 \tan \theta}{1 - \tan^2 \theta} + \tan \theta}{1 - \frac{2 \tan \theta}{1 - \tan^2 \theta} \tan \theta} \\
 &= \frac{2 \tan \theta + \tan \theta(1 - \tan^2 \theta)}{(1 - \tan^2 \theta) - 2 \tan^2 \theta} \\
 &= \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}
 \end{aligned}$$

$$3\theta = 2\theta + \theta$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\text{Replace } \tan 2\theta \text{ by } \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\text{Multiply numerator and denominator by } (1 - \tan^2 \theta)$$

$$\text{Combine}$$

Half-angle identities

A further set of important identities is the **half-angle identities**.

Half-angle identities

$$[4] \quad \sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

$$[5] \quad \cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

$$[6-a] \quad \tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$$

$$[6-b] \quad \tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha}$$

$$[6-c] \quad \tan \frac{\alpha}{2} = \frac{1 - \cos \alpha}{\sin \alpha}$$

We verify identity [5] as follows:

$$2 \cos^2 \theta - 1 = \cos 2\theta \quad \text{Identity [2-c] above}$$

$$2 \cos^2 \theta = 1 + \cos 2\theta$$

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

$$\cos^2 \frac{\alpha}{2} = \frac{1 + \cos \alpha}{2} \quad \text{Replace } \theta \text{ by } \frac{\alpha}{2}$$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}} \quad \text{Take the square root of each member}$$

The verification of the remaining identities is left for the exercises.

The choice of plus or minus in identities [4], [5], and [6-a] depends on the quadrant in which the angle in question terminates (using the ASTC rule from section 2-3). It is only possible to determine the quadrant if we have information about the measure of the angle. To see why, consider figure 5-1.

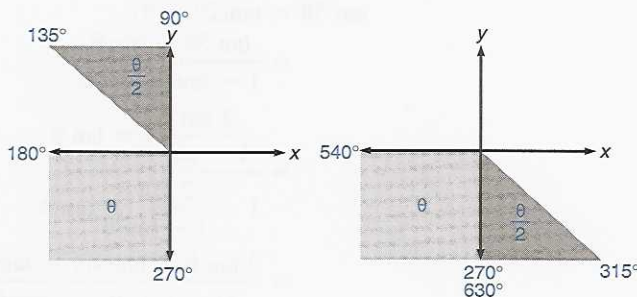


Figure 5-1

If $180^\circ \leq \theta \leq 270^\circ$ then $90^\circ \leq \frac{\theta}{2} \leq 135^\circ$; in this case, θ terminates in quadrant III and $\frac{\theta}{2}$ terminates in quadrant II. However, if $540^\circ \leq \theta \leq 630^\circ$, then $270^\circ \leq \frac{\theta}{2} \leq 315^\circ$. In this case, θ also terminates in quadrant III but $\frac{\theta}{2}$ terminates in quadrant IV.

These identities have applications such as those shown in example 5-3 B.

■ **Example 5-3 B**

1. Use the fact that 22.5° is one half of 45° to find the exact value of $\sin 22.5^\circ$.

$$\begin{aligned}\sin 22.5^\circ &= \sin \frac{45^\circ}{2} \\ &= \pm \sqrt{\frac{1 - \cos 45^\circ}{2}} && \sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}; \alpha = 45^\circ \\ &= \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} && \text{We know } \sin 22.5^\circ > 0, \text{ so choose plus} \\ &= \sqrt{\frac{2 - \sqrt{2}}{4}} && \frac{1}{2} \left(1 - \frac{\sqrt{2}}{2} \right) = \frac{1}{2} \left(\frac{2 - \sqrt{2}}{2} \right) \\ &= \frac{\sqrt{2 - \sqrt{2}}}{2}\end{aligned}$$

2. $\cos \theta = \frac{3}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$. Find the exact value for $\cos \frac{\theta}{2}$.

Since $\frac{3\pi}{2} < \theta < 2\pi$, $\frac{3\pi}{4} < \frac{\theta}{2} < \pi$, so $\frac{\theta}{2}$ terminates in quadrant II, where the cosine function is negative.

$$\begin{aligned}\cos \frac{\theta}{2} &= -\sqrt{\frac{1 + \cos \theta}{2}} && \text{Choose minus since } \frac{\theta}{2} \text{ terminates in quadrant II} \\ & && \text{where } \cos \frac{\theta}{2} < 0 \\ &= -\sqrt{\frac{1 + \frac{3}{5}}{2}} && \text{Replace } \cos \theta \text{ with } \frac{3}{5} \\ &= -\sqrt{\frac{\frac{1}{2} \cdot \frac{8}{5}}{2}} = -\sqrt{\frac{4}{5}} = -\sqrt{\frac{20}{25}} \\ &= -\frac{2\sqrt{5}}{5}\end{aligned}$$

It is important to understand how to rewrite identities with different forms of the argument. For example, the following identities are all the same; the argument of each is shown in different forms.

[1]	$\sin 2\alpha = 2 \sin \alpha \cos \alpha$	Identity [1] of the double-angle identities
	$\sin 4\alpha = 2 \sin 2\alpha \cos 2\alpha$	Replace α in [1] by 2α
	$\sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}$	Replace α in [1] by $\frac{\alpha}{2}$

Example 5-3 C illustrates.

■ Example 5-3 C

Rewrite each expression as an expression of the form $a \sin x$, $a \cos x$, or $a \tan x$, for appropriate values of a and x .

1. $2 \sin 4\theta \cos 4\theta$

Compare

$$\begin{aligned} [1] \quad 2 \sin \alpha \cos \alpha &= \sin 2\alpha \\ 2 \sin 4\theta \cos 4\theta & \end{aligned}$$

We can see that we should replace α by 4θ in identity [1] to obtain $2 \sin 4\theta \cos 4\theta$. Then,

$$\begin{aligned} 2 \sin \alpha \cos \alpha &= \sin 2\alpha && \text{Identity [1]} \\ 2 \sin 4\theta \cos 4\theta &= \sin 2(4\theta) && \text{Replace by } 4\theta \\ &= \sin 8\theta \end{aligned}$$

Thus, $2 \sin 4\theta \cos 4\theta = \sin 8\theta$.

2. $\frac{4 \tan 2\theta}{1 - \tan^2 2\theta}$

$$\frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \tan 2\alpha \quad \text{Identity [3]}$$

$$\frac{4 \tan \alpha}{1 - \tan^2 \alpha} = 2 \tan 2\alpha \quad \text{Multiply each member by 2}$$

$$\frac{4 \tan 2\theta}{1 - \tan^2 2\theta} = 2 \tan 4\theta \quad \text{Replace } \alpha \text{ by } 2\theta$$

3. $\cos^2 80^\circ - \sin^2 80^\circ$

Compare

$$\begin{aligned} \cos^2 \alpha - \sin^2 \alpha &= \cos 2\alpha && \text{Identity [2] of the double-angle identities} \\ \cos^2 80^\circ - \sin^2 80^\circ & \end{aligned}$$

Since 80° replaces α , we know that $\cos 2\alpha$ becomes $\cos 2(80)^\circ = \cos 160^\circ$. Thus, $\cos^2 80^\circ - \sin^2 80^\circ = \cos 160^\circ$. ■

A similar idea is illustrated in example 5-3 D.

■ Example 5-3 D

Find a value of θ for which the statement $\sin 110^\circ = 2 \sin \theta \cos \theta$ is true, then rewrite the statement replacing θ by this value.

Compare

$$\begin{aligned} \sin 110^\circ &= 2 \sin \theta \cos \theta \\ \sin 2\theta &= 2 \sin \theta \cos \theta \end{aligned}$$

Let $2\theta = 110^\circ$, so $\theta = 55^\circ$.

The statement becomes $\sin 110^\circ = 2 \sin 55^\circ \cos 55^\circ$. ■

The identities of this and the previous sections may be combined to verify new identities.

■ Example 5-3 E

Verify the following identities.

$$1. \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$$

It is best to work with the right member since it is more complicated.

$$\begin{aligned} \frac{2 \tan \theta}{1 + \tan^2 \theta} &= 2 \cdot \frac{\tan \theta}{\sec^2 \theta} & \tan^2 \alpha + 1 &= \sec^2 \alpha \\ &= 2 \cdot \tan \theta \cos^2 \theta & \cos \alpha &= \frac{1}{\sec \alpha} \\ &= 2 \cdot \frac{\sin \theta}{\cos \theta} \cos^2 \theta \\ &= 2 \sin \theta \cos \theta & \frac{1}{\cos \theta} \cdot \cos^2 \theta &= \cos \theta \\ &= \sin 2\theta & \sin 2\alpha &= 2 \sin \alpha \cos \alpha \end{aligned}$$

$$2. \sin 4\theta = 8 \sin \theta \cos^3 \theta - 4 \sin \theta \cos \theta$$

Although the right member is more complicated, it is easier to begin with $\sin 4\theta$ and expand this expression.

$$\begin{aligned} \sin 4\theta &= \sin[2(2\theta)] \\ &= 2 \sin 2\theta \cos 2\theta & \text{Use } \sin 2\alpha &= 2 \sin \alpha \cos \alpha, \text{ with } \alpha = 2\theta \\ &= 2(2 \sin \theta \cos \theta)(2 \cos^2 \theta - 1) \\ &= 4 \sin \theta \cos \theta (2 \cos^2 \theta - 1) \\ &= 8 \sin \theta \cos^3 \theta - 4 \sin \theta \cos \theta \end{aligned}$$

Mastery points

Can you

- Write the double-angle and half-angle identities?
- Use the double-angle and half-angle identities to find exact values of $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$, $\sin \frac{\theta}{2}$, $\cos \frac{\theta}{2}$, $\tan \frac{\theta}{2}$?
- Use the double-angle and half-angle identities to derive new identities and to verify given identities?
- Rewrite certain identities as a trigonometric function of $k\theta$, k an integer?

Exercise 5-3

Use the identities of this section to rewrite each expression as an expression of the form $a \sin x$, $a \cos x$, or $a \tan x$, for appropriate values of a and x .

1. $2 \sin \frac{\pi}{4} \cos \frac{\pi}{4}$

2. $2 \sin 52^\circ \cos 52^\circ$

3. $\cos^2 3\pi - \sin^2 3\pi$

4. $2 \cos^2 5\pi - 1$

5. $1 - 2 \sin^2 \frac{\pi}{10}$

6. $\frac{2 \tan \frac{\pi}{6}}{1 - \tan^2 \frac{\pi}{6}}$

7. $\frac{6 \tan 10^\circ}{1 - \tan^2 10^\circ}$

8. $8 \cos^2 \frac{\pi}{2} - 4$

9. $2 \sin 6\theta \cos 6\theta$

10. $4 \sin 2\theta \cos 2\theta$

11. $6 \cos^2 5\theta - 3$

12. $8 \cos^2 3\theta - 4$

13. $\frac{10 \tan 3\theta}{1 - \tan^2 3\theta}$

14. $\frac{8 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$

15. $2 - 4 \sin^2 7\theta$

16. $\frac{1}{2} - \sin^2 2\theta$

17. $3 \cos^2 3\theta - 3 \sin^2 3\theta$

18. $2 \cos^2 \frac{\theta}{2} - 2 \sin^2 \frac{\theta}{2}$

Find a value of θ for which each statement is true.

19. $\sin 140^\circ = 2 \sin \theta \cos \theta$

20. $\sin \theta \cos \theta = \frac{1}{2} \sin \frac{\pi}{5}$

21. $\cos \frac{5\pi}{6} = \cos^2 \theta - \sin^2 \theta$

22. $2 \tan 86^\circ = \frac{4 \tan \theta}{1 - \tan^2 \theta}$

23. $3 \cos 70^\circ = 6 \cos^2 \theta - 3$

24. $\cos 560^\circ = 1 - 2 \sin^2 \theta$

25. $\sin 10^\circ = \sqrt{\frac{1 - \cos \theta}{2}}$

26. $\tan \theta = \sqrt{\frac{1 - \cos 46^\circ}{1 + \cos 46^\circ}}$

27. $\cos \theta = \sqrt{\frac{1}{2} \left(1 + \cos \frac{\pi}{4} \right)}$

28. $\sin \frac{\pi}{6} = \sqrt{\frac{1}{2} (1 - \cos \theta)}$

29. $\tan \frac{2\pi}{5} = \frac{1 - \cos \theta}{\sin \theta}$

30. $\cos 40^\circ = \sqrt{\frac{1 + \cos \theta}{2}}$

Find the exact value of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$ for each of the following.

31. $\sin \theta = \frac{3}{5}, 0 < \theta < \frac{\pi}{2}$

32. $\sin \theta = -\frac{12}{13}, \pi < \theta < \frac{3\pi}{2}$

33. $\cos \theta = -\frac{4}{5}, \frac{\pi}{2} < \theta < \pi$

34. $\tan \theta = -\frac{3}{4}, \frac{3\pi}{2} < \theta < 2\pi$

35. $\csc \theta = -\frac{8}{5}, \pi < \theta < \frac{3\pi}{2}$

36. $\tan \theta = \frac{5}{12}, \pi < \theta < \frac{3\pi}{2}$

Find the exact value of $\sin \frac{\theta}{2}$, $\cos \frac{\theta}{2}$, and $\tan \frac{\theta}{2}$ for each of the following.

37. $\sec \theta = -\frac{5}{2}, \pi < \theta < \frac{3\pi}{2}$

38. $\tan \theta = -\sqrt{15}, \frac{\pi}{2} < \theta < \pi$

39. $\cot \theta = -2, \frac{3\pi}{2} < \theta < 2\pi$

40. $\cos \theta = \frac{1}{4}, \frac{3\pi}{2} < \theta < 2\pi$

Use the half-angle identities to find the exact value of $\sin \theta$, $\cos \theta$, and $\tan \theta$ for the following values of θ .

41. 15° , or $\frac{\pi}{12}$

42. 22.5° , or $\frac{\pi}{8}$

43. 75° , or $\frac{5\pi}{12}$

Use the sum/difference identities (from section 5-2) and the results of problems 41 and 42 to compute the exact value of the following. Observe that $37.5^\circ = 15^\circ + 22.5^\circ$.

44. $\sin 37.5^\circ$

47. Find $\sin 7.5^\circ$; see problem 41.

45. $\cos 37.5^\circ$

46. $\tan 37.5^\circ$

48. Find $\cos 7.5^\circ$; see problem 41.

Verify the following identities.

49. $\sin 2\theta + 1 = (\sin \theta + \cos \theta)^2$

52. $\cot \theta = \frac{1 + \cos 2\theta}{\sin 2\theta}$

55. $\cot \theta - \tan \theta = \frac{2 \cos 2\theta}{\sin 2\theta}$

57. $\sin 2\theta - 4 \sin^3 \theta \cos \theta = \sin 2\theta \cos 2\theta$

59. $\csc^2 \theta = \frac{2}{1 - \cos 2\theta}$

62. $\cot 4\theta = \frac{1 - \tan^2 2\theta}{2 \tan 2\theta}$

65. $\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$

68. $\sec^2 \theta - \cos^2 \frac{\theta}{2} = \tan^2 \theta + \sin^2 \frac{\theta}{2}$

70. $\sin^2 \frac{\theta}{2} - \cos^2 \frac{\theta}{2} = -\cos \theta$

73. $\sin^2 \frac{\theta}{2} = \frac{\csc \theta - \cot \theta}{2 \csc \theta}$

76. $\tan \frac{\theta}{2} + \cot \frac{\theta}{2} = \frac{2}{\sin \theta}$

78. Show that $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$.

79.  Find an identity for $\cos 3\theta$ in terms of $\cos \theta$. See problem 78.

80.  Find identities for (a) $\sin 4\theta$ in terms of $\sin \theta$ and $\cos \theta$ and for (b) $\cos 4\theta$ in terms of $\cos \theta$.

81.  Find identities for (a) $\sin 5\theta$ in terms of $\sin \theta$ and for (b) $\cos 5\theta$ in terms of $\cos \theta$.

82. Finding the center of gravity of a certain solid involves the expression $\frac{3}{16}a \left(\frac{1 - \cos 2\alpha}{1 - \cos \alpha} \right)$. Show that this is equivalent to $\frac{3}{8}a(1 + \cos \alpha)$.

83. Show that $\tan \frac{\alpha}{2} = \frac{1 - \cos \alpha}{\sin \alpha}$ (identity [6-c]). Do this as follows. Let $\frac{\alpha}{2} = \theta$, so that $\alpha = 2\theta$. Replace $\frac{\alpha}{2}$ and α in the identity. Then, simplify the right member; the most direct route will use $\cos 2\theta = 1 - 2 \sin^2 \theta$.

50. $\cos 2\theta + 2 \sin^2 \theta = 1$

53. $\frac{1 + \cos 2\theta}{1 - \cos 2\theta} = \cot^2 \theta$

60. $\frac{2 \cos^3 \theta}{1 - \sin \theta} = 2 \cos \theta + \sin 2\theta$

63. $2 \csc 2\theta \sin \theta \cos \theta = 1$

66. $\frac{\csc \theta - \cot \theta}{1 + \cos \theta} = \csc \theta \tan^2 \frac{\theta}{2}$

69. $\cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} = \frac{\sin^2 \theta}{4}$

71. $\tan^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} = \frac{\cos^2 \theta + 3}{2 + 2 \cos \theta}$

74. $4 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2} = \sin^2 \theta$

77. $2 \cos^2 \frac{\theta}{2} - \cos \theta = 1$

51. $\cos^4 \theta - \sin^4 \theta = \cos 2\theta$

54. $\tan 2\theta = \frac{2 \tan \theta}{2 - \sec^2 \theta}$

56. $2 \csc 2\theta = \tan \theta + \cot \theta$

58. $\cos 4\theta = 1 - 8 \sin^2 \theta \cos^2 \theta$

61. $\tan 2\theta = \frac{2(\tan \theta + \tan^3 \theta)}{1 - \tan^4 \theta}$

64. $\sec 2\theta = \frac{1}{1 - 2 \sin^2 \theta}$

67. $\cos^2 \frac{\theta}{2} = \frac{1 - \cos^2 \theta}{2 - 2 \cos \theta}$

72. $\tan^2 \frac{\theta}{2} = \frac{2}{1 + \cos \theta} - 1$

75. $\frac{1 + \sec \theta}{\sec \theta} = 2 \cos^2 \frac{\theta}{2}$

84. Show that $\tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha}$ (identity [6-b]). See the previous problem.

85.  a. Use the identity for $\sin \frac{\theta}{2}$ with $\theta = 30^\circ$ to find the exact value of $\sin 15^\circ$.

b. Use $\alpha = 45^\circ$, $\beta = 30^\circ$ and $\sin(\alpha - \beta)$ to find the exact value of $\sin 15^\circ$.

c. Show that the values in (a) and (b) are the same. You may find useful the principle that if $a > 0$ and $b > 0$, then $a^2 = b^2$ implies that $a = b$.

86. a. Find $\tan 15^\circ$ with the identity for $\tan \frac{\alpha}{2}$ (half-angle identity [6-a]), with $\alpha = 30^\circ$.

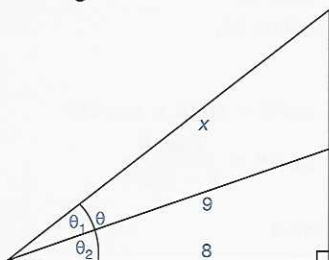
b. Rewrite $\tan 15^\circ$ as $\frac{\sin 15^\circ}{\cos 15^\circ}$ and use identities [4] and [5], with $\alpha = 30^\circ$ to compute $\tan 15^\circ$.

c. Show that the values in parts a and b are the same.

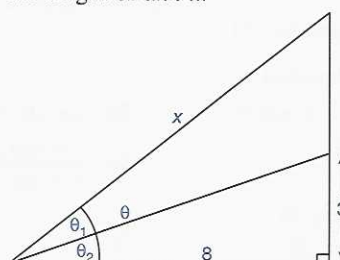
87. Verify half-angle identities [5] and [6-a].

88. Verify the double-angle identities [2-a], [2-b], [2-c], and [3].

89. In the figure, $\theta_1 = \theta_2$. Use the identity for $\cos \frac{\theta}{2}$ to find the length of side x .



90. In the figure, $\theta_1 = \theta_2$. Use the identity for $\tan \frac{\theta}{2}$ to find the length of side x .



The following four identities are important in some situations because they relate the sums and differences of trigonometric expressions to the products of trigonometric expressions. Verify each identity.

91. $\sin 2\alpha + \sin 2\beta = 2 \sin(\alpha + \beta) \cdot \cos(\alpha - \beta)$
 93. $\cos 2\alpha + \cos 2\beta = 2 \cos(\alpha + \beta) \cdot \cos(\alpha - \beta)$
 (Hint: Convert everything to cosine.)

92. $\sin 2\alpha - \sin 2\beta = 2 \sin(\alpha - \beta) \cdot \cos(\alpha + \beta)$
 94. $\cos 2\alpha - \cos 2\beta = -2 \sin(\alpha + \beta) \cdot \sin(\alpha - \beta)$
 (Hint: Convert everything to cosine.)

95. Professor Gilbert Strang of the Massachusetts Institute of Technology has shown¹ an interesting relationship between the identity $\cot 2\theta = \frac{1}{2} \left(\cot \theta - \frac{1}{\cot \theta} \right)$ and the subject of chaotic behavior in iterative systems. Verify this identity.

¹"A Chaotic Search for i ," *The College Mathematics Journal*, Vol. 22, No. 1, January 1991.

5-4 Conditional trigonometric equations

Conditional trigonometric equations were introduced in section 1-4, and were revisited several times in chapter 2, as well as in section 5-0. In this section, we examine these equations in a more general way, and examine using the graphing calculator to find approximate solutions.

Remember that whenever we compute an inverse trigonometric function to solve an equation we use the *absolute value of the argument*, which gives us the *reference angle* of the answer.

Primary solutions

In this section we solve for values that are in both degree and radian measure. We determine all solutions that fall between $0^\circ \leq x < 360^\circ$ or, in radian measure, $0 \leq x < 2\pi$. We call such solutions **primary solutions**. Example 5-4 A illustrates.

■ Example 5-4 A

Find all primary solutions for the following trigonometric equations. Find the solutions in degrees (nearest tenth) and radians (four decimal places).

1. $5 \sin \alpha = -2$

$$5 \sin \alpha = -2$$

$$\sin \alpha = -\frac{2}{5}$$

Since $\sin \alpha < 0$ all solutions are in quadrants III and IV.

$$\alpha' = \sin^{-1} \frac{2}{5}$$

$$\alpha' \approx 23.6^\circ \text{ or } 0.4115 \text{ radians}$$

As noted earlier we use $|\sin^{-1} \frac{2}{5}|$

Degree mode, radian mode

Degrees:

$$\alpha \approx 180^\circ + 23.6^\circ \text{ or } 360^\circ - 23.6^\circ$$

Radians:

$$\alpha \approx \pi + 0.4115 \text{ or } 2\pi - 0.4115$$

Thus, in degrees $\alpha \approx 203.6^\circ$ or 336.4° ; in radians $\alpha \approx 3.5531$ or 5.8717 .

2. $\cos \theta = -\frac{1}{2}$

$$\cos \theta = -\frac{1}{2}$$

$$\theta' = \cos^{-1} \frac{1}{2}$$

$$\theta' = 60^\circ \text{ or } \frac{\pi}{3} \text{ radians}$$

$\cos \theta < 0$ so all solutions are in quadrants II and III

Use the absolute value of $-\frac{1}{2}$

Exact values, obtained from the unit circle, figure 2-18

$$\theta = 180^\circ \pm 60^\circ \text{ or } \pi \pm \frac{\pi}{3}$$

In quadrant II $\theta = 180^\circ - \theta'$; in quadrant III $\theta = 180^\circ + \theta'$

$$\theta = 120^\circ \text{ or } 240^\circ \text{ (degrees) or } \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} \text{ (radians).}$$

3. $2 \cos^2 \theta - \cos \theta - 1 = 0$

The left member is quadratic in the variable $\cos \theta$. It can be factored. If this is difficult to see, try substitution as illustrated in section 5-0.

$$2 \cos^2 \theta - \cos \theta - 1 = 0$$

$$(2 \cos \theta + 1)(\cos \theta - 1) = 0$$

$$2 \cos \theta + 1 = 0 \text{ or } \cos \theta - 1 = 0 \quad \text{Zero factor property}$$

$$\cos \theta = -\frac{1}{2}$$

$$\cos \theta = 1$$

$$\theta' = \cos^{-1} \frac{1}{2}$$

$$\theta' = \cos^{-1} 1$$

$$\theta' = 60^\circ \text{ or } \frac{\pi}{3}$$

$$\theta' = 0^\circ \text{ or } 0 \text{ (radians)}$$

$$\theta = 120^\circ \text{ or } 240^\circ \text{ (degrees)}$$

$$\theta = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} \text{ (radians); } \theta = 0^\circ \text{ or } 0.$$

In degrees, the solutions are 0° , 120° , and 240° and in radians they are 0 ,

$$\frac{2\pi}{3}, \text{ and } \frac{4\pi}{3}.$$

4. $\tan^2 x + 4 \tan x = 1$

$$\tan^2 x + 4 \tan x - 1 = 0$$

This is quadratic, but it will not factor. Solve it using the quadratic formula, as presented in section 5-0.

$$\begin{aligned}\tan x &= \frac{-4 \pm \sqrt{4^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{-4 \pm \sqrt{20}}{2} = \frac{-4 \pm 2\sqrt{5}}{2} \\ &= -2 \pm \sqrt{5}\end{aligned}$$

$$\begin{aligned}a &= 1, b = 4, c = -1 \text{ in} \\ &\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\end{aligned}$$

$$\sqrt{20} = \sqrt{4 \cdot 5} = 2\sqrt{5}$$

$$\tan x = -2 + \sqrt{5}$$

$$\begin{aligned}x' &= \tan^{-1}(-2 + \sqrt{5}) \\ &\approx 13.3^\circ, 0.2318 \text{ (radians)} \\ x &\text{ is in quadrants I or III since} \\ &\quad -2 + \sqrt{5} > 0. \\ x &\approx 13.3^\circ \text{ or } 180^\circ + 13.3^\circ \\ &\approx 0.2318 \text{ or } \pi + 0.2318\end{aligned}$$

$$\tan x = -2 - \sqrt{5}$$

$$\begin{aligned}x' &= \tan^{-1}|-2 - \sqrt{5}| \\ &= \tan^{-1}(2 + \sqrt{5}) \\ &\approx 76.7^\circ, 1.3390 \text{ (radians)} \\ x &\text{ is in quadrants II or IV since} \\ &\quad -2 - \sqrt{5} < 0. \\ x &\approx 180^\circ - 76.7^\circ \text{ or } 360^\circ - 76.7^\circ \\ &\approx \pi - 1.3390 \text{ or } 2\pi - 1.3390\end{aligned}$$

Thus, $x \approx 13.3^\circ, 193.3^\circ, 103.3^\circ, \text{ and } 283.3^\circ$ or $0.2318, 3.3734, 1.8026, \text{ and } 4.9442$.

Using a graphing calculator to help solve an equation

How the solutions of an equation relate to its graph

To see the relationship between solving an equation and its graph, consider the equation $3 \sin x = 2$. This is equivalent to solving $3 \sin x - 2 = 0$. Now, consider the function $y = 3 \sin x - 2$. Solving $3 \sin x - 2 = 0$ is equivalent to finding all values of x that make $y = 0$ in the function $y = 3 \sin x - 2$. Figure 5-2 shows the graph of this function for the interval 0 to 2π (graphed with the calculator in radian mode).

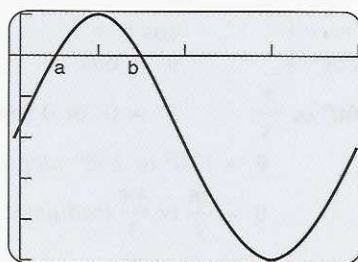


Figure 5-2

Y=	3	SIN	X	T	-	2,
RANGE 0,6.28,1.57,-5,1,1						

The x -coordinates of the points where the curve crosses the x -axis, marked a and b , are $x \approx 0.7297$ or 2.4119 . Because the value of y is zero at these values of x , we also refer to these x -values as **zeros** of the function.

Thus, if we are solving an equation in which one member is zero, the solutions correspond to the x -intercepts of the graph of the nonzero member of the equation. The graph in figure 5-2 shows that there are two primary solutions.

Using the TI-81 trace function to find approximations to solutions

The Trace function in the calculator can be used to find an approximate value of a solution. For example, graph the function as shown above, then select **TRACE**. A blinking box appears on the function toward the center of the screen. By using the left and right cursor keys $\leftarrow$ and $\rightarrow$ we can cause the cursor to trace the function, all the while indicating its x - and y -coordinates at the bottom of the screen. By “tracing” to the point b we see that x is about 2.286 and y is about 0.057. Of course if we were at the exact solution, y would be zero.

By zooming in (using **ZOOM** 2) and reselecting **TRACE** we can get a better approximation to the actual value, and by repeatedly zooming in again and tracing again we can obtain more and more accurate values for x .

This method of finding approximations to solutions is tedious and inefficient. We next show a much better way to find approximations to solutions.

The TI-81 and Newton's method

There are numeric methods for finding solutions to equations quickly and with great accuracy by using a programmable calculator. One can write a program that searches for a zero of a function. This is useful when the function is well behaved around the zero. For our purposes here, by well behaved we mean that one continuous, smooth line could be used to draw the graph of the function near the zero in question. A good method is called Newton's method. The Texas Instruments TI-81 calculator handbook presents a program called **NEWTON** that implements this method.

Figure 5-3 illustrates how Newton's method gets closer and closer to a root. Assume a function f has a zero at c in figure 5-3. Suppose x_1 is a value of x near c . The program uses the line that is tangent to (i.e., just touches) the function f at the point $(x_1, f(x_1))$ to locate the point x_2 , which is closer to c . The program then uses the line that is tangent to the function f at the point $(x_2, f(x_2))$ to locate the point x_3 , which is even closer to c . The program repeats this until the difference between the last x -value and the newest x -value is less than a predetermined error value.

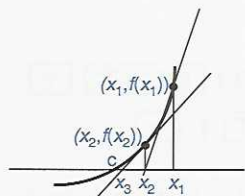


Figure 5-3

The algebraic way in which the program discovers the tangent line at each step is left for a course in the calculus. With a little background in this subject, it is not hard to understand. The program NEWTON can be entered into the calculator as follows:

PRGM **ENTER**

Program edit mode

A-LOCK

2nd **ALPHA** (Alpha lock)

Enter the keys that correspond to the word N E W T O N. For example, T is over the **4** key.

ENTER

Use this after entering the name NEWTON.

Now type in the program as shown.

Program

:(Xmax - Xmin)/100→D

:Lbl 1

:X-Y₁/NDeriv(Y₁,D)→R

:If abs (X-R)≤abs (X/1E10)

:Goto 2

:R→X

:Goto 1

:Lbl 2

:Disp "ROOT="

:Disp R

Keystroke guide

Xmax is in VARS RNG.

Xmin is in VARS RNG.

→D is **STO** **x⁻¹**.

Lbl is in PRGM CTL.

Y₁ is in Y-VARS.

NDeriv is **MATH** 8.

“,D” is **ALPHA** **.** **ALPHA** **x⁻¹**.

→R is **STO** **×**.

If is in **PRGM** CTL.

≤ is in TEST (**2nd** **MATH**).

1E10 is 1 **EE** 10.

Goto is in **PRGM** CTL.

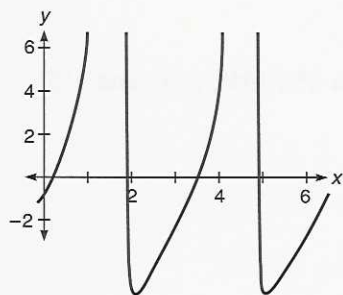
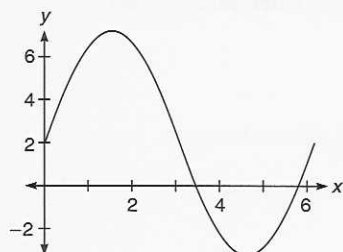
Use **ALPHA** **×** **STO** **X|T**.

Use **PRGM** I/O 1 **A-LOCK** **+**

R O O T **TEST** 1 **+**

Example 5-4 B illustrates using this program NEWTON to find approximate solutions to a trigonometric equation.

Example 5-4 B



Solve the following problems using the programmable calculator.

1. $5 \sin \alpha = -2$

This is equivalent to $5 \sin \alpha + 2 = 0$; find the zeros of the function $y = 5 \sin x + 2$.

Graph $y = 5 \sin x + 2$. This is shown above, with $X_{\min} = -0.1$, $X_{\max} = 6.3$, $Y_{\min} = -3$, $Y_{\max} = 7$. The calculator is in radian mode. Use the trace feature to position the cursor near the zero between 3 and 4. Now execute the program NEWTON. To do this, select **PRGM**, select the number that corresponds to the NEWTON program, and use **ENTER** to execute the program. The value 3.5531095 appears. This is one of the zeros.

Graph the function again, select trace, position the cursor near the second zero, and run the program NEWTON again. The value 5.871668461 appears. This is an approximation to the second zero.

Repeating these steps in degree mode will find approximations to the zeros in degree mode. Of course $X_{\min}$ should be something like -10° , and $X_{\max}$ about 360° . The results displayed, in degrees, are 203.5781785° and 336.4218215° .

2. $\tan^2 x + 4 \tan x = 1$

This is equivalent to $\tan^2 x + 4 \tan x - 1 = 0$. Graph $y = \tan^2 x + 4 \tan x - 1$ with $X_{\min} = -0.1$, $X_{\max} = 7$, $Y_{\min} = -3$, $Y_{\max} = 6.3$. This would be entered as $Y_1 = (\tan X)^2 + 4 \tan X - 1$. The graph is shown in the figure.

Select trace, position the cursor near the first zero, and run the program NEWTON. The value 0.2318238045 appears. This is an approximation to the first zero.

Regraph the function and repeat the first step at each zero. The values that appears are 1.802620131, 3.373416458, and 4.944212785.

As in part 1 of the example, redo the problem in degree mode to obtain the results in degrees. The values displayed are 13.28252559° , 103.2825256° , 193.2825256° , and 283.2825256° .

Using identities to help solve an equation

When an expression involves more than one trigonometric function we often use identities to rewrite the equation in terms of a single trigonometric function. This is illustrated in example 5-4 C.

■ **Example 5-4 C**

Find all primary solutions for the following trigonometric equations. Find the solutions in degrees (nearest tenth) and radians (four decimal places).

1. $\tan \theta - \cot \theta = 0$

$$\tan \theta - \cot \theta = 0$$

$$\tan \theta - \frac{1}{\tan \theta} = 0 \quad \cot \theta = \frac{1}{\tan \theta} \text{ where } \tan \theta \neq 0$$

$$\tan^2 \theta - 1 = 0$$

Multiply each term by $\tan \theta$

$$\tan^2 \theta = 1$$

$$\tan \theta = \pm 1$$

When $\tan \theta = 1$, θ' is 45° or $\frac{\pi}{4}$ (see table 2-1), so using this fact and the

ASTC rule for $\tan \theta > 0$ we obtain $\theta = 45^\circ$ or $180^\circ + 45^\circ$, or $\frac{\pi}{4}$ or

$$\pi + \frac{\pi}{4}.$$

When $\tan \theta = -1$, $\theta' = 45^\circ$ or $\frac{\pi}{4}$, but $\theta = 180^\circ - 45^\circ$ or

$$360^\circ - 45^\circ, \text{ or in radians } \pi - \frac{\pi}{4} \text{ or } 2\pi - \frac{\pi}{4}.$$

Thus, the primary solutions in degrees are 45° , 135° , 225° , and 315° and in radians are $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, and $\frac{7\pi}{4}$.

2. $2 \cos^2 x - 3 \sin x - 3 = 0$

$$2 \cos^2 x - 3 \sin x - 3 = 0$$

$$2(1 - \sin^2 x) - 3 \sin x - 3 = 0 \quad \cos^2 \theta = 1 - \sin^2 \theta$$

$$2 - 2 \sin^2 x - 3 \sin x - 3 = 0$$

$$-2 \sin^2 x - 3 \sin x - 1 = 0$$

$$2 \sin^2 x + 3 \sin x + 1 = 0$$

$$(2 \sin x + 1)(\sin x + 1) = 0$$

$$2 \sin x + 1 = 0$$

$$2 \sin x = -1$$

$$\sin x = -\frac{1}{2}$$

$$x = 210^\circ, 330^\circ, \text{ or } \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\sin x + 1 = 0$$

$$\sin x = -1$$

$$x = 270^\circ \text{ or } \frac{3\pi}{2}$$

The primary solutions are 210° , 270° , 330° or $\frac{7\pi}{6}$, $\frac{3\pi}{2}$, $\frac{11\pi}{6}$.

Finding all solutions to a trigonometric equation

We need to take note of the fact that there are an infinite number of solutions to the equations we solved in the preceding problems. Because the trigonometric functions are periodic the set of all solutions can be found by adding all integral values of the appropriate period (2π or π) to the solution.

This can be illustrated for part 3 of example 5-4 A, where we found that the primary solutions to $\cos \theta = -\frac{1}{2}$ are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ (in radians). However, the cosine function is 2π -periodic, which means that $\cos(\theta + 2k\pi) = \cos \theta$ for any value of θ and for integer values of k . Thus, the actual set of all radian-valued solutions for this problem is $\frac{2\pi}{3} + 2k\pi$ and $\frac{4\pi}{3} + 2k\pi$, k any integer.

This idea is illustrated in figure 5-4.

We use this periodicity for solving trigonometric equations where the coefficient of the argument is not 1. In this situation, it is easier to find all solutions than to find just the primary solutions. This is illustrated in example 5-4 D.

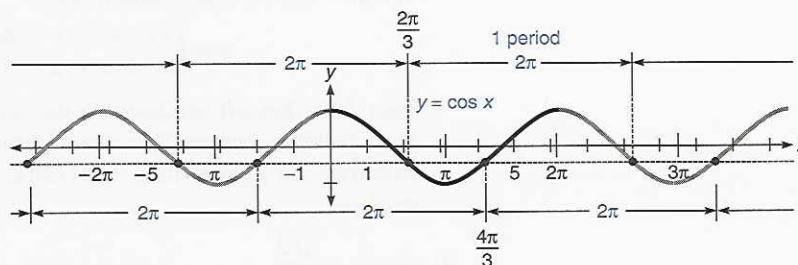


Figure 5-4

■ Example 5-4 D

Find *all* solutions for the following trigonometric equations. Find the solutions in degrees (nearest tenth) and radians (four decimal places).

1. $3 \sin 2x = -2$

$$3 \sin 2x = -2$$

$$\sin 2x = -\frac{2}{3}$$

Since $\sin 2x < 0$, $2x$ is in quadrants III or IV.

$$(2x)' = \sin^{-1} \frac{2}{3}$$

$$(2x)' \approx 41.8^\circ \text{ or } 0.7297 \text{ radians}$$

- Note**
1. Do not divide both members by 2 at this point. It is necessary to find all solutions for $2x$ *before* dividing by 2.
 2. Although we show the intermediate values above as 41.8° and 0.7297 , it is important to keep the maximum accuracy of the calculator up to the last step of the problem. Thus, the calculations that follow are actually performed with the values $\boxed{41.8103149}$ and $\boxed{0.7297276562}$. All calculators have the capability to store at least one value in memory, which should be used to avoid tedious and error-prone reentry of values.

$$2x \approx \begin{cases} 180^\circ + 41.8^\circ \text{ or } 360^\circ - 41.8^\circ \text{ (degrees)} \\ \pi + 0.7297 \text{ or } 2\pi - 0.7297 \text{ (radians)} \end{cases} \quad \begin{array}{l} 2x \text{ is an angle in} \\ \text{quadrants III or IV} \end{array}$$

$$2x \approx \begin{cases} 221.8^\circ, 318.2^\circ \text{ (degrees)} \\ 3.8713, 5.5535 \text{ (radians)} \end{cases} \quad \begin{array}{l} \text{Primary solutions} \\ \text{for } 2x \end{array}$$

To describe all solutions we add multiples of the period of the sine function, 360° or 2π .

$$2x \approx \begin{cases} 221.8^\circ + k \cdot 360^\circ, 318.2^\circ + k \cdot 360^\circ \\ 3.8713 + 2k\pi, 5.5535 + 2k\pi \end{cases}$$

We now divide each solution by 2.

$$x \approx \begin{cases} 110.9^\circ + k \cdot 180^\circ, 159.1^\circ + k \cdot 180^\circ \\ 1.9357 + k\pi, 2.7768 + k\pi \end{cases}$$

This describes *all* solutions to the equation. To find primary solutions for x we would compute the values above for $k = 0$ and $k = 1$. If $k = 2$ the solutions are greater than 360° (2π), and if k is negative the solutions are negative.

$$2. \tan \frac{x}{2} = \frac{\sqrt{3}}{3}$$

$$\tan \frac{x}{2} = \frac{\sqrt{3}}{3}$$

$$\left(\frac{x}{2}\right)' = \tan^{-1} \frac{\sqrt{3}}{3}$$

$$\left(\frac{x}{2}\right)' = 30^\circ, \frac{\pi}{6}$$

$$\frac{x}{2} = 30^\circ, 210^\circ, \text{ or } \frac{\pi}{6}, \frac{7\pi}{6} \quad \text{These are the primary solutions for } \frac{x}{2}$$

The tangent function is π -periodic. Thus, we add integer multiples of 180° (π) to obtain all solutions.

$$\frac{x}{2} = \begin{cases} 30^\circ + k \cdot 180^\circ, 210^\circ + k \cdot 180^\circ \text{ (degrees)} \\ \frac{\pi}{6} + k\pi, \frac{7\pi}{6} + k\pi \text{ (radians)} \end{cases}$$

This describes all solutions. However, $210^\circ - 30^\circ = 180^\circ$, and similarly

$\frac{7\pi}{6} - \frac{\pi}{6} = \pi$, so the solutions can be described more compactly.

$$\frac{x}{2} = 30^\circ + k \cdot 180^\circ, \text{ or } \frac{\pi}{6} + k\pi$$

$$x = 60^\circ + k \cdot 360^\circ \text{ or } \frac{\pi}{3} + 2k\pi \quad \text{Multiply each member by 2}$$

All solutions for x are $x = 60^\circ + k \cdot 360^\circ$ or $\frac{\pi}{3} + 2k\pi$. ■

Equations involving more than one multiple of the angle

If an equation mixes multiples of values with the values themselves, such as θ and 2θ in example 5-4 E, we can eliminate the multiple value with an appropriate identity.

■ Example 5-4 E

Solve $\sin 2\theta - \sin \theta = 0$; find primary solutions.

$$\sin 2\theta - \sin \theta = 0$$

$$2 \sin \theta \cos \theta - \sin \theta = 0 \quad \sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\sin \theta(2 \cos \theta - 1) = 0$$

$$\sin \theta = 0 \text{ or } 2 \cos \theta - 1 = 0$$

$$\sin \theta = 0$$

$$\theta = 0^\circ, 180^\circ, \text{ or } 0, \pi \text{ (radians)}$$

$$2 \cos \theta - 1 = 0$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = 60^\circ, 300^\circ, \text{ or } \frac{\pi}{3}, \frac{5\pi}{3} \text{ (radians)}$$

The solutions are $0^\circ, 60^\circ, 180^\circ, 300^\circ$ or $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ (radians). ■

Mastery points

Can you

- Solve linear and quadratic trigonometric equations?
- Solve trigonometric equations involving multiple angles?
- Solve trigonometric equations by applying an appropriate identity?

Exercise 5-4

Find all primary solutions to the following trigonometric equations. Leave answers in both degrees and radians. All answers should be exact.

1. $\tan \theta + 1 = 0$

5. $\sqrt{3} \tan \theta - 1 = 0$

9. $3 \sin^2 \theta - 3 = 0$

2. $\sin \theta - 1 = 0$

6. $\cot \theta + \sqrt{3} = 0$

10. $3 \csc^2 \theta = 3$

3. $2 \cos \theta - 1 = 0$

7. $\csc \theta + 2 = 0$

11. $\sec^2 \theta = 1$

4. $2 \cos \theta + 1 = 0$

8. $\sec \theta - 2 = 0$

12. $\tan^2 \theta - 1 = 0$

13. $(\cos \theta - 1)(\sin \theta + 1) = 0$
 14. $(\sec \theta + 2)(\csc \theta - 2) = 0$
 15. $(2 \cos^2 \theta - 1)(\cot \theta - 1) = 0$
 16. $(3 \tan^2 \theta - 1)(\sqrt{3} \sec \theta - 2) = 0$
 17. $\sin^2 \theta - \sin \theta = 0$
 18. $\cos^2 \theta + \cos \theta = 0$
 19. $\tan^2 \theta - \sqrt{3} \tan \theta = 0$
 20. $\cos^2 \theta - \frac{1}{2} \cos \theta = 0$
 21. $2 \sin^2 \theta + \sin \theta - 1 = 0$
 22. $\cos^2 \theta + 2 \cos \theta + 1 = 0$
 23. $2 \sin^3 \theta - \sin \theta = 0$
 24. $2 \cos^2 \theta + 3 \cos \theta = 2$
 25. $2 \sin \theta \cos \theta - \sin \theta = 0$
 26. $2 \sin \theta \cos \theta + \cos \theta = 0$
 27. $\sqrt{3} \tan \theta \cot \theta + \cot \theta = 0$
 28. $2 \tan^2 \theta \cos \theta - \tan^2 \theta = 0$
 29. $\tan x \cot x = 0$
 30. $\sin x \cos x = 0$
 31. $2 \sin x - \csc x + 1 = 0$
 32. $2 \cos x + \sec x - 3 = 0$
 33. $\tan x + \cot x = -2$
 34. $2 - \sin x - \csc x = 0$
 35. $2 \sin^2 x - \cos x = 1$
 36. $2 \cos^2 x - 3 \sin x = 3$
 37. $4 \tan^2 x = 3 \sec^2 x$
 38. $4 \cot^2 x - 3 \csc^2 x = 0$
 39. $\sin^2 x - \cos^2 x = 0$
 40. $\cot^2 x + \csc^2 x = 0$
 41. $2 \tan^2 x \sin x = \tan^2 x$
 42. $\sin^2 x \cos x - \cos x = 0$

Solve the following equations using the quadratic formula if necessary and the calculator. Find the primary solutions in both radians and degrees. Round radian answers to hundredths, and degree answers to tenths.

43. $6 \sin^2 x - 2 \sin x - 1 = 0$
 44. $3 \cos^2 x + \cos x - 2 = 0$
 45. $\cot^2 x - 3 \cot x - 2 = 0$
 46. $\tan^2 x + 5 \tan x + 2 = 0$
 47. $\sec^2 x - 2 \sec x - 4 = 0$
 48. $2 \csc^2 x - \csc x - 5 = 0$
 49. $\tan x + 2 \sec x = 3$
 50. $3 \cot x - \csc x - 1 = 0$

Find all solutions to the following trigonometric equations, both in degrees and in radians.

51. $\cos x = \frac{1}{2}$
 52. $\sin x = 1$
 53. $\cot x = -\sqrt{3}$
 54. $\cos x = -\frac{\sqrt{3}}{2}$
 55. $\sin x = -\frac{\sqrt{2}}{2}$
 56. $\tan x = -1$
 57. $\tan x = 1$
 58. $\sec x = \frac{2}{\sqrt{3}}$
 59. $\csc x = 2$
 60. $\tan \frac{x}{2} = 1$
 61. $\sin \frac{x}{2} = \frac{\sqrt{3}}{2}$
 62. $\sin 3x = 0$
 63. $\cos 3x = -1$
 64. $\sec \frac{x}{2} = 1$
 65. $3 \cot 2x = \sqrt{3}$
 66. $2 \sin 3x = -1$
 67. $2 \cos 4x = -1$
 68. $-\sqrt{3} \tan 5x = 1$
 69. $2 \cos 2x + 1 = 0$
 70. $\tan 2\theta - 1 = 0$
 71. $\cot 2\theta - \sqrt{3} = 0$
 72. $2 \cos 3\theta = -1$
 73. $2 \sin 2\theta = 1$
 74. $\sin \frac{\theta}{3} = \frac{\sqrt{3}}{2}$
 75. $\sec 3\theta = 2$
 76. $\csc 2\theta = -\frac{2\sqrt{3}}{3}$
 77. $\sqrt{3} \tan \frac{\theta}{4} = 1$
 78. $\cot \frac{\theta}{3} = \frac{\sqrt{3}}{3}$

Find the primary solutions to the following trigonometric equations, both in radians and degrees. Find solutions in radians to hundredths, and in degrees to the nearest tenth of a degree, where necessary.

79. $\cos 2\theta + \sin \theta = 0$
 80. $\cos 2\theta - \cos \theta = 0$
 81. $\sin 2\theta + \sin \theta = 0$
 82. $\cos^2 \theta - \sin^2 \theta = 1$
 83. $\cos 2\theta = 1 - \sin \theta$
 84. $\cos 2\theta = \cos \theta - 1$
 85. $\sin \frac{\theta}{2} = \tan \frac{\theta}{2}$
 86. $\sin \frac{\theta}{2} = \cos \theta$
 87. $2 \sec \theta = \csc^2 \frac{\theta}{2}$
 88. $\sin^2 \frac{\theta}{2} = \cos \theta$
 89. $\tan \frac{\theta}{2} = \cos \theta - 1$
 90. $\cot \theta - \tan \frac{\theta}{2} = 0$
 91. $\sin 2\theta - \cos \theta = \cos^2 \theta$

In the mathematical modeling of an aerodynamics problem the following equation arises:

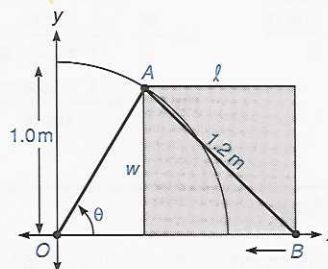
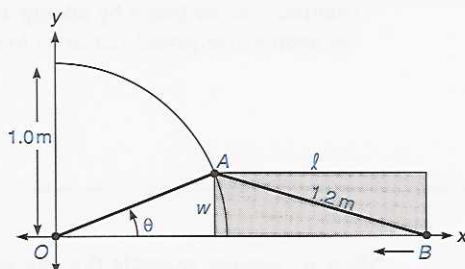
$$y = x \cos A \cos B - x^2 \cos A \sin B - x^3 \sin A$$

Problems 92 and 93 use this equation.

92.  If $A = 0.855$, $B = 1.052$, and $y = 0$, solve for x to the nearest 0.01.

93.  If $B = 0.7$, $x = 2$, and $y = -8$, find A to the nearest 0.01. Find the least nonnegative solution(s).

 A mechanical device is constructed as shown in the diagram. The arm OA moves through angle θ , from 0° to 90° . Two positions are shown. Point A moves along a circle of radius 1.0 meters, and point B moves horizontally only. The distance AB is fixed by arm AB at 1.2 meters. The area of the shaded rectangle is the product of its length and width, $A = \ell w$.



94. Show that $A = \sin \theta \sqrt{1.44 - \sin^2 \theta}$. (The units are square meters.)

96. Find θ when $A = 0.5 \text{ m}^2$. Round the answer to the nearest 0.1° .

95. Find A , to the nearest 0.01 m^2 , when $\theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ$, and 90° .

Chapter 5 summary

- A **trigonometric identity** is a trigonometric equation that is true for all permissible replacements of the variable for which each member is defined.
- To verify that a trigonometric equation is an identity we must show that each member of the equation is equivalent to the same expression.
- To show that an equation is not an identity find a value for the variable for which the statement is not true. This value is called a counter example.

Reciprocal identities

$$\csc \theta = \frac{1}{\sin \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad \cot \theta = \frac{1}{\tan \theta}$$

$$\sin \theta = \frac{1}{\csc \theta}, \quad \cos \theta = \frac{1}{\sec \theta}, \quad \tan \theta = \frac{1}{\cot \theta}$$

Tangent and cotangent identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

Fundamental identity of trigonometry

$$\sin^2 \theta + \cos^2 \theta = 1$$

Pythagorean identities

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\sec^2 \theta = \tan^2 \theta + 1$$

$$\csc^2 \theta = \cot^2 \theta + 1$$

Useful forms

$$\sin^2 \theta = 1 - \cos^2 \theta$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

$$\tan^2 \theta = \sec^2 \theta - 1$$

$$\sec^2 \theta - \tan^2 \theta = 1$$

$$\cot^2 \theta = \csc^2 \theta - 1$$

$$\csc^2 \theta - \cot^2 \theta = 1$$

Sum and difference identities for sine and cosine

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

- When the sum of two angles is 90° , or $\frac{\pi}{2}$ radians, the angles are said to be complementary.

Cofunction identities

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta \quad \cot\left(\frac{\pi}{2} - \theta\right) = \tan \theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta \quad \sec\left(\frac{\pi}{2} - \theta\right) = \csc \theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta \quad \csc\left(\frac{\pi}{2} - \theta\right) = \sec \theta$$

Double-angle identities

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

$$\cos 2\alpha = 1 - 2 \sin^2 \alpha$$

$$\cos 2\alpha = 2 \cos^2 \alpha - 1$$

• **Half-angle identities**

$$\begin{aligned}\sin \frac{\alpha}{2} &= \pm \sqrt{\frac{1 - \cos \alpha}{2}} & \tan \frac{\alpha}{2} &= \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} \\ \cos \frac{\alpha}{2} &= \pm \sqrt{\frac{1 + \cos \alpha}{2}} & \tan \frac{\alpha}{2} &= \frac{\sin \alpha}{1 + \cos \alpha} \\ & & \tan \frac{\alpha}{2} &= \frac{1 - \cos \alpha}{\sin \alpha}\end{aligned}$$

• **Primary solutions** are solutions that fall between $0^\circ \leq x \leq 360^\circ$ or, in radian measure, $0 \leq x < 2\pi$.

• The trigonometric functions are periodic, so the set of all solutions can be found by adding all integral values of the appropriate period (2π or π) to the solution.

Chapter 5 review

[5-1] Show that the trigonometric expression on the left is equivalent to the simplified expression on the right.

- $\frac{\cot \theta}{\cos \theta}$; $\csc \theta$
- $\sec \theta \tan \theta$; $\sec^2 \theta \sin \theta$
- $\frac{\tan^2 \theta}{\sec^2 \theta - 1}$; $\sin^4 \theta \csc^4 \theta$
- $\frac{\csc^2 \theta - 1}{\sec^2 \theta - 1}$; $\cot^4 \theta$
- $\frac{\csc \theta \tan \theta}{\sin \theta}$; $\csc \theta \sec \theta$
- $\sin^2 \theta - \cos^2 \theta$; $2 \sin^2 \theta - 1$

Obtain an equivalent expression involving only the sine and cosine functions. Simplify the resulting expression.

- $\csc \theta - \sec \theta$
- $\tan \theta + \cot \theta$
- $\frac{\sec \theta}{\tan \theta - \cot \theta}$
- $\frac{1 - \cot \theta}{\csc \theta + 1}$
- $\frac{\sec^2 \theta - 1}{\sec^2 \theta}$
- $\frac{1 - \cot^2 \theta}{\csc^2 \theta - 1}$

Verify that each equation is an identity.

- $\csc x - \tan x \cot x = \csc x - 1$
- $\sin^2 x + \sin^2 x \cot^2 x = 1$
- $\csc^2 x - \sec^2 x = \csc^2 x \sec^2 x (\cos^2 x - \sin^2 x)$
- $\frac{1}{\csc x - \cot x} = \frac{\sin x}{1 - \cos x}$
- $\tan x - 1 = \sec x (\sin x - \cos x)$
- $\frac{\csc^2 x - 1}{\sin^2 x} = \cos^2 x \csc^4 x$
- $\frac{1}{1 + \csc x} + \frac{1}{1 - \csc x} = -2 \tan^2 x$
- $\sin^2 x + \sin^2 x \cos^2 x = 1 - \cos^4 x$
- $\tan^4 x + \tan^2 x = \frac{\sec^2 x}{\cot^2 x}$
- $\frac{1 - \cot x}{1 + \csc x} = \frac{\sin x - \cos x}{\sin x + 1}$

Show by counter example that the following equations are not identities.

- $\sin \theta + \cos \theta = 1$
- $\tan \theta - \sin \theta \cos \theta = 0$
- $\frac{1}{\cot \theta - \csc \theta} = \sec \theta$

[5-2] Use the sum and difference formulas to find the exact value of the following.

- $\cos \frac{\pi}{12}$
- $\tan \left(-\frac{\pi}{12} \right)$
- $\sin 105^\circ$
- $\tan(-15^\circ)$

- Given $\sin \alpha = \frac{3}{4}$ and $\cos \beta = -\frac{5}{6}$, α and β lie in quadrant II, find
a. $\sin(\alpha - \beta)$ b. $\tan(\alpha + \beta)$

- Given $\cos \alpha = -\frac{\sqrt{3}}{2}$ and $\sin \beta = -\frac{1}{4}$, α lies in quadrant II and β lies in quadrant III, find
a. $\cos(\alpha + \beta)$ b. $\tan(\alpha - \beta)$

- Given $\sin \alpha = -\frac{5}{12}$ and $\cos \beta = \frac{8}{17}$, α lies in quadrant IV and β lies in quadrant I, find
a. $\sec(\alpha + \beta)$ b. $\cot(\alpha - \beta)$

Using the sum and difference formulas, verify each of the following identities.

- $\frac{\sin(\alpha + \beta)}{\cos(\alpha - \beta)} = \frac{\cot \alpha + \cot \beta}{\cot \alpha \cot \beta + 1}$
- $\cos \left(\frac{3\pi}{2} + \theta \right) = \sin \theta$
- $\sin \left(\frac{\pi}{4} - \theta \right) = \frac{\sqrt{2}}{2} (\cos \theta - \sin \theta)$
- $\frac{\cos(\alpha + \beta)}{\sin \alpha \cos \beta} = \cot \alpha - \tan \beta$

37. Using the identity $\cot \alpha = \frac{1}{\tan \alpha}$ and the identity $\tan(\alpha + \beta)$, show that $\cot(\theta + \pi) = \cot \theta$.

[5-3] Using the double-angle identities, find angle θ that makes the following statements true.

38. $\cos \theta = \cos^2 62^\circ - \sin^2 62^\circ$

39. $\sin \theta = 2 \sin 5\pi \cos 5\pi$

40. $\tan \theta = \frac{2 \tan \frac{7\pi}{12}}{1 - \tan^2 \frac{7\pi}{12}}$

41. $\cos 24^\circ = 1 - 2 \sin^2 \theta$

42. Express $6 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$ as a single trigonometric function of a constant k times θ .

43. Express $\frac{4 \tan 4\theta}{1 - \tan^2 4\theta}$ as a single trigonometric function of a constant k times θ .

44. Given $\cos \theta = -\frac{5}{12}$, θ lies in quadrant III, find the exact value of a. $\tan 2\theta$ b. $\sin 2\theta$

45. Given $\sin \theta = \frac{4}{5}$, θ lies in quadrant II, find the exact value of a. $\cos 2\theta$ b. $\tan 2\theta$

46. Given $\tan \theta = -\frac{5}{4}$, θ lies in quadrant IV, find the exact value of a. $\csc 2\theta$ b. $\cot 2\theta$

Verify the following identities.

47. $\sin 2x - \cos x = \cos x(2 \sin x - 1)$

48. $1 + \cos 2x = 2 \cos^2 x$

49. $\frac{\cos 2x}{2 - 4 \sin^2 x} = \frac{1}{2}$

50. $\tan 2x = \frac{2 \cot x}{\cot^2 x - 1}$

51. $\sin 2x - \cos 2x = 2 \cos x(\sin x - \cos x) + 1$

Using the half-angle identities, find the exact value of the following.

52. $\tan 22.5^\circ$

53. $\cos(-15^\circ)$

54. $\sin \frac{5\pi}{12}$

55. $\cos \frac{3\pi}{8}$

56. Given $\cos x = \frac{12}{13}$, $0 < x < \frac{\pi}{2}$, find

a. $\sin \frac{x}{2}$ b. $\tan \frac{x}{2}$

57. Given $\sin x = -\frac{2}{3}$, $\pi < x < \frac{3\pi}{2}$, find

a. $\sec \frac{x}{2}$ b. $\cot \frac{x}{2}$

58. Find $\sin \frac{\theta}{2}$ if $\cos \theta = -\frac{7}{18}$ and $\pi < \theta < \frac{3\pi}{2}$.

Verify the following equations are identities.

59. $\cot \frac{\theta}{2} = \frac{1 + \cos \theta}{\sin \theta}$

60. $\sec^2 \frac{\theta}{2} - \tan^2 \frac{\theta}{2} = 1$

61. $\tan \frac{\theta}{2} \csc^2 \frac{\theta}{2} = \frac{2}{\sin \theta}$

- [5-4] Solve the following conditional equations for $0 \leq x \leq \frac{\pi}{2}$.

62. $2 \sin x - 1 = 0$

63. $3 \cot^2 x - 1 = 0$

64. $(\sin x - 1)(2 \cos x - 1) = 0$

65. $(4 \sin^2 x - 1)(\sec x - 2) = 0$

Solve the following conditional equations for $0^\circ \leq \theta < 360^\circ$.

66. $\cot^2 \theta - \cot \theta = 0$

67. $\sec^2 \theta - 4 = 0$

68. $2 \cos^2 \theta - \cos \theta - 1 = 0$

69. $2 \sin \theta - \csc \theta + 1 = 0$

70. $2 \cot \theta \cos \theta = \cot^2 \theta$

71. $2 \sin^2 \theta - 3 \cos \theta = 3$

Find all solutions in radians to the following equations. Use the quadratic formula and calculator where necessary (round such answers to the nearest hundredth).

72. $\cos^2 x - 1 = 0$

73. $\tan^2 x - 3 \tan x - 3 = 0$

74. $\sin x - 2 \csc x = 5$

75. $\sec^2 x - \sec x = 2$

Solve each of the following equations for $0 \leq x < 2\pi$ (primary solutions, radians).

76. $2 \sin 4x = 1$

77. $2 \cos \frac{x}{2} - \sqrt{3} = 0$

78. $\sqrt{3} \tan \frac{x}{3} + 1 = 0$

79. $\sin^2 \frac{x}{4} = \frac{1}{2}$

Solve each of the following equations for $0^\circ \leq \theta < 360^\circ$.

80. $2 \cos 5\theta = 1$

81. $3 \tan^2 \frac{\theta}{4} = 9$

82. $\tan 6\theta - \cot 6\theta = 0$

83. $\cos \theta + \sin 2\theta = 0$

Find all solutions in radians to the following equations. Find solutions to the nearest hundredth, where necessary.

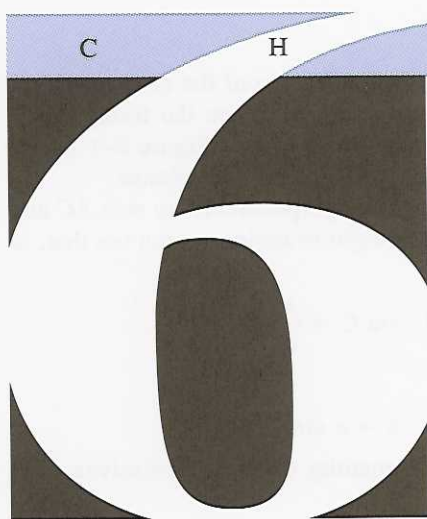
84. $\cos \frac{x}{2} - \sin x = 0$

85. $(\cot 4x - \sqrt{3})(\csc 3x + 2) = 0$

86. $3 \sin^2 2x - \sin 2x - 2 = 0$

Chapter 5 test

1. Show that the expression $\csc^2 x \sin x \cos x$ is equivalent to $\cot x$.
2. Write the expression $\frac{\csc x - \sec x}{\tan x + \cot x}$ as an expression in sine and cosine and simplify.
3. Show by counter example that the equation $\cot x - 2 \tan x = 1$ is not an identity.
4. Given $\cos \alpha = -\frac{1}{2}$ and $\sin \beta = \frac{15}{17}$, α lies in quadrant III and β lies in quadrant II, find $\sin(\alpha + \beta)$.
5. Given $\cos x = -\frac{8}{17}$, x lies in quadrant II, find $\sin 2x$.
6. Given $\csc x = \frac{5}{3}$, $\frac{\pi}{2} < x < \pi$, find $\tan \frac{x}{2}$.
7. Using the appropriate half-angle formula, find $\sec 22.5^\circ$.
8. Verify the following identities.
 - a. $\frac{1 + \cot \theta}{\csc \theta} = \sin \theta + \cos \theta$
 - b. $\frac{\cos^2 x - 1}{\sin^2 x} = -1$
 - c. $\cos\left(\theta - \frac{3\pi}{2}\right) = -\sin \theta$
 - d. $\cos 2x - \sin 2x = -1 + 2 \cos x(\cos x - \sin x)$
9. Solve the equation $4 \cos^2 x - 1 = 0$ for $0 \leq x < 2\pi$.
10. Solve the equation $(\cot \theta - \sqrt{3})(\sec \theta + 2) = 0$ for $0^\circ \leq \theta < 360^\circ$.
11. Find all primary solutions, in radians, to the equation $6 \sin^2 x + 5 \sin x - 1 = 0$. (Use the quadratic formula and the calculator if necessary.)
12. Solve the equation $\sin^2 5\theta - 1 = 0$ for $0^\circ \leq \theta < 360^\circ$.
13. Find all primary solutions, in radians, to the equation $\sec^2 \frac{x}{4} - 2 = 0$.
14. If $\sin^2 x = \frac{1 + \sqrt{1 - m^2}}{2}$, find $\sin 2x$. (Hint: Use $\sin^2 2x = 4 \sin^2 x \cos^2 x$.)



Oblique Triangles and Vectors

In this chapter we examine several applications of the trigonometric functions. We begin with the law of sines and the law of cosines. These are theorems that permit the solution of triangles which are not right triangles. (Recall that we examined the solution of right triangles in section 1–3.) We then introduce the subject of vectors, which has wide application in science and engineering, and which, in a generalized form called linear algebra, has applications in the social and economic sciences as well.

6–1 The law of sines

A triangle in which none of the angles is a right angle is called an **oblique** triangle. One method for solving certain oblique triangles is the **law of sines**. The following paragraphs develop this law.

First, we observe that *at least two of the angles in every triangle are acute*. If only one angle were acute (less than 90°) then the other two would be obtuse or right (greater than or equal to 90°). This is impossible since the sum of these two angles would be greater than or equal to 180° , and thus the sum of all three angles would be greater than 180° .

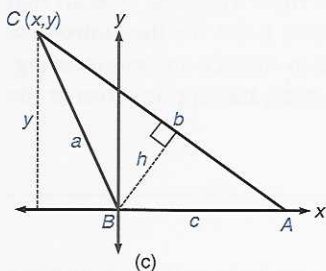
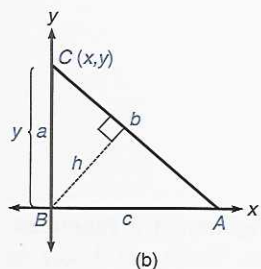
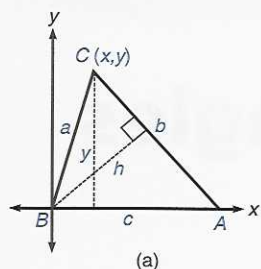


Figure 6-1

Now we consider any triangle ABC , and label two of the acute angles A and C . Angle B may be acute, obtuse, or right. We place the triangle in a coordinate system so that angle B is in standard position. Figure 6-1 shows sketches for the cases where B is (a) acute, (b) right, and (c) obtuse.

From vertex B we construct a line segment perpendicular to side AC and label this line h . From what we know about right triangles we can see that, in all three cases,

$$\sin A = \frac{h}{c} \quad \text{and} \quad \sin C = \frac{h}{a}$$

If we solve for h in each, we obtain

$$h = c \sin A \quad \text{and} \quad h = a \sin C$$

Since $c \sin A$ and $a \sin C$ equal the same quantity (h) they themselves must be equal. Thus,

$$\frac{c \sin A}{ac} = \frac{a \sin C}{ac} \quad \text{Divide both members by } ac$$

$$\frac{\sin A}{a} = \frac{\sin C}{c} \quad \text{Remove common factors}$$

We now extend the relation above to include angle B and side b . Let (x, y) be the coordinates of the vertex of angle C . From what we know about the trigonometric functions for any angle in standard position (chapter 2), we see that

$$\sin B = \frac{y}{a} \quad \text{or} \quad y = a \sin B$$

From what we know about right triangles,

$$\sin A = \frac{y}{b} \quad \text{or} \quad y = b \sin A$$

Thus, $a \sin B = y = b \sin A$, so

$$\frac{a \sin B}{a} = \frac{b \sin A}{b}$$

Putting these results together we have the law of sines.

The law of sines

In any triangle ABC ,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Concept

The ratio of the sine of an angle to the length of the side opposite that angle is the same for all angles in any triangle.

Note We only use two of the three ratios at a time.

In the rest of this chapter we always assume side a is opposite angle A , side b is opposite angle B , and side c is opposite angle C .

■ Example 6-1 A

Solve the triangle ABC . Round off answers to tenths.

$$a = 13.2, A = 21.3^\circ, B = 61.4^\circ$$

It is a good idea to make a table of values:

$$a: 13.2 \quad A: 21.3^\circ$$

$$b: ? \quad B: 61.4^\circ$$

$$c: ? \quad C: ?$$

We can find C first.

$$\begin{aligned} C &= 180^\circ - A - B && \text{The sum of the measure of all three angles is } 180^\circ \\ &= 180^\circ - 21.3^\circ - 61.4^\circ \\ &= 97.3^\circ \end{aligned}$$

Now we fill in the law of sines.

$$\begin{aligned} \frac{\sin A}{a} &= \frac{\sin B}{b} = \frac{\sin C}{c} \\ \frac{\sin 21.3^\circ}{13.2} &= \frac{\sin 61.4^\circ}{b} = \frac{\sin 97.3^\circ}{c} \end{aligned}$$

To use the law of sines we must always know one of the three ratios completely.

In this case, we know the first ratio, so we use it to solve the other two.

Using the first and second ratios:

$$\begin{aligned} \frac{\sin 21.3^\circ}{13.2} &= \frac{\sin 61.4^\circ}{b} \\ b \sin 21.3^\circ &= 13.2 \sin 61.4^\circ && \text{Multiply each member by } 13.2b \\ b &= \frac{13.2 \sin 61.4^\circ}{\sin 21.3^\circ} \approx 31.9 && \text{Divide each member by } \sin 21.3^\circ \end{aligned}$$

Using the first and third ratios:

$$\begin{aligned} \frac{\sin 21.3^\circ}{13.2} &= \frac{\sin 97.3^\circ}{c} \\ c \sin 21.3^\circ &= 13.2 \sin 97.3^\circ && \text{Multiply each member by } 13.2c \\ c &= \frac{13.2 \sin 97.3^\circ}{\sin 21.3^\circ} \approx 36.0 && \text{Divide each member by } \sin 21.3^\circ \end{aligned}$$

Thus we have solved the triangle:

$$a: 13.2 \quad A: 21.3^\circ$$

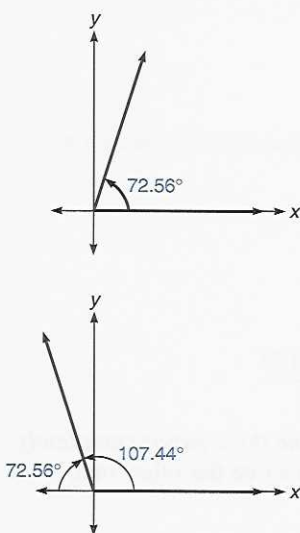
$$b: 31.9 \quad B: 61.4^\circ$$

$$c: 36.0 \quad C: 97.3^\circ.$$

The ambiguous case

If we are given only one of the two angles of a triangle it is possible to get two different solutions to the problem. The reason for this is shown in example 6-1 B. When we use the law of sines to solve a triangle *for which only one angle is known* we call this the **ambiguous case**.

Example 6-1 B



Solve each triangle. Round off answers to tenths.

1. $a = 28.5$, $b = 30.0$, $A = 65^\circ$.

$$a: 28.5 \quad A: 65^\circ$$

Make a table of values

$$b: 30.0 \quad B: ?$$

$$c: ? \quad C: ?$$

$$\frac{\sin 65^\circ}{28.5} = \frac{\sin B}{30.0} = \frac{\sin C}{c}$$

Fill values into the law of sines

$$\frac{\sin 65^\circ}{28.5} = \frac{\sin B}{30.0}$$

Use the first two ratios to find angle B

$$30.0 \sin 65^\circ = 28.5 \sin B$$

$$\frac{30.0 \sin 65^\circ}{28.5} = \sin B$$

A reference angle for angle B is found with the inverse sine function.

$$B' = \sin^{-1}\left(\frac{30.0 \sin 65^\circ}{28.5}\right) \approx 72.56^\circ$$

Since angle B is in a triangle, we know its measure is between 0° and 180° . Thus, using B' as a reference angle B could be either 72.56° or $180^\circ - 72.56^\circ = 107.44^\circ$. See the figure.

At this point we must divide the problem into two cases: the case where $B \approx 72.56^\circ$ and the one where $B \approx 107.44^\circ$.

Case 1: $B \approx 72.56^\circ$

$$a: 28.5 \quad A: 65^\circ$$

$$b: 30.0 \quad B: 72.56^\circ$$

$$c: ? \quad C: ?$$

$$C \approx 180^\circ - 65^\circ - 72.56^\circ \approx 42.44^\circ$$

We can use the value of angle C to find c .

$$\begin{aligned}\frac{\sin 65^\circ}{28.5} &\approx \frac{\sin 42.44^\circ}{c} \\ c \sin 65^\circ &\approx 28.5 \sin 42.44^\circ \\ c &\approx \frac{28.5 \sin 42.44^\circ}{\sin 65^\circ} \approx 21.2\end{aligned}$$

Thus, $C \approx 42.4^\circ$, $c \approx 21.2$.

Case 2: $B \approx 107.44^\circ$

$$\begin{aligned}a: 28.5 & \quad A: 65^\circ \\ b: 30.0 & \quad B: 107.44^\circ \\ c: ? & \quad C: ?\end{aligned}$$

$$C \approx 180^\circ - 65^\circ - 107.44^\circ \approx 7.56^\circ$$

$$\begin{aligned}\frac{\sin 65^\circ}{28.5} &\approx \frac{\sin 7.56^\circ}{c} \\ c \sin 65^\circ &\approx 28.5 \sin 7.56^\circ \\ c &\approx \frac{28.5 \sin 7.56^\circ}{\sin 65^\circ} \approx 4.1\end{aligned}$$

Thus, $C \approx 7.6^\circ$, $c \approx 4.1$.

We can summarize these two solutions in two tables.

Case 1

$$\begin{aligned}a: 28.5 & \quad A: 65^\circ \\ b: 30.0 & \quad B: 72.6^\circ \\ c: 21.2 & \quad C: 42.4^\circ\end{aligned}$$

Case 2

$$\begin{aligned}a: 28.5 & \quad A: 65^\circ \\ b: 30.0 & \quad B: 107.4^\circ \\ c: 4.1 & \quad C: 7.6^\circ\end{aligned}$$

Figure 6-2 shows the two triangles. The last figure shows both triangles together, where we can see why the ambiguous case was possible—with the given information ($a = 28.5$, $b = 30.0$, $A = 65^\circ$), side a could be in one of two positions, giving two possible triangles.

As a final check on our work, we observe that the sum of the angles in each case is 180° , and that in each case the longest side is opposite the largest angle and the shortest side is opposite the smallest angle. These are facts that should be true for any triangle.

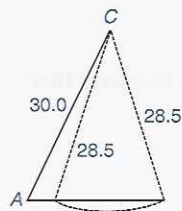
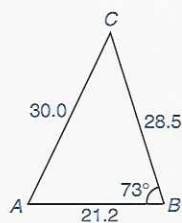


Figure 6-2

$$2. \ b = 51.2, \ c = 32.1, \ B = 6.1^\circ$$

$$a: ? \quad A: ?$$

Make a table of values

$$b: 51.2 \quad B: 6.1^\circ$$

$$c: 32.1 \quad C: ?$$

$$\frac{\sin A}{a} = \frac{\sin 6.1^\circ}{51.2} = \frac{\sin C}{32.1}$$

Fill values into the law of sines

$$\sin C = \frac{32.1 \sin 6.1^\circ}{51.2}$$

Using the last two ratios

$$C' \approx 3.82^\circ, \text{ which we will round to } 3.8^\circ \text{ in the final answer.}$$

$$C \approx 3.82^\circ \text{ or } 180^\circ - 3.82^\circ \approx 176.18^\circ.$$

Case 1: $C \approx 3.82^\circ$

$$a: ? \quad A: ?$$

$$b: 51.2 \quad B: 6.1^\circ$$

$$c: 32.1 \quad C: 3.82^\circ$$

$$A \approx 180^\circ - 3.82^\circ - 6.1^\circ = 170.08^\circ$$

$$\frac{\sin 170.08^\circ}{a} = \frac{\sin 6.1^\circ}{51.2}$$

Use the first two ratios

$$a = \frac{51.2 \sin 170.08^\circ}{\sin 6.1^\circ} \approx 83.00$$

This finishes case 1: $A \approx 170.08^\circ$, $a \approx 83.0$.**Case 2:** $C \approx 176.18^\circ$

$$a: ? \quad A: ?$$

$$b: 51.2 \quad B: 6.1^\circ$$

$$c: 32.1 \quad C: 176.18^\circ$$

$$A \approx 180^\circ - 176.18^\circ - 6.1^\circ = -2.28^\circ$$

Since an angle in a triangle cannot have negative measure this case does not produce a solution.

Thus, the solution to case 1 is the only solution.

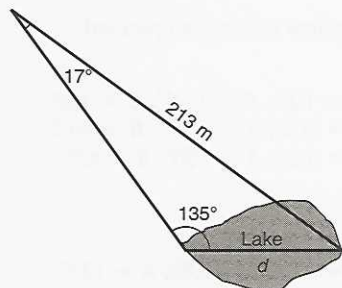
$$a: 83.0 \quad A: 170.1^\circ$$

$$b: 51.2 \quad B: 6.1^\circ$$

$$c: 32.1 \quad C: 3.8^\circ$$

The law of sines is used in many applications of trigonometry.

Example 6-1 C



Solve the following problems using the law of sines.

1. A surveyor made the measurements shown in the figure to measure the distance d across a lake. Find the distance to the nearest meter.

Using the law of sines, we know that

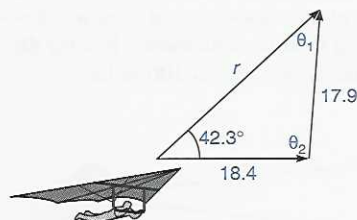
$$\frac{\sin 135^\circ}{213} = \frac{\sin 17^\circ}{d}$$

$$\frac{213d}{1} \cdot \frac{\sin 135^\circ}{213} = \frac{213d}{1} \cdot \frac{\sin 17^\circ}{d}$$

$$d \sin 135^\circ = 213 \sin 17^\circ$$

$$d = \frac{213 \sin 17^\circ}{\sin 135^\circ} \approx 88.1 \text{ meters}$$

Thus, to the nearest meter the distance across the lake is 88 meters.



2. The figure illustrates the following situation. A hang glider is flying at 18.4 mph pointed due east, with the wind blowing at 17.9 mph in the direction shown. The result is that the hang glider travels in a direction 42.3° north of east, with a ground speed of r mph. Assuming that θ_1 is acute, find the ground speed r and the direction of the wind, θ_2 .

Using the law of sines, we know

$$\frac{\sin 42.3^\circ}{17.9} = \frac{\sin \theta_1}{18.4} = \frac{\sin \theta_2}{r}$$

$$\sin \theta_1 = \frac{18.4 \sin 42.3^\circ}{17.9}$$

$$\theta_1 \approx 43.77^\circ$$

$$\theta_1 = \theta_1' \approx 43.77^\circ$$

θ_1 is an acute angle

$$\theta_2 \approx 180^\circ - 42.3^\circ - 43.77^\circ = 93.93^\circ$$

We can now find r .

$$\frac{\sin 42.3^\circ}{17.9} = \frac{\sin 93.93^\circ}{r}$$

$$r = \frac{17.9 \sin 93.93^\circ}{\sin 42.3^\circ} \approx 26.5$$

Thus, the hang glider is traveling with a ground speed of 26.5 mph, and the wind is blowing in the direction 93.9° north of west (or 3.9° east of north).

Mastery points

Can you

- State and use the law of sines to solve oblique triangles?
- Recognize and solve the ambiguous case when using the law of sines?

Exercise 6-1

In the following problems round answers to the same number of decimal places as the data, unless otherwise specified.

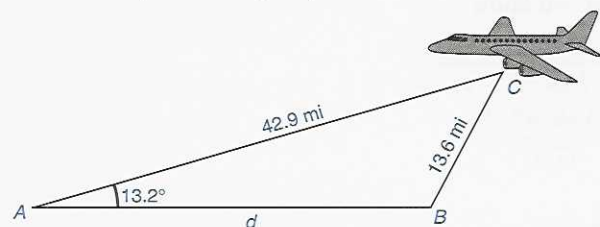
Solve the following oblique triangles using the law of sines.

1. $a = 12.5$, $A = 35^\circ$, $B = 49^\circ$
2. $b = 17.1$, $B = 100^\circ$, $C = 10^\circ$
3. $a = 1.25$, $B = 13.6^\circ$, $C = 132^\circ$
4. $c = 9.04$, $A = 51.6^\circ$, $B = 40.0^\circ$
5. $b = 92.5$, $A = 47^\circ$, $B = 100^\circ$
6. $c = 10.2$, $A = 16.7^\circ$, $B = 89.2^\circ$
7. $a = 0.452$, $A = 67.6^\circ$, $C = 91.8^\circ$
8. $b = 0.508$, $B = 13.1^\circ$, $C = 5.2^\circ$
9. $c = 5.00$, $A = 100^\circ$, $B = 45^\circ$
10. $a = 10.9$, $B = 76.9^\circ$, $C = 100^\circ$

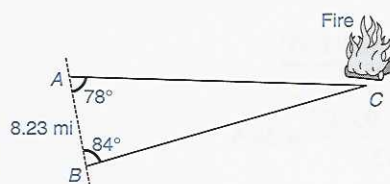
Solve the following oblique triangles using the law of sines.

11. $a = 12.5$, $b = 13.2$, $B = 49^\circ$
12. $b = 37.1$, $c = 21.3$, $B = 100^\circ$
13. $a = 4.25$, $c = 2.86$, $A = 132^\circ$
14. $c = 9.04$, $a = 21.3$, $C = 10.0^\circ$
15. $b = 92.5$, $c = 98.6$, $B = 43.7^\circ$
16. $c = 10.2$, $a = 16.7$, $A = 89.2^\circ$
17. $a = 4$, $b = 22$, $A = 30^\circ$
18. $a = 0.452$, $c = 0.606$, $C = 91.8^\circ$
19. $b = 6.35$, $c = 4.29$, $C = 42.3^\circ$
20. $b = 0.508$, $c = 1.09$, $C = 5.2^\circ$
21. $c = 5.00$, $b = 8.00$, $B = 45.0^\circ$
22. $a = 10.9$, $c = 16.9$, $C = 100.0^\circ$

23. Ground-based radar at point A determines that the angle of elevation to an aircraft 42.9 miles away is 13.2° . Radar at point B is on a straight line between a point on the ground directly below the aircraft and the radar at A and determines that the same aircraft is 13.6 miles away from point B . To the nearest 0.1 mile, find the distance from A to B . (See the diagram.)

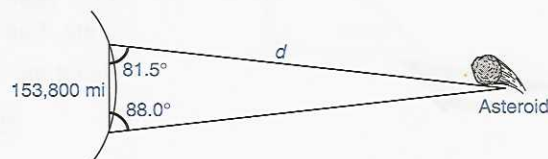


24. Two forest rangers sight a fire. Their reports are plotted on a map and yield the results shown in the diagram.

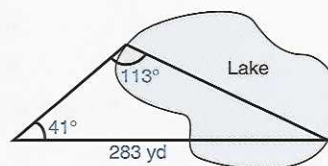


If the locations of the rangers are 8.23 miles apart, how far is the fire from Station A , to the nearest 0.1 mile?

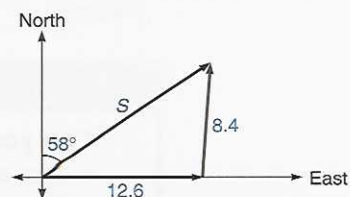
25. The diagram shows a situation in which astronomers made measurements at two locations of a new, slow-moving asteroid. Using their measurements, find the distance d to the asteroid, to the nearest 100 miles.



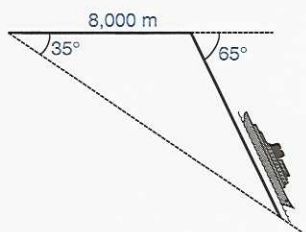
26. A surveyor made the measurements shown in the diagram. Find the distance across the lake, to the nearest foot.



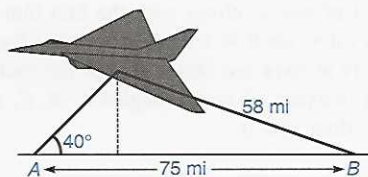
27. The diagram illustrates a situation in which a ship is moving at 12.6 knots heading due east. It is moving through a current moving east of north at 8.4 knots. The result is that the ship is moving at an angle of 58° east of north. Find the true speed S of the ship.



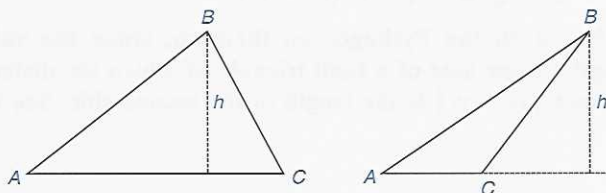
28. A ship travels due east to a point 8,000 meters from its starting point. It then turns toward the south through an angle of 65° and proceeds until it crosses a line of sight from the starting position to itself that is 35° south of east. At this point, how far is the ship from its starting point, to the nearest 10 meters?



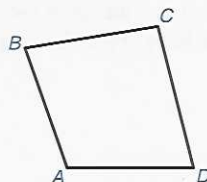
29. Two cities are 75 miles apart. An aircraft that is between the two cities is being tracked from radar in each city. City A's radar shows that the aircraft is at an angle of elevation of 40° ; city B's radar shows that the slant distance of the plane to city B is 58 miles and that the angle of elevation there is less than 40° . What is the slant distance d from the plane to city A, to the nearest mile?



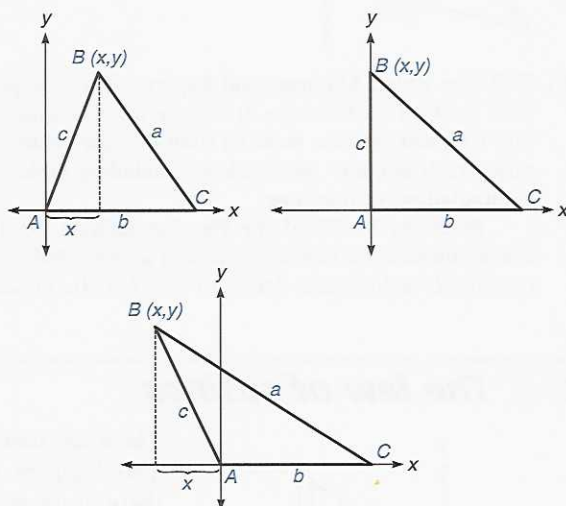
30. Find the height of the aircraft in problem 29, to the nearest 100 feet (1 mile = 5,280 feet). Use the diagram for help.
31.  Show that in any oblique triangle ABC , if h is the altitude of the triangle relative to side b , then an expression for h is $h = \frac{b \cdot \tan A \cdot \tan C}{\tan A + \tan C}$. Use the diagrams for help.



32. Quadrilateral $ABCD$ is shown in the diagram. $AB = 17.3$, $AD = 18.9$, angle $A = 110^\circ$, angle $ABD = 52^\circ$, angle $BDC = 41^\circ$, angle $C = 93^\circ$. Find the length of CD to the nearest 0.1. (Hint: Draw diagonal BD .)



33.  Show that in any triangle ABC , (1) $a = b \cos C + c \cos B$, (2) $b = c \cos A + a \cos C$, and (3) $c = a \cos B + b \cos A$. (Hint: The diagram shows angle A in standard position when the angle is acute, right, or obtuse.)



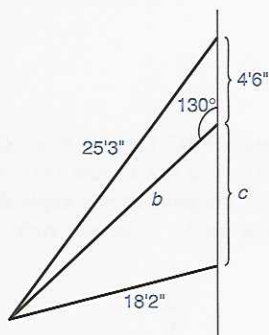
Show why the statements

$$\cos A = \frac{x}{c} \text{ and } \cos C = \frac{b-x}{a}$$

are true in each case, and then use them to show that (2) is true.

34. An Army observation point is 325 yards northeast (i.e., 45° north of east) of a second point. At this point a tank is sighted on a line of sight 37° south of east. The same tank is sighted at the second point along a line of sight 18° north of east. To the nearest yard, how far is the tank from the first observation point?

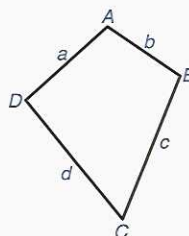
35.  Recall that the formula for the area of a triangle is $\frac{1}{2}bh$ (one half the product of the base and height). Use the formula to show that the area of any triangle ABC is $\frac{1}{2}bc \sin A$.
36. The figure shows three wires that are attached from a common point to the side of a building. Find the lengths of b and c , to the nearest inch.



37.  The Rhind Mathematical Papyrus is an Egyptian work on mathematics. It dates to the sixteenth century B.C., and contains material from the nineteenth century B.C. It contains 84 problems, including tables for manipulations of fractions.

Problems 51–53 of the Papyrus include the following formula for finding the area of a four-sided figure like $ABCD$ in the figure: $\frac{1}{2}(a + c) \times \frac{1}{2}(b + d)$. (Observe

that $\frac{1}{2}(a + c)$ is the average length of the two sides a and c ; the same is true for $\frac{1}{2}(b + d)$.) This formula is inaccurate. Except for rectangles, it gives an answer that is too large. It was used for the purpose of taxing land, which shows that there is not always an economic incentive to get the correct answer.



Problem 35 shows that the area of any triangle ABC is $\frac{1}{2}bc \sin A$. It is also $\frac{1}{2}ac \sin B$, or $\frac{1}{2}ab \sin C$. Geometrically, this is one-half the product of two sides and the sine of the angle between those two sides.

- a. Use this result to show that the area of the four-sided figure can be described by

$$\frac{1}{4}(ab \sin A + ad \sin D + bc \sin B + cd \sin C)$$

- b. Use the result of part a, along with the fact that for $0 < \theta < 180^\circ$, $0 < \sin \theta < 1$ to show that the Egyptian formula is always too large, except for rectangles, when it is exact. (Assume angles A , B , C , and D are all less than 180° .)

6-2 The law of cosines

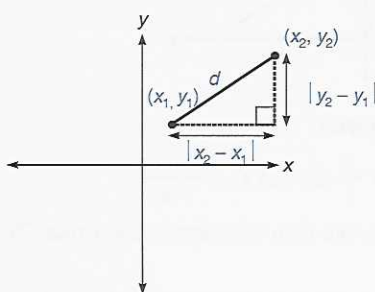


Figure 6-3

There are some oblique triangles that cannot be solved using the law of sines. This happens when we do not know any of the three ratios completely. In these cases we use the *law of cosines*. To develop this law, we will use the **distance formula** of analytic geometry: If (x_1, y_1) and (x_2, y_2) are two points in the x - y coordinate plane, then the distance d between them is defined as

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This definition is based on the Pythagorean theorem, since the value $|x_2 - x_1|$ is the length of one side of a right triangle of which the distance d is the hypotenuse, and $|y_2 - y_1|$ is the length of the second side. See figure 6-3.

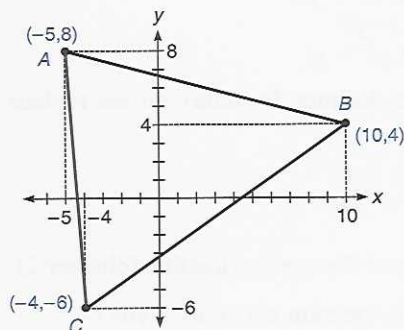
■ Example 6-2 A

1. Find the distance between the points $(-4, 7)$ and $(2, 10)$.

Using $(x_1, y_1) = (-4, 7)$ and $(x_2, y_2) = (2, 10)$, we have

$$\begin{aligned} d &= \sqrt{[2 - (-4)]^2 + (10 - 7)^2} \\ &= \sqrt{36 + 9} \\ &= \sqrt{45} \\ &= \sqrt{9 \cdot 5} \\ &= 3\sqrt{5} \end{aligned}$$

Note The points could have been used in the reverse order with the same result. That is, $(x_1, y_1) = (2, 10)$ and $(x_2, y_2) = (-4, 7)$.



2. Assume that an optics table is coordinatized in the usual rectangular coordinate system. Three mirrors are located on the table at points $A(-5, 8)$, $B(10, 4)$, and $C(-4, -6)$. Find the distance traversed by a laser beam traveling from A to B to C and back to A , both exactly and to the nearest 0.1. See the figure.

We use the distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. To find the distance from A to B , we can let A be the first point and B be the second. Then the formula becomes

$$AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

We fill in the given values.

$$\begin{aligned} AB &= \sqrt{[10 - (-5)]^2 + (4 - 8)^2} \\ &= \sqrt{15^2 + (-4)^2} \\ &= \sqrt{241} \approx 15.52 \text{ (to two decimal places)} \end{aligned}$$

To find the distance from A to C , we can let A be the first point and C be the second.

$$\begin{aligned} AC &= \sqrt{(x_C - x_A)^2 + (y_C - y_A)^2} \\ &= \sqrt{[(-4) - (-5)]^2 + [(-6) - 8]^2} \\ &= \sqrt{1^2 + (-14)^2} \\ &= \sqrt{197} \approx 14.04 \end{aligned}$$

To find the distance from B to C , we can let B be the first point and C be the second.

$$\begin{aligned} BC &= \sqrt{(x_C - x_B)^2 + (y_C - y_B)^2} \\ &= \sqrt{[(-4) - 10]^2 + [(-6) - 4]^2} \\ &= \sqrt{(-14)^2 + (-10)^2} \\ &= \sqrt{296} = \sqrt{4(74)} = 2\sqrt{74} \approx 17.20 \end{aligned}$$

The exact total distance is

$$\begin{aligned} AB + AC + BC &= \sqrt{241} + \sqrt{197} + 2\sqrt{74} \text{ (exactly)} \\ &\approx 46.8 \end{aligned}$$

We now use the distance formula in our development of the law of cosines.

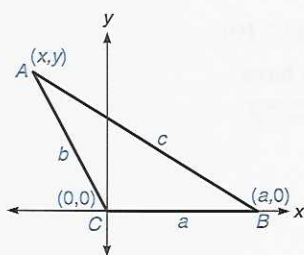


Figure 6-4

Let $\triangle ABC$ be any triangle. Put angle C in standard position, and call (x,y) the point at vertex A , as illustrated in figure 6-4.

Note The figure shows C as an obtuse angle. The algebraic statements that follow do not use this fact, however. Thus, they would also apply if C were acute or right.

Now apply the distance formula to distance c .

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Use $(x_2, y_2) = (a, 0)$ and $(x_1, y_1) = (x, y)$.

$$\begin{aligned} c &= \sqrt{(a - x)^2 + (0 - y)^2} \\ c^2 &= (a - x)^2 + (-y)^2 \\ &= a^2 - 2ax + x^2 + y^2 \end{aligned}$$

We know that $x^2 + y^2 = b^2$ (also by the distance formula), so we replace $x^2 + y^2$ in the above equation by b^2 .

$$\begin{aligned} c^2 &= a^2 - 2ax + b^2 \\ &= a^2 + b^2 - 2ax \end{aligned}$$

We know that $\cos C = \frac{x}{b}$ by the definition of the cosine function (chapter 2).

Thus, $x = b \cos C$, and we replace x in the equation above by $b \cos C$.

$$c^2 = a^2 + b^2 - 2ab \cos C$$

The equation $c^2 = a^2 + b^2 - 2ab \cos C$ is called the **law of cosines** for angle C . If we had put angle A or angle B in standard position, we would have arrived at two other versions of this law. All three versions are:

The law of cosines

Law of cosines for angle A : $a^2 = b^2 + c^2 - 2bc \cos A$

Law of cosines for angle B : $b^2 = a^2 + c^2 - 2ac \cos B$

Law of cosines for angle C : $c^2 = a^2 + b^2 - 2ab \cos C$

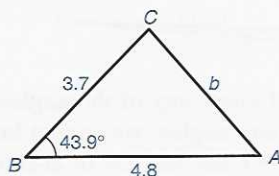
Concept

The law of cosines states that in any triangle the square of the length of one side equals the sum of the squares of the lengths of the other two sides less twice the product of these lengths and the cosine of the angle opposite the first side.

When solving oblique triangles, the law of cosines should be used whenever the law of sines cannot be used.

■ **Example 6-2 B**

1. Solve oblique triangle ABC if $a = 3.7$, $c = 4.8$, and angle $B = 43.9^\circ$. See the figure.



Observe that we do not know any of the three ratios of the law of sines completely: we know the length of side a but not the measure of angle A , the length of side c but not the measure of angle C , and the measure of angle B but not the length of side b . This indicates that we must use the law of cosines to help solve the problem.

Since we know angle B , we use the form of the law of cosines in which angle B appears:

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac \cos B \\ &= 3.7^2 + 4.8^2 - 2(3.7)(4.8)(\cos 43.9^\circ) \\ b &= \sqrt{3.7^2 + 4.8^2 - 2(3.7)(4.8)(\cos 43.9^\circ)} \quad \boxed{\text{CS 1}} \\ &\approx 3.337068251 \dots \end{aligned}$$

(which we will round off to 3.337 for now and to 3.3 in the final answer).

Now we use the law of sines to find one of the angles, either A or C . It is best to find angle A first, because we know angle A is not the largest angle in the triangle (angle C is) and, therefore, angle A must be acute. This will eliminate the ambiguous case.

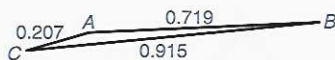
$$\begin{aligned} \frac{3.7}{\sin A} &\approx \frac{3.337}{\sin 43.9^\circ} \\ 3.337(\sin A) &\approx 3.7(\sin 43.9^\circ) \\ \sin A &\approx \frac{3.7(\sin 43.9^\circ)}{3.337} \\ A' &\approx 50.2^\circ \end{aligned}$$

(remember, A' is the reference angle for angle A). We know that A is acute, so we do not have to worry about the supplement of A (which we normally do whenever we find an angle using the law of sines), so $A \approx 50.2^\circ$. Also, $C = 180^\circ - A - B \approx 180^\circ - 50.2^\circ - 43.9^\circ \approx 85.9^\circ$. Since we now know all three angles and all three sides of the triangle, we are done.

$$\begin{aligned} A &\approx 50.2^\circ, B = 43.9^\circ, C \approx 85.9^\circ \\ a &= 3.7, b = 3.3, c = 4.8 \end{aligned}$$

Note As illustrated in the last example, it is never necessary to use the law of cosines more than once to solve a triangle. We complete the solution with the law of sines.

2. Solve oblique triangle ABC if $a = 0.915$, $b = 0.207$, and $c = 0.719$. See the figure.



We do not know any of the angles, so we cannot use the law of sines. (Without any angles, we cannot know any of the three ratios in the law of sines.) We use the law of cosines to find one of the angles.

It is best to find the largest angle first; this will be angle A since it is opposite the longest side a . The reason for finding the largest angle first is explained in the note below. We must use the form of the law of cosines that includes angle A .

$$a^2 = b^2 + c^2 - 2bc \cos A$$

We often solve for $\cos A$ before we use the law of cosines.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$2bc \cos A = b^2 + c^2 - a^2$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Substituting values, we get

$$\begin{aligned}\cos A &= \frac{0.207^2 + 0.719^2 - 0.915^2}{2(0.207)(0.719)} \\ &\approx -0.9320. \\ A &\approx 158.74^\circ\end{aligned}$$

CS 2

Now we know the measure of the largest angle: $A \approx 158.74^\circ$. We can now use the law of sines to find another angle. We must use the ratio

$\frac{a}{\sin A}$ since A is the only angle measure we know. To find angle B we use

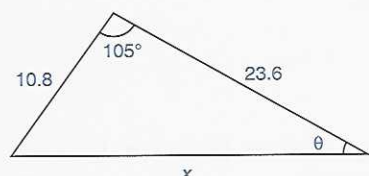
$$\frac{0.915}{\sin 158.74^\circ} \approx \frac{0.207}{\sin B}$$

to find that

$$B' \approx 4.70^\circ.$$

Since we have already found the largest angle A , we know that the remaining two angles are acute, and so we do not have to worry about the supplement of B' ; thus, $B \approx 4.70^\circ$ and $C \approx 180^\circ - 158.7^\circ - 4.7^\circ = 16.6^\circ$. The triangle is solved because we now know all three sides (which were given) and all three angles. $A \approx 158.7^\circ$, $B \approx 4.7^\circ$, $C \approx 16.6^\circ$.

Note There is no ambiguous case for the law of cosines because the range of the inverse cosine function includes all angles from 0° to 180° . If an angle is not acute, its cosine will be negative (as above), and then we know the angle is between 90° and 180° . Thus, *when finding an angle of a triangle using the law of cosines (as above), it is best to find the largest angle first*. If the largest angle is obtuse we will find this out directly; either way, the remaining angles are acute and, therefore, easy to find with the law of sines.



3. A sheet metal worker must make the triangular pattern shown in the figure. Find the angle of inclination θ and the length of side x .

Using the law of cosines, we see that

$$\begin{aligned} x^2 &= 10.8^2 + 23.6^2 - 2(10.8)(23.6) \cos 105^\circ \\ &\approx 805.54 \\ x &\approx 28.38 \end{aligned}$$

We can now use the law of sines to find θ .

$$\begin{aligned} \frac{28.38}{\sin 105^\circ} &\approx \frac{10.8}{\sin \theta} \\ 28.38 \sin \theta &\approx 10.8 \sin 105^\circ \\ \sin \theta &\approx \frac{10.8 \sin 105^\circ}{28.38} \approx 0.3676 \\ \theta' &\approx 21.6^\circ \end{aligned}$$

θ is acute because it is not the largest angle in the triangle, so $\theta = 21.6^\circ$.

4. Find the measure of angle ACB in part 2 of example 6-2 A to the nearest 0.1° . Recall that we have a triangle whose vertices were the points $A(-5, 8)$, $B(10, 4)$, and $C(-4, -6)$. See the figure.

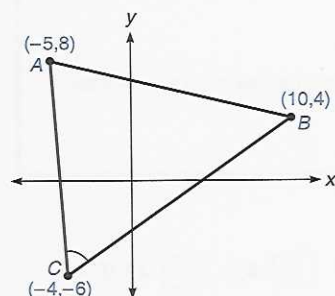
In part 2 of example 6-2 A we used the distance formula to establish

$$\begin{aligned} AB &= \sqrt{241} \approx 15.52 \\ AC &= \sqrt{197} \approx 14.04 \\ BC &= 2\sqrt{74} \approx 17.20 \end{aligned}$$

To find angle ACB , we use the law of cosines with angle C .

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos C \\ \cos C &= \frac{a^2 + b^2 - c^2}{2ab} \\ &= \frac{(2\sqrt{74})^2 + (\sqrt{197})^2 - (\sqrt{241})^2}{2(2\sqrt{74})(\sqrt{197})} \\ &= \frac{296 + 197 - 241}{4(\sqrt{74})(\sqrt{197})} \approx 0.5218 \\ C &\approx 58.5^\circ \end{aligned}$$

CS 3



■ Example 6-2 C

In triangle ABC , $a = 4$, $b = 6$, $c = 12$. Solve the triangle.

We must use the law of cosines to solve this triangle. If we wish to find the largest angle, which is C , we use

$$\begin{aligned}\cos C &= \frac{a^2 + b^2 - c^2}{2ab} \\ &= \frac{4^2 + 6^2 - 12^2}{2(4)(6)} \\ &= \frac{-92}{48} \\ &\approx -1.9167\end{aligned}$$

Note that this result shows that there is no angle C , since $|\cos x| \leq 1$ for any value of x . Thus, there is no solution to this triangle. ■

Example 6-2 C illustrates a situation in which the law of cosines shows us that there is no solution. There is no solution because $a + b < c$. It is a fact that any two sides of a triangle must have a total length greater than the third side. This is often called the **triangle inequality**. Any time we are given the lengths of three sides of a triangle, it is a good idea to check that the sum of the lengths of any two sides is greater than the length of the remaining side.

Mastery points

Can you

- State the three forms of the law of cosines?
- Use the law of cosines to solve triangles when the law of sines cannot be used?
- Use the distance formula?

Calculator steps

- $\textcircled{A}$ 3.7 x^2 $+$ 4.8 x^2 $-$ 2 $\times$ 3.7 $\times$ 4.8 $\times$ 43.9
 COS $=$ $\sqrt{x}$
 $\textcircled{P}$ 3.7 x^2 4.8 x^2 $+$ 2 ENTER 3.7 $\times$ 4.8 $\times$ 43.9
 COS $\times$ $-$ $\sqrt{x}$
 TI-81 $\sqrt{}$ (3.7 x^2 $+$ 4.8 x^2 $-$ 2 $\times$ 3.7 $\times$ 4.8
 COS 43.9 $)$ ENTER

2. (A) $.207 \times^2 + .719 \times^2 - .915 \times^2 = \div 2 \div .207 \div .719 = \text{INV} \text{COS}$
 (P) $.207 \times^2 .719 \times^2 + .915 \times^2 - 2 \div .207 \div .719 \div \text{COS}^{-1}$
 TI-81 $\text{cos}^{-1} ((.207 \times^2 + .719 \times^2 - .915 \times^2) \div (2 \times .207 \times .719)) \text{ENTER}$
3. (A) $296 + 197 - 241 = \div 4 \div 74 \sqrt{x} \div 197 \sqrt{x} = \text{INV} \text{COS}$
 (P) $296 \text{ENTER} 197 + 241 - 4 \div 74 \sqrt{x} \div 197 \sqrt{x} \div \text{COS}^{-1}$
 TI-81 $\text{cos}^{-1} ((296 + 197 - 241) \div (4 \sqrt{74} \sqrt{197})) \text{ENTER}$

Exercise 6-2

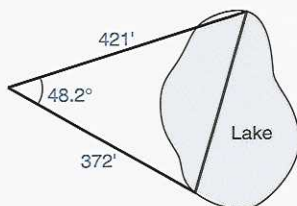
Find the distance between the following points.

1. $(3, -4), (-2, 6)$
2. $(3, -1), (4, -8)$
3. $(-5, -2), (10, -1)$
4. $(-2, -3), (-4, -1)$
5. $(0, -3), (-9, -12)$
6. $(10, -1), (2, 0)$

Solve the following oblique triangles. You will have to use the law of cosines as the first step. Round your answers to the same number of decimal places as the data.

7. $a = 3.2, b = 5.9, C = 39.4^\circ$
8. $a = 4.9, b = 3.2, C = 78.2^\circ$
9. $b = 61.3, c = 23.9, A = 124.0^\circ$
10. $b = 123.0, c = 89.4, A = 19.5^\circ$
11. $a = 31.4, c = 17.0, B = 100.3^\circ$
12. $a = 67.25, c = 13.56, B = 76.30^\circ$
13. $a = 23.5, b = 19.4, c = 35.0$
14. $a = 61.7, b = 80.0, c = 102.0$
15. $a = 0.214, b = 0.399, c = 0.500$
16. $a = 1.03, b = 0.98, c = 1.75$
17. $a = 13.2, b = 5.9, C = 139.4^\circ$
18. $a = 14.9, b = 13.2, C = 45.0^\circ$
19. $b = 61.3, c = 43.9, A = 24.5^\circ$
20. $b = 23.9, c = 89.4, A = 79.5^\circ$
21. $a = 30.0, c = 20.0, B = 112.0^\circ$
22. $a = 6.72, c = 1.55, B = 76.35^\circ$
23. $a = 235, b = 194, c = 354$
24. $a = 61.7, b = 80.0, c = 42.0$
25. $a = 0.21, b = 0.49, C = 1.50^\circ$
26. $a = 10.0, b = 13.9, c = 17.5$

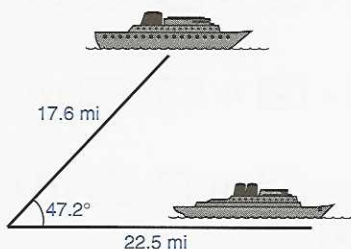
27. A surveyor made the measurements shown in the diagram to calculate the distance across a lake.



Compute the distance to the nearest 0.1 foot.

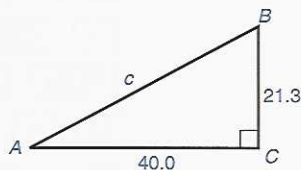
28. Calculate the measure of the smallest angle in a triangle whose sides have measure 22.1 cm, 32.6 cm, and 40.5 cm.
29. Calculate the measure of the largest angle in a triangle whose sides have measure 12.3 in., 16.2 in., and 19.0 in.
30. A numerically controlled laser cloth cutter is being set up to cut a triangular pattern. The vertices of the triangle are at $A(2, 5)$, $B(4, 8)$, and $C(5, 12)$.
 - a. Find the length of side AB .
 - b. Determine the measure of the three angles to the nearest 0.1° .

31. In the same situation as in problem 30, the vertices of another piece of triangular cloth are determined to be at $A(0,5)$, $B(2,3)$, and $C(8,4)$. Determine the measure of the largest of the three angles A , B , or C , to the nearest 0.1° .
32. Two ships are being tracked by radar. One ship is determined to be 17.6 miles from the radar, while the second is 22.5 miles from the radar. The lines of sight from the radar to the two ships form an angle of 47.2° . (See the diagram.) Find the distance between the two ships to the nearest 0.1 mile.

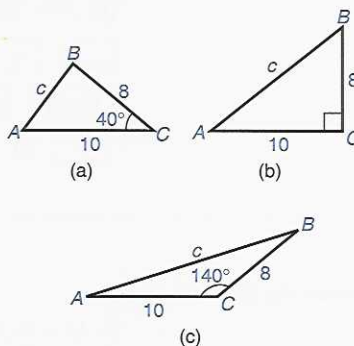


33. In the situation described in problem 32, what would be the angle formed by the two lines of sight to the ships if the ships were 31.5 miles apart?
34. A ship leaves a harbor heading due east and travels 17.3 km. It then turns north through a 33° angle and travels for another 22.0 km. How far is the ship from its starting point, to the nearest kilometer?
35. A plane takes off and travels southeast (45° south of east) for 27 miles, then turns due south and travels for 16 miles. How far is it from its starting position, to the nearest mile?
36. The points $(5,3)$, $(-2,1)$, and $(1,-4)$ form a triangle. Find the measure of the smallest angle in this triangle, to the nearest 0.1° .
37. Find the measure of the largest angle in the triangle of problem 36, to the nearest 0.1° .
38. Two observers are 87 meters apart, and a range finder shows that a certain building is 111 meters from one observer and 114 meters from the other. What is the angle formed by the two lines of sight from the building to the observers, to the nearest degree?

39. The triangle in the diagram is a right triangle. Can the law of cosines be used to find the length of c ? Compare using the law of cosines to using the Pythagorean theorem.



40. The diagram shows three triangles in which angle C is (a) acute, (b) right, and (c) obtuse.



If we use the Pythagorean theorem and apply it to the right triangle in (b) and apply the law of cosines to angle C in all three situations, the resulting equations are

$$c^2 = a^2 + b^2 \text{ and} \\ c^2 = a^2 + b^2 - 2ab \cos C$$

We can view the expression $-2ab \cos C$ as a “correction factor” to the Pythagorean theorem that will give the correct result, even when angle C is not 90° .

Solve for side c in each case, using the law of cosines, and discuss how the correction factor “knows” when to increase the length of side c (as in [c]) and when to decrease it (as in [a]).

6-3 Vectors

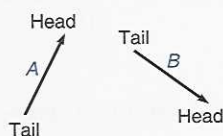


Figure 6-5

If we know that a plane flying over a certain spot is flying at 100 mph, we cannot tell where it will be in 1 hour without also knowing its direction. This combination of speed and direction is called *velocity*. Many other natural phenomena are described by a magnitude and direction: forces of all types, accelerations, alternating voltage in electricity theory are all examples. A conceptual tool used to describe two such pieces of information is the **vector**. It is noteworthy that the concept of vectors extends into an area of mathematics called *linear algebra*, which has applications in every field of knowledge, from physics and economics to medicine and sociology.

We imagine a vector as a directed line segment. That is, a finite portion of a straight line that is considered to be pointing in one direction. Our representation of vectors would commonly be called “arrows.” Examples of vectors are shown in figure 6-5. Observe that we use the terms **head** and **tail** to describe the “end” and “beginning” of a vector, respectively, and that we often use capital letters, such as A and B , to name a vector.

One way to describe a vector is by specifying its length and direction. The length is called the **magnitude** of the vector, and for a vector A we denote its magnitude by $|A|$ (the same notation as that for absolute value of a real number). The **direction** of a vector is specified by an angle, θ , usually specified in degrees, measured in the same way as angles in standard position. When we specify a vector this way we say it is in **polar form**.

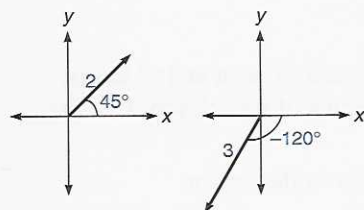


Figure 6-6

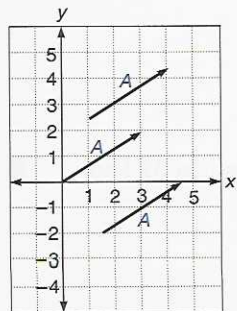


Figure 6-7

Polar form of a vector

A vector A in polar form is the ordered pair $A = (|A|, \theta_A)$

$|A|$ is the magnitude of vector A ; $|A| \geq 0$.

θ_A is the direction of vector A .

Note The polar form of a vector is not unique; all coterminal values of θ_A are equivalent.

Figure 6-6 illustrates the vectors $(2, 45^\circ)$ and $(3, -120^\circ)$.

A vector can also be specified by using its *relative rectangular coordinates*. For a vector in standard position this is equivalent to specifying the coordinates of its head (see figure 6-8). If a vector is not in standard position, its relative rectangular coordinates are relative to the coordinates of the vector's tail. We refer to these coordinates as describing the **rectangular form** of a vector.

Rectangular form of a vector

A vector A in rectangular form is $A = (A_x, A_y)$, where A_x is called the **horizontal component** of the vector and A_y is called the **vertical component** of the vector.

Figure 6-7 illustrates the vector $A(3, 2)$, shown in three different positions.

Converting polar form to rectangular form

There is a useful relation for converting the polar form of a vector to its rectangular form. If we recall the definitions of section 2-2 for an angle in standard position, we can see the following.

To convert polar form to rectangular form

Given vector $A = (|A|, \theta_A) = (A_x, A_y)$,

$$A_x = |A| \cos \theta_A \quad \text{and} \quad A_y = |A| \sin \theta_A$$

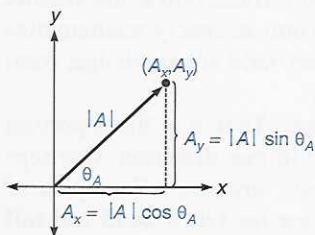
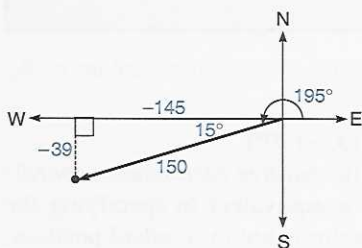
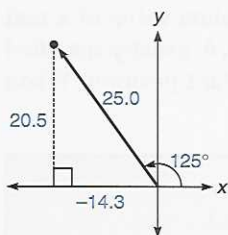


Figure 6-8

Example 6-3 A



This is true because, by the definitions of section 5-2, $\cos \theta_A = \frac{A_x}{|A|}$ and $\sin \theta_A = \frac{A_y}{|A|}$. Figure 6-8 illustrates this relationship.

Example 6-3 A illustrates converting a vector in polar to form to its rectangular form.

Convert from polar to rectangular form.

1. Convert the polar form of the vector to the rectangular form (approximate to the nearest tenth). $A = (25.0, 125^\circ)$.

$$A_x = |A| \cos \theta_A = 25.0 \cos 125^\circ \approx -14.3$$

$$A_y = |A| \sin \theta_A = 25.0 \sin 125^\circ \approx 20.5$$

Thus, the rectangular form is $(-14.3, 20.5)$ (see the figure).

2. An aircraft is moving in the direction 15° south of west at 150 knots. Find the east-west and north-south components of its velocity V to the nearest knot and interpret the results.

As we see in the figure the aircraft's velocity is the vector $V = (150, 195^\circ)$. The east-west component is

$$\begin{aligned} V_x &= |V| \cos \theta_V \\ &= 150 \cos 195^\circ \approx -145 \end{aligned}$$

The north-south component is

$$\begin{aligned} V_y &= |V| \sin \theta_V \\ &= 150 \sin 195^\circ \approx -39 \end{aligned}$$

Thus, the aircraft is moving west at 145 knots and south at 39 knots. ■

Converting rectangular form to polar form

When we convert from rectangular to polar form we always give the direction of the vector θ_V so that it has the smallest possible absolute value. This means *we will choose θ_V so that $-180^\circ < \theta_V \leq 180^\circ$* . One reason for doing this is that this is the result obtained from electronic calculators (see the discussion following example 6-3 B).

Examining figure 6-8 shows that, for a given vector $V = (V_x, V_y) = (|V|, \theta_V)$, $|V| = \sqrt{V_x^2 + V_y^2}$, and $\tan \theta'_V = \frac{V_y}{V_x}$ if $V_x \neq 0$. The angle $\theta'_V = \tan^{-1} \frac{V_y}{V_x}$ is only the reference angle and θ_V depends on the quadrant in which the vector occurs.

Reference angles obtained with the inverse tangent function fall in the range $-90^\circ < \theta' < 90^\circ$ (quadrant I and quadrant IV). If $V_x > 0$ this is the value of θ_V , since that angle should be in quadrant I or quadrant IV.

If $V_x < 0$, θ_V can be obtained by adding or subtracting 180° to or from θ'_V . If $\theta'_V < 0$, add 180° , and if $\theta'_V > 0$, subtract 180° . This can be incorporated into a rule as follows.

To convert rectangular form to polar form

Given vector $V = (V_x, V_y) = (|V|, \theta_V)$. Then,

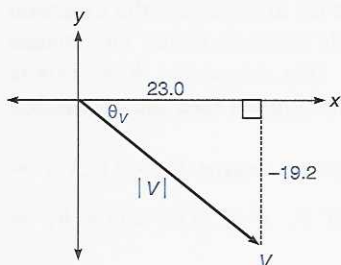
$$|V| = \sqrt{V_x^2 + V_y^2}, \theta'_V = \tan^{-1} \frac{V_y}{V_x} \text{ if } V_x \neq 0, \text{ and}$$

$$\theta_V = \begin{cases} \theta'_V & \text{if } V_x > 0 \\ \theta'_V - 180^\circ & \text{if } \theta'_V > 0 \\ \theta'_V + 180^\circ & \text{if } \theta'_V < 0 \end{cases} \quad \text{if } V_x < 0$$

Note If $V_x = 0$ then θ_V is 90° if $V_y > 0$, and -90° if $V_y < 0$. This is clear from a sketch of the vector.

The examples and the exercises will make clear why the rule works when $V_x < 0$. Example 6-3 B illustrates finding the polar form of a vector when its rectangular form is known.

Example 6-3 B



Convert to polar form.

1. The horizontal component of a force vector is 23.0 pounds to the right; its vertical component is 19.2 pounds down. Find the force vector V , to the nearest 0.1 pound.

The vector is $(23.0, -19.2)$ in rectangular form. We can find $|V|$ using the Pythagorean theorem:

$$\begin{aligned}|V|^2 &= 23.0^2 + (-19.2)^2 \\ |V| &\approx 30.0\end{aligned}$$

We now find θ'_V the reference angle for θ_V .

$$\begin{aligned}\theta'_V &= \tan^{-1}\left(\frac{-19.2}{23.0}\right) \approx -39.9^\circ \\ \theta_V &= \theta'_V, \quad V_x > 0\end{aligned}$$

Thus, $V \approx (30.0, -39.9^\circ)$.

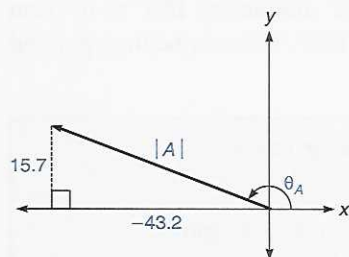
2. Vector $A = (-43.2, 15.7)$. Find the polar form of A .

$$\begin{aligned}|A| &= \sqrt{A_x^2 + A_y^2} \\ &= \sqrt{(-43.2)^2 + (-15.7)^2} \approx 46.0 \quad \text{Nearest tenth}\end{aligned}$$

$$\theta'_A = \tan^{-1}\left(\frac{15.7}{-43.2}\right) \approx -20.0^\circ$$

$$\theta_A \approx -20.0^\circ + 180^\circ \approx 160.0^\circ \quad A_x < 0, \theta'_A < 0$$

Thus, $A \approx (46.0, 160.0^\circ)$.



Using special calculator keys

Most engineering/scientific calculators are programmed to perform the conversions of examples 6-3 A and 6-3 B. These calculators have keys marked “ $R \rightarrow P$ ” or simply “ $\rightarrow P$ ” (rectangular to polar conversion) and “ $P \rightarrow R$ ” or “ $\rightarrow R$ ” (polar to rectangular conversion), or something equivalent. The results are stored in locations referred to as x and y . Typical keystrokes are illustrated here. Part 1 of example 6-3 A would be done as follows. (The TI-81 is discussed below.)

$$A = (25.0, 125^\circ)$$

$$25 \quad [P \rightarrow R] \quad 125 \quad [=]$$

$$\text{Display: } [-14.33941091]$$

$$[x \leftrightarrow y]$$

$$\text{Display: } [20.47880111]$$

Thus $A \approx (-14.3, 20.5)$.

Part 1 of example 6-3 B would be done in the following way.

$$V = (23.0, -19.2)$$

$$23 \quad [R \rightarrow P] \quad 19.2 \quad [+/-] \quad [=]$$

$$\text{Display: } [29.96064085]$$

$$[x \leftrightarrow y]$$

$$\text{Display: } [-39.855454183]$$

Thus, $V \approx (30.0, -39.9^\circ)$.

The TI-81 uses the values X , Y , R , θ . (Y , R , θ are $\boxed{\text{ALPHA}}$ 1, $\boxed{\times}$, and 3, respectively.) It also uses the two MATH functions “ $\boxed{R} \rightarrow \boxed{P}$ ” (Rectangular to polar) and “ $\boxed{P} \rightarrow \boxed{R}$ ” (Polar to rectangular).

Example 6-3 A, part 1:

$\boxed{\text{MODE}}$ $\boxed{\text{Deg}}$ $\boxed{\text{ENTER}}$ Make sure the calculator is in degree mode.

$\boxed{\text{MATH}}$ 2 25 $\boxed{\text{ALPHA}}$ $\boxed{.}$ 125 $\boxed{)}$

$\boxed{\text{ENTER}}$

Display: $\boxed{-14.33941091}$

$\boxed{\text{ALPHA}}$ 1 $\boxed{\text{ENTER}}$

Display: $\boxed{20.47880111}$

Example 6-3 B, part 1:

$\boxed{\text{MATH}}$ 1 23 $\boxed{\text{ALPHA}}$ $\boxed{.}$ $\boxed{(-)}$ 19

$\boxed{.}$ 2 $\boxed{)}$ $\boxed{\text{ENTER}}$

Display: $\boxed{29.96064085}$

$\boxed{\text{ALPHA}}$ 3 $\boxed{\text{ENTER}}$

Display: $\boxed{-39.855454183}$

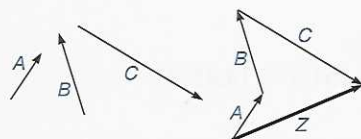


Figure 6-9

Addition of vectors

It has been shown experimentally that most natural phenomena that are described by vectors combine as if they were connected tail to head in a series. The result is a vector with its tail at the tail of the first vector in the series, and its head at the head of the last vector in the series. The resulting vector is called the **resultant vector**. This is illustrated in figure 6-9, where vectors A , B , and C combine into the resultant vector Z . This idea is the basis for the definition of addition of vectors which we develop here.

Example 6-3 C illustrates that this process is equivalent to summing all the horizontal components and, separately, the vertical components. This is the basis for the definition of the addition of two vectors.

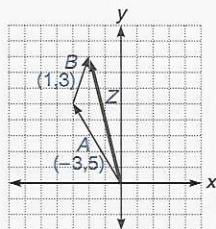
Vector sum

Let Z be the resultant (vector sum) of two vectors $A = (A_x, A_y)$ and $B = (B_x, B_y)$. Then we say $Z = A + B$, where

$$Z_x = A_x + B_x \quad \text{and} \quad Z_y = A_y + B_y$$

Observe that this definition describes how to add two vectors whose rectangular form is known. When vectors are known in polar form they must first be converted to rectangular form. This is illustrated in example 6-3 C.

Example 6-3 C



Find the vector sum of the given vectors.

1. $A = (-3, 5)$ and $B = (1, 3)$

$$Z_x = A_x + B_x = -3 + 1 = -2$$

$$Z_y = A_y + B_y = 5 + 3 = 8$$

$$Z = (-2, 8)$$

This is illustrated in the figure.

2. $A = (2, -\frac{1}{2})$, $B = (-6, 3\frac{1}{2})$, $C = (3, -2)$

$$Z_x = A_x + B_x + C_x = 2 + (-6) + 3 = -1$$

$$Z_y = A_y + B_y + C_y = -\frac{1}{2} + 3\frac{1}{2} + (-2) = 1$$

$$Z = (-1, 1)$$

3. $V = (13.8, 40.2^\circ)$, $W = (20.9, 164.6^\circ)$

Note that these vectors are in polar form. See the figure.

$$Z_x = V_x + W_x$$

$$= |V| \cos \theta_V + |W| \cos \theta_W$$

$$= 13.8 \cos 40.2^\circ + 20.9 \cos 164.6^\circ \approx -9.6093$$

$$Z_y = V_y + W_y$$

$$= |V| \sin \theta_V + |W| \sin \theta_W$$

$$= 13.8 \sin 40.2^\circ + 20.9 \sin 164.6^\circ \approx 14.4574$$

Thus, $Z \approx (-9.6, 14.5)$ (to nearest tenth).

We should find Z in polar form, since V and W were given in this form.

$$Z = \sqrt{Z_x^2 + Z_y^2} = \sqrt{(-9.6093)^2 + 14.4589^2} \approx 17.4 \quad (\text{nearest } 0.1)$$

$$\theta'_Z = \tan^{-1} \frac{Z_y}{Z_x} \approx \tan^{-1} \left(\frac{14.4589}{-9.6093} \right) \approx -56.4^\circ$$

$$\theta_Z = \theta'_Z + 180^\circ \approx 123.6^\circ \quad \text{Since } Z_x < 0, \text{ and } \theta'_Z \text{ is negative, we add } 180^\circ$$

Thus, in polar form, $Z \approx (17.4, 123.6^\circ)$.

4. Three forces are acting on a point, where F_1 is 25 pounds acting in the direction 30° , F_2 is 40 pounds acting in the direction 100° , and F_3 is 50 pounds acting in the direction -40° . Find the resultant force acting on the point.

Find the resultant of the three vectors $A(25, 30^\circ)$, $B(40, 100^\circ)$, and $C(50, -40^\circ)$.

$$Z_x = F_{1x} + F_{2x} + F_{3x}$$

$$= 25 \cos 30^\circ + 40 \cos 100^\circ + 50 \cos(-40^\circ) \approx 53.01$$

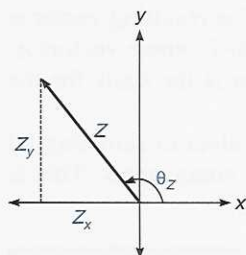
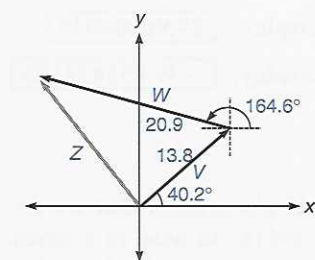
$$Z_y = F_{1y} + F_{2y} + F_{3y}$$

$$= 25 \sin 30^\circ + 40 \sin 100^\circ + 50 \sin(-40^\circ) \approx 19.75$$

$$|Z| = \sqrt{Z_x^2 + Z_y^2}$$

$$= \sqrt{53.01^2 + 19.75^2} \approx 56.6 \text{ pounds}$$

$$\tan \theta'_Z = \frac{Z_y}{Z_x} = \frac{19.75}{53.01}; \theta'_Z \approx 20.4^\circ$$



Since Z_x and Z_y are both positive, the resultant is in the first quadrant, so $\theta_Z = \theta'_Z \approx 20.4^\circ$.

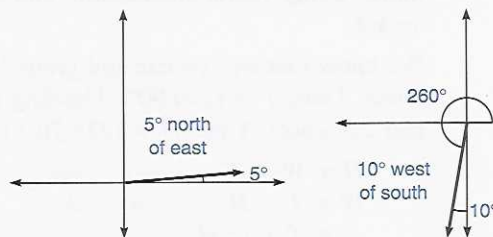
Thus, $Z \approx (56.6, 20.4^\circ)$. ■

 Problem 76 in the exercises presents a program that adds two or more vectors given in polar form.

Example 6-3 D illustrates a general principle of navigating aircraft (and, analogously, ships at sea). The speed of the aircraft relative to the air is called the airspeed, and the direction in which the aircraft is pointed is its heading. The airspeed and heading combine into the heading vector, H . The wind vector, W , is the speed and direction of the wind. If we add these two vectors we get the true course and ground speed of the plane, $T = H + W$. T is the speed and direction relative to the earth's surface.

■ Example 6-3 D

An aircraft is flying with heading 5° north of east and airspeed 123 knots. The wind is blowing at 32 knots in the direction 10° west of south. Find the true course and ground speed of the aircraft (see the figure).



$$H = (123, 5^\circ); W = (32, 260^\circ)$$

$$T = H + W$$

$$T_x = H_x + W_x$$

$$= |H| \cos \theta_H + |W| \cos \theta_W$$

$$= 123 \cos 5^\circ + 32 \cos 260^\circ \approx 116.98$$

$$T_y = H_y + W_y$$

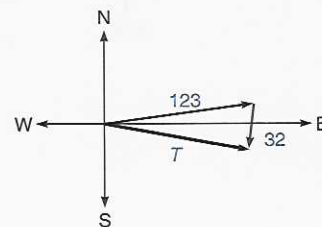
$$= |H| \sin \theta_H + |W| \sin \theta_W$$

$$= 123 \sin 5^\circ + 32 \sin 260^\circ \approx -20.79$$

$$\theta'_T = \tan^{-1} \frac{T_y}{T_x} = \tan^{-1} \left(\frac{-20.79}{116.98} \right) \approx -10^\circ$$

$$\theta_T = \theta'_T \text{ since } T_x > 0, |T| = \sqrt{T_x^2 + T_y^2} = \sqrt{116.98^2 + (-20.79)^2} \approx 119.$$

Thus, the ground speed of the aircraft is 119 knots and the true course is 10° south of east. ■



To nearest degree

The zero vector and the opposite of a vector

It is useful to define a zero vector and the opposite of a vector. The zero vector is defined so its length is zero. The direction does not matter. The opposite of a vector is defined to have equal length but opposite direction. We do this by adding or subtracting 180° . Both definitions are in terms of polar form.

Zero vector

The vector $0 = (0, \theta)$, where θ is any angle, is the zero vector.

Opposite of a vector

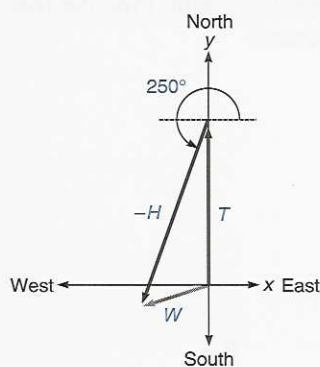
Given a vector $V = (|V|, \theta_V)$, then $-V$ means its opposite, and $-V = (|V|, \theta_V \pm 180^\circ)$.

In computing the opposite we generally choose whichever value, $\theta_V + 180^\circ$ or $\theta_V - 180^\circ$, has the smallest absolute value. It is easy to show that for any vector V ,

$$V + (-V) = 0$$

Example 6-3 E illustrates uses of the zero vector and the opposite of a vector.

■ **Example 6-3 E**



1. An aircraft's on-board inertial navigation computer shows that the aircraft is traveling due north at 200 knots with respect to the ground, and that the aircraft is headed 20° east of north with an airspeed of 225 knots. Using vector subtraction, find the wind vector W . Interpret this vector.

We know that true course and ground speed T are due north and 200 knots. Thus, $T = (200, 90^\circ)$. Heading and airspeed are 20° east of north and 225 knots. Thus, $H = (225, 70^\circ)$ (see the figure).

$$H + W = T \quad \text{Aircraft heading} + \text{wind} = \text{true course}$$

$$W = T - H \quad \text{Solve}^1 \text{ for } W$$

$$= T + (-H)$$

$$W = (200, 90^\circ) + (225, 250^\circ) \quad -H = (225, 70^\circ + 180^\circ)$$

$$W_x = 200 \cos 90^\circ + 225 \cos 250^\circ \approx -76.95$$

$$W_y = 200 \sin 90^\circ + 225 \sin 250^\circ \approx -11.43$$

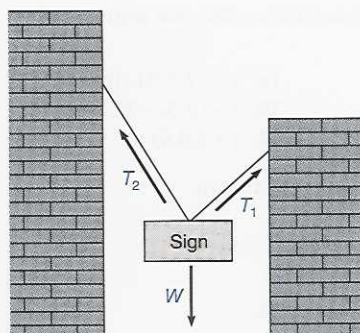
$$|W| = \sqrt{W_x^2 + W_y^2} \approx 77.80$$

$$\theta'_W \approx \tan^{-1} \left(\frac{-11.43}{-76.95} \right) \approx -8.4^\circ$$

$$\theta_W = \theta'_W - 180^\circ \approx -8.4^\circ + 180^\circ \quad W_x < 0 \text{ and } \theta'_W > 0 \\ \approx -171.6^\circ$$

Thus, $W \approx (77.8, -171.6^\circ)$. This tells us that the wind is blowing in a direction 8.4° south of west at 77.8 knots.

¹We are assuming we can solve a vector-valued equation as we solve real-valued equations. In fact, it can be proved that this is valid.



2. A large sign is suspended between two buildings by two wires, as in the figure. One wire acts at an angle of 45° above the horizontal and has a tension (force) T_1 , of 400 pounds. If the sign weighs 800 pounds (vector W), compute a vector that describes T_2 , the tension and direction of the second wire, to the nearest unit.

We use a fact from physics to describe the situation. Since the sign is motionless, all the forces acting on it must be balanced, or add to zero. Thus, we proceed as follows.

$$T_1 + T_2 + W = 0 \quad \text{All forces balanced}$$

$$T_2 = -T_1 - W \quad \text{Solve for } T_2$$

$$T_1 = (400, 45^\circ), \text{ so } -T_1 = (400, 45^\circ + 180^\circ) = (400, 225^\circ)$$

$$W = (800, 270^\circ), \text{ so } -W = (800, 270^\circ - 180^\circ) = (800, 90^\circ)$$

$$T_{2x} = -T_{1x} + (-W_x)$$

$$= 400 \cos 225^\circ + 800 \cos 90^\circ \approx -282.84$$

$$T_{2y} = -T_{1y} + (-W_y)$$

$$= 400 \sin 225^\circ + 800 \sin 90^\circ \approx 517.16$$

$$|T_2| = \sqrt{(-282.84)^2 + 517.16^2} \approx 589 \text{ pounds.}$$

$$\theta'_{T_2} \approx \tan^{-1} \left(\frac{517.16}{-282.84} \right) \approx -61.3^\circ$$

$$\theta_{T_2} = 180^\circ + \theta'_{T_2} \quad T_{2x} < 0, \theta'_{T_2} < 0$$

$$\approx 118.7^\circ$$

Thus, to the nearest unit $T_2 \approx (589, 119^\circ)$. ■

Mastery points

Can you

- Find the horizontal and vertical components of a vector?
- Find the magnitude and direction of a vector when given the horizontal and vertical components?
- Add or subtract vectors?
- Apply vectors to navigation and force problems?

Exercise 6-3

Convert each vector from its polar to its rectangular form. Leave all answers to the nearest tenth unless the reference angle is 30° , 45° , or 60° .

1. $(40, 30^\circ)$

2. $(15.2, 33.6^\circ)$

3. $(100.0, 122.3^\circ)$

4. $(4.2, 97.3^\circ)$

5. $(10.0, 200.0^\circ)$

6. $(18, 120^\circ)$

7. $(25, 300^\circ)$

8. $(82.0, 341.9^\circ)$

9. $(6, -45^\circ)$

10. $(5.9, 59.2^\circ)$

11. $(7.8, -264.3^\circ)$

12. $(20.0, -333.0^\circ)$

Convert each vector to polar form. Round to the nearest tenth unless an exact form is possible (if the reference angle is 30° , 45° , or 60°).

- | | | | |
|----------------------|-------------------|------------------------------|----------------------|
| 13. $(3.0, 4.0)$ | 14. $(31.2, 6.9)$ | 15. $(-3.0, 5.2)$ | 16. $(-12.5, 31.0)$ |
| 17. $(\sqrt{3}, -2)$ | 18. $(3, -3)$ | 19. $(5, -10)$ | 20. $(-12.5, -20.3)$ |
| 21. $(-6.8, 3.4)$ | 22. $(-8, 4)$ | 23. $(-\sqrt{2}, -\sqrt{8})$ | 24. $(-1, 0.4)$ |

Add the following vectors. Leave the resultant in rectangular form. Round the resultant to the nearest tenth.

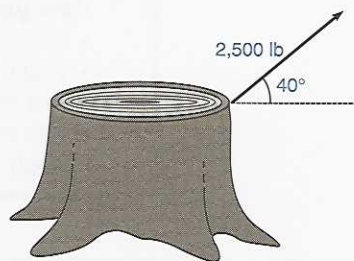
- | | |
|---|---|
| 25. $(-3, 8)$, $(2, 12)$ | 26. $(4, 3\frac{1}{2})$, $(-1, -2\frac{1}{2})$ |
| 27. $(\sqrt{2}, 5)$, $(\sqrt{8}, 1)$, $(\sqrt{50}, -6)$ | 28. $(5, -13)$, $(-\frac{1}{2}, 8)$, $(7\frac{1}{2}, -1)$ |

Add the following vectors. Leave the resultant in polar form. Round the resultant to the nearest tenth.

- | | |
|--|--|
| 29. $(30.0, 30^\circ)$, $(15.2, 33.6^\circ)$ | 30. $(100, 0^\circ)$, $(4.2, 97.3^\circ)$ |
| 31. $(10.0, 200^\circ)$, $(29.3, 250^\circ)$ | 32. $(13.7, 300.0^\circ)$, $(82.0, 341.9^\circ)$ |
| 33. $(3.2, -45.0^\circ)$, $(5.9, -59.2^\circ)$ | 34. $(37.9, -100.5^\circ)$, $(69.2, -170.1^\circ)$ |
| 35. $(41.9, -213.0^\circ)$, $(7.7, -264.3^\circ)$ | 36. $(20.0, -333.0^\circ)$, $(39.2, -359.0^\circ)$ |
| 37. $(3.5, 19.2^\circ)$, $(2.7, 83.1^\circ)$, $(4.3, 145.7^\circ)$ | 38. $(7.1, 13.8^\circ)$, $(6.2, 131^\circ)$, $(10.4, 215^\circ)$ |
| 39. $(15.3, 311^\circ)$, $(20.9, 117^\circ)$, $(13.2, 83^\circ)$ | 40. $(3.2, 19.5^\circ)$, $(5.1, 45.0^\circ)$, $(6.0, 180^\circ)$ |
| 41. $(3.5, -25^\circ)$, $(6.8, 25^\circ)$, $(4.2, 50^\circ)$ | 42. $(25, -30^\circ)$, $(25, -60^\circ)$, $(25, -100^\circ)$ |

43. An aircraft is moving in the direction 25° north of west at 150 knots. Find the east-west and north-south components of its velocity to the nearest knot and interpret the results.
44. An aircraft is moving in the direction 35° east of south at 120 knots. Find the east-west and north-south components of its velocity to the nearest knot and interpret the results.
45. An aircraft is moving in the direction 15° north of east at 200 knots. Find the east-west and north-south components of its velocity to the nearest knot and interpret the results.
46. A rocket is climbing with a speed of 825 knots and an angle of climb of 58.6° . (The angle of climb is the angle measured from the ground to its flight path.) Find the horizontal and vertical components of the rocket's velocity, to the nearest knot.
47. An aircraft is traveling in a direction 30° west of north. Its speed is 456 knots. Find the east-west and north-south components of its velocity, to the nearest knot. (Remember that 30° west of north corresponds to the angle 120° .)
48. At the location of a ship, the Gulf Stream ocean current is moving to the northwest at a speed of 8.2 knots. Find the east-west and north-south components of its velocity, to the nearest knot.

49. A ship has left an east-coast harbor and has been sailing in a direction 32° north of east for 2.5 hours, at a speed of 18 knots. (a) How far north of the harbor has it gone, to the nearest nautical mile? (b) How far east of the harbor has it gone, to the nearest nautical mile?
50. A force is acting on a tree stump at a 40° angle of elevation. See the diagram. If the force is 2,500 pounds, find its vertical and horizontal components, to the nearest pound.



51. A 2,250 pound force is pulling on a sled loaded with lumber, at an angle of elevation of 33° . If the sled will not move until the horizontal component of the force exceeds 1,900 pounds, will the sled move?
52. A sled loaded with lumber will not move until the horizontal component of the applied force is 1,200 pounds or more. If a winch being used to move the sled can apply a maximum force of 1,700 pounds, what is the largest angle of elevation at which the winch can act on the sled and move it?

53. Consider a force of 1,000 pounds acting at an angle of elevation of 15° on a point.
- Compute the horizontal and vertical components of the force.
 - Double the force to 2,000 pounds and recompute the horizontal and vertical components. Do they double also?
 - Double the angle of elevation to 30° (keep the force at 1,000 pounds). Recompute the horizontal and vertical components. Do they double also?
54. A plane is flying over Minneapolis with a ground speed of 200 miles per hour and true course due east. After 1 hour it turns to a true course of 60° south of east, maintaining the same ground speed. After flying for an additional half-hour, the navigator notes its position on a map. How far and in what direction is the plane from Minneapolis, to the nearest unit?
55. A plane is flying over Orlando with ground speed 135 miles per hour, and true course 23° north of east. After 1 hour it turns to a true course of 40° south of west, maintaining the same ground speed. After flying for an additional hour the navigator notes its position on a map. How far and in what direction is the plane from Orlando, to the nearest unit?
56. Two forces are acting on a point, 12.6 pounds in the direction 123° and 15.8 pounds in the direction 211° . Compute the magnitude and direction of the resultant force to the nearest tenth.
57. Two forces are acting on a point, 2.6 newtons in the direction 18.3° and 15.8 newtons in the direction -86.2° . Compute the magnitude and direction of the resultant force, to the nearest tenth.
58. Three forces are acting on a point: 27.6 newtons in the direction 18.3° , 32.1 newtons at 223.0° , and 46.8 newtons at -30.0° . Find the resultant force acting on the point, to the nearest 0.1 newton.
59. Three forces are acting on a point: 199 pounds at 19.0° , 175 pounds at 131.0° , and 96 pounds at 130.0° . Find the resultant force acting on the point, to the nearest 0.1 pound.
60. A ship leaves its harbor traveling 10° north of east. After 1 hour it turns to the direction 40° south of east. After 2 more hours, it turns to the direction 15° west of south. The ship travels for 1 half hour more and then stops. The ship has maintained a steady speed of 16 knots (nautical miles per hour) for the entire trip. How many nautical miles, and in what direction, is the ship from its starting position, to the nearest knot?
61. A ship leaves its harbor traveling 15° west of south. After 1 hour it turns to the direction 34° south of west. After 2 more hours, it turns to the direction 10° north of west. The ship travels for 1 half hour more and then stops. The ship has maintained a steady speed of 20 knots for the entire trip. How many nautical miles, and in what direction, is the ship from its starting position, to the nearest knot?
62. An aircraft is flying with an airspeed of 123 knots and a heading of 30° west of north. The wind is blowing in the direction 15° south of west at 26 knots. Add the heading and wind vectors to find the aircraft's true course and ground speed, to the nearest integer.
63. If the wind in problem 62 now shifts to 35 knots in the direction 10° west of south, find the aircraft's new true course and ground speed, to the nearest integer.
64. A ship is traveling through an ocean current that flows in the direction 5° east of north at 7.2 knots. The ship's heading is 10° north of west, and its speed relative to the water is 19.6 knots. Add these two vectors to find the ship's true course and speed, to the nearest tenth.
65. A ship is traveling through an ocean current that flows in the direction 15° west of north at 7.2 knots. The ship's heading is 10° north of east, and its speed relative to the water is 26.1 knots. Add these two vectors to find the ship's true course and speed, to the nearest tenth.
66. The voltage in an alternating current circuit adds vectorially. If one voltage E_1 is 122 volts with phase angle 30° and a second voltage E_2 is 86 volts with phase angle 21° , find the magnitude and phase angle of the resultant voltage E_T to the nearest unit.
67. (Refer to problem 66.) In an AC circuit E_1 is 240 volts at -45° and E_2 is 115 volts at $+45^\circ$. Find the resultant E_T to the nearest unit.
68. An aircraft has a ground speed of 135 knots and true course 35° east of north. If its heading is due north and airspeed is 120 knots, find the direction and speed of the wind, to the nearest unit.
69. An aircraft has a ground speed of 80 knots and true course 15° west of north. If the wind is directly from the northeast at 12 knots, find the plane's heading and airspeed, to the nearest unit.
70. A ship is traveling at 14 knots, relative to the water, with heading 20° south of west. If the true speed and direction of the ship is 12 knots due west, find the speed and direction of the ocean current, to the nearest tenth.

71. The ocean current in a certain area is 6.4 knots with direction 8° east of south. A ship in the area is traveling with true direction of 12 knots at 25° east of north. Find the ship's heading and speed relative to the water, to the nearest tenth.
72. Two cables support a 1-ton (2,000 pound) sign between two buildings. One of the cables has a tension of 1,500 pounds and acts at an angle of 33° above the horizontal. The other cable is attached to the other building. Find the tension in the other cable as well as its direction relative to the horizontal, to the nearest unit.
73. Two cables support a sign between two buildings. The tension and direction of one cable is 456 pounds at 63° above the horizontal. If the sign weighs 650 pounds, find the tension in the other cable, as well as its direction relative to the horizontal, to the nearest unit.

74.  Prove that vector addition is commutative. That is, if A and B are vectors, then $A + B = B + A$. (Hint: Real number addition is commutative, and the horizontal and vertical components of a vector are real values.)
75.  Prove that vector addition is associative. That is, if A , B , and C are vectors, then $(A + B) + C = A + (B + C)$. (Hint: Real number addition is associative, and the horizontal and vertical components of a vector are real values.)
76.  Write a program for a computer or programmable calculator that will add two or more vectors when given in polar form.

Chapter 6 summary

- **The law of sines**

In any triangle ABC , $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

- **The ambiguous case** When the law of sines is applied to a situation in which only one of the two angles of a triangle is known, it is possible to get two different solutions to the problem. This is called the ambiguous case.

- **The law of cosines** For any triangle ABC ,

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

- **Polar form of a vector** A vector A in polar form is the ordered pair $A = (|A|, \theta_A)$.

$|A|$ is the magnitude of vector A ; $|A| \geq 0$.

θ_A is the direction of vector A .

- **Rectangular form of a vector** A vector A in rectangular form is $A = (A_x, A_y)$, where A_x is the horizontal component of the vector and A_y is the vertical component of the vector.

- **To convert polar form to rectangular form**

Given vector $A = (|A|, \theta_A) = (A_x, A_y)$,

$$A_x = |A| \cos \theta_A$$

$$A_y = |A| \sin \theta_A$$

- **To convert rectangular form to polar form**

Given vector $V = (V_x, v_y) = (|V|, \theta_V)$. Then,

$$|V| = \sqrt{V_x^2 + V_y^2}, \theta_V = \tan^{-1} \frac{V_y}{V_x} \text{ if } V_x \neq 0, \text{ and}$$

$$\theta_V = \begin{cases} \theta_V' & \text{if } V_x > 0 \\ \theta_V' - 180^\circ & \text{if } \theta_V' > 0 \\ \theta_V' + 180^\circ & \text{if } \theta_V' < 0 \end{cases} \text{ if } V_x < 0$$

- **Vector sum** Let $Z = (Z_x, Z_y)$ be the resultant (vector sum) of two vectors $A = (A_x, A_y)$ and $B = (B_x, B_y)$. Then, $Z = A + B$, and $Z_x = A_x + B_x$ and $Z_y = A_y + B_y$.

- With regard to an aircraft, $T = H + W$, where

T is the vector of ground speed and true course

H is the vector of airspeed and aircraft heading

W is the vector of the speed and direction of the wind.

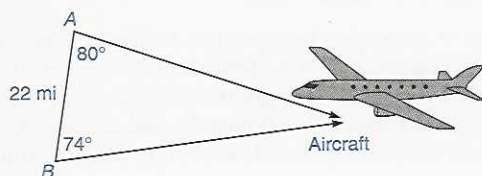
- **Zero vector** The vector $0 = (0, \theta)$, where θ is any angle, is the zero vector.

- **Opposite of a vector** Given a vector $V = (|V|, \theta_V)$, then $-V = (|V|, \theta_V \pm 180^\circ)$.

Chapter 6 review

[6-1] Solve the following oblique triangles using the law of sines. Round answers to the nearest tenth.

- $a = 10.6$, $A = 47.9^\circ$, $B = 10.3^\circ$
- $b = 3.55$, $B = 23.8^\circ$, $C = 5.2^\circ$
- $a = 10.0$, $b = 13.0$, $B = 79.0^\circ$
- $a = 12.6$, $c = 7.0$, $C = 32.7^\circ$
- A lost pilot is given a position report by triangulation with two radar sites. The situation is shown in the diagram. Find the distance of the aircraft from radar site A, to the nearest mile.



[6-2] Solve the following oblique triangles, to the nearest tenth. You will have to use the law of cosines as the first step.

- $a = 4.1$, $b = 6.8$, $C = 29.4^\circ$
 - $b = 60.0$, $c = 20.0$, $A = 92.1^\circ$
 - $a = 21.4$, $c = 27.0$, $B = 112^\circ$
 - $a = 43.5$, $b = 17.8$, $c = 35.0$
 - $a = 31.7$, $b = 80.0$, $c = 105$
 - A technician is setting up a numerically controlled grinding machine. A triangular pattern is to be ground and, therefore, must be coordinatized. The vertices of the triangle are at $A(-2,5)$, $B(4,7)$, and $C(5,-2)$. Solve the resulting triangle. Round answers to the nearest 0.1.
 - A ship leaves a harbor heading due west and travels 23.3 km. It then turns north through a 63° angle and travels for another 10.0 km. How far is the ship from its starting point, to the nearest kilometer?
- [6-3]
- Convert the vector $(27.2, 29.0^\circ)$, to rectangular form. Round to the nearest 0.1.
 - The horizontal and vertical components of a vector are 19.6 and 30.5, respectively. Find the magnitude and direction of the vector.

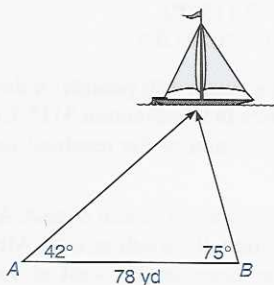
- A rocket climbs with a speed of 450 knots and an angle of climb of 34.6° . (The angle of climb is the angle measured from the ground to its flight path.) Find the horizontal and vertical components of the rocket's velocity, to the nearest knot.
- A force is acting on a cart at a 13.5° angle of elevation (13.5° measured from the ground up to the force vector). If the force is 256 pounds, find the vertical and horizontal components, to the nearest pound.

Add the following vectors. Round the resultant to the nearest tenth.

- $(33.0, 14.7^\circ)$, $(15.2, 33.6^\circ)$
- $(10.2, 112.3^\circ)$, $(4.2, 19.3^\circ)$
- $(3.5, 29.2^\circ)$, $(1.7, 43.1^\circ)$, $(4.3, 115.0^\circ)$
- $(7.1, 13.8^\circ)$, $(6.6, 142.0^\circ)$, $(11.9, 215.0^\circ)$
- Two forces are acting on a point, 126 pounds in the direction 223° and 158 pounds in the direction 311° . Compute the magnitude and direction of the resultant force, to the nearest unit.
- A ship leaves its harbor traveling 15° south of east. After 1 hour it turns to the direction 40° south of east. After 2 more hours it turns to the direction 25° west of south. The ship travels for 1 half-hour more and then stops. The ship has maintained a steady speed of 18 knots for the entire trip. How many nautical miles and in what direction is the ship from its starting position, to the nearest integer?
- An aircraft is flying with an airspeed of 195 knots and heading of 24° west of north. The wind is blowing in the direction 25° west of south at 19 knots. Add the heading and wind vectors to find the aircraft's true course and ground speed, to the nearest integer.
- An aircraft's true course and ground speed are 120° at 225 knots. The wind is blowing in a direction 15° south of east at 30 knots. Find the heading and airspeed of the aircraft, to the nearest unit.

Chapter 6 test

1. In triangle ABC , $b = 22.6$, $A = 13.5^\circ$, and $C = 82.1^\circ$. Solve this triangle. Round answers to the nearest tenth.
2. In triangle ABC , $b = 22.6$, $c = 24.0$, and $C = 62.1^\circ$. Solve this triangle. Round answers to the nearest tenth.
3. In triangle ABC , $a = 25.9$, $c = 16.2$, and $B = 100^\circ$. Solve this triangle. Round answers to the nearest tenth.
4. In triangle ABC , $a = 2.55$, $b = 3.12$, and $c = 4.00$. Solve this triangle. Round answers to the nearest tenth.
5. The distance to a boat on a lake is being found by triangulation from two points on shore. The situation is shown in the diagram. Find the distance to the boat from site B , to the nearest yard.
6. Given the three points $A(6,8)$, $B(-3,5)$, and $C(10,-4)$, find the angle formed by line segments AB and BC , to the nearest 0.1° .
7. Convert the vector $(2, 30^\circ)$ to rectangular form. Leave the answer in exact form.
8. The horizontal and vertical components of a vector are 4.0 and 5.0. Find the magnitude and direction of the vector, to the nearest tenth.
9. Add the vectors $(5.4, 19.0^\circ)$ and $(8.0, 123^\circ)$. Round the resultant to the nearest tenth.
10. A sign is suspended between two buildings. One cable from which the sign is suspended has a tension of 535 pounds and acts at an angle of elevation of 62° . If the weight of the sign is 1,000 pounds, find the tension and angle of elevation in the other cable, to the nearest unit.





Complex Numbers and Polar Coordinates

In this chapter we examine more applications of the trigonometric functions. We first define complex numbers, and then show how trigonometry is useful in representing these numbers as well as computing with them.

We then introduce polar coordinates, a system of coordinates that is useful in describing many situations where periodic motion is involved.

7-1 Complex numbers

Complex numbers were developed hundreds of years ago, but they became much more important about 100 years ago when they began to be used to model physical phenomena, particularly in the study of electricity. Their development and use centers on the number i .

The i stands for the word *imaginary*; and represents $\sqrt{-1}$. We learned in the past that the square root of a negative number is not defined since, for example, if $i = \sqrt{-1}$, then $i^2 = -1$, and we know that the square of any real number is positive or zero. Nevertheless, equations like $x^2 = -1$ produce the “solution” $\sqrt{-1}$. To the mathematicians of the sixteenth and later centuries these numbers were bothersome. They keep occurring when solving equations, yet, to these people, such numbers could not exist. René Descartes coined the term “imaginary” to describe solutions to equations that involved the square roots of negative numbers, and this term has persisted to this day. (In fact, the term “real” number was coined in reaction to the word “imaginary.”)

It turned out, however, that the study of certain physical laws almost demanded the existence of $\sqrt{-1}$, and today the existence of a number such as i is no longer doubted. Since the property $i^2 = -1$ is incompatible with the properties of the real number system, a new number system was introduced to incorporate these imaginary numbers. This system is the complex number system, and we now study its properties.

The square root of -1

$$i = \sqrt{-1}$$

Rectangular form of a complex number

A **complex number** is an expression of the form $a + bi$, where a and b are real numbers. a is called the *real part*, and b is called the *imaginary part* of the number. $a + bi$ is called the **rectangular form** of a complex number.

$2 + 3i$ is an example of a complex number; 2 is its real part and 3 is its imaginary part.

Example 7-1 A

Identify the real and imaginary parts of the following complex numbers:

- a. $5 + 7i$ b. $2 - i$ c. 4 d. $-6i$

- a. Real part: 5; imaginary part: 7.
 b. Real part: 2; imaginary part: -1 . $2 - i = 2 + (-1)i$
 c. If we represent 4 as $4 + 0i$, we see that the real part is 4 and the imaginary part is 0.
 d. Represent $-6i$ as $0 - 6i$ to see that the real part is 0 and the imaginary part is -6 . A complex number in which the real part is zero is usually called a pure imaginary number. ■

Complex conjugate

$a - bi$ is called the **complex conjugate** of $a + bi$.

Example 7-1 B

Form the complex conjugate of each complex number.

1. $5 + 7i$
 $5 - 7i$ is the complex conjugate of $5 + 7i$.
2. $4 - 5i$
 $4 + 5i$ is the complex conjugate of $4 - 5i$.
3. 7
 We rewrite 7 as $7 + 0i$, whose complex conjugate is $7 - 0i$ or 7.
 Thus, 7 is the complex conjugate of 7.
4. $-9i$
 We rewrite $-9i$ as $0 - 9i$, whose complex conjugate is $0 + 9i$ or $9i$. Thus, $9i$ is the complex conjugate of $-9i$. ■

An algebra exists for the complex numbers. Rules have been established for the addition, subtraction, multiplication, and division of complex numbers. These operations obey the rules of the real number system if we treat i as a

symbol having the property $i^2 = -1$. The definitions of these operations are given and illustrated in the following paragraphs. First, however, we must define what it means for two complex numbers to be equal.

Equality of complex numbers

The complex numbers $a + bi$ and $c + di$ are *equal* if and only if $a = c$ and $b = d$.

The complex numbers are not ordered; that is, a given complex number is never said to be greater than or less than another complex number.

We now define and illustrate addition and subtraction.

Addition and subtraction of complex numbers

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Concept

To add or subtract two complex numbers, add or subtract the real parts and the imaginary parts separately.

■ Example 7-1 C

Perform the indicated operations and simplify.

1. $(5 + 4i) + (3 + 2i)$

$$(5 + 4i) + (3 + 2i) = 5 + 3 + 4i + 2i = 8 + 6i$$

2. $(5 + 4i) - (3 + 2i)$

$$(5 + 4i) - (3 + 2i) = 5 - 3 + 4i - 2i = 2 + 2i$$

Multiplication of complex numbers

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Concept

This definition is equivalent to our usual rules for the multiplication of real number expressions if we replace i^2 by -1 .

$$(a + bi)(c + di) \quad \text{Multiply as if real expressions}$$

$$= ac + adi + bci + bdi^2$$

$$= ac + adi + bci - bd \quad \text{Remember that } i^2 = -1$$

$$= (ac - bd) + (ad + bc)i$$

■ Example 7-1 D

Perform the indicated operations and simplify.

1. $(5 + 4i)(3 + 2i)$

$$\begin{aligned} (5 + 4i)(3 + 2i) &= 15 + 10i + 12i + 8i^2 \\ &= 15 + 22i + 8(-1) \\ &= 15 + 22i - 8 \\ &= 7 + 22i \end{aligned}$$

2. $(-2 + 5i)(3 - 5i)$

$$\begin{aligned} (-2 + 5i)(3 - 5i) &= -6 + 10i + 15i - 25i^2 \\ &= -6 + 25i - 25(-1) \\ &= -6 + 25i + 25 \\ &= 19 + 25i \end{aligned}$$

3. $5i(6 - 4i)$

$$\begin{aligned} 5i(6 - 4i) &= 30i - 20i^2 \\ &= 30i + 20 \\ &= 20 + 30i \end{aligned}$$

Note We always write the real part of a complex number first. ■

Division of complex numbers

$$\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$$

Concept

The above result is obtained by multiplying the numerator and denominator of the quotient by the conjugate of the denominator.

$$\begin{aligned} \frac{a + bi}{c + di} &= \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} \\ &= \frac{ac - adi + bci - bd^2}{c^2 - cdi + cdi - d^2i^2} \\ &= \frac{ac + bd + (bc - ad)i}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i \end{aligned}$$

Example 7-1 E

Perform the indicated operations and simplify each expression.

1. $\frac{5 + 4i}{3 + 2i}$

$$\begin{aligned} \frac{5 + 4i}{3 + 2i} &= \frac{5 + 4i}{3 + 2i} \cdot \frac{3 - 2i}{3 - 2i} = \frac{15 - 10i + 12i - 8i^2}{9 - 6i + 6i - 4i^2} \\ &= \frac{15 + 2i + 8}{9 + 4} \\ &= \frac{23 + 2i}{13} \\ &= \frac{23}{13} + \frac{2}{13}i \end{aligned}$$

Note A complex number must have two parts, a real and an imaginary part. This is why we performed the last step in the previous part.

$$2. \frac{-2 + 5i}{3 - 4i}$$

$$\begin{aligned} \frac{-2 + 5i}{3 - 4i} &= \frac{-2 + 5i}{3 - 4i} \cdot \frac{3 + 4i}{3 + 4i} = \frac{-6 - 8i + 15i - 20}{9 + 12i - 12i + 16} \\ &= \frac{-26 + 7i}{25} \\ &= -\frac{26}{25} + \frac{7}{25}i \end{aligned}$$

$$3. \frac{5i}{6 - 4i}$$

$$\begin{aligned} \frac{5i}{6 - 4i} &= \frac{5i}{6 - 4i} \cdot \frac{6 + 4i}{6 + 4i} = \frac{30i - 20}{36 + 16} = -\frac{20}{52} + \frac{30}{52}i \\ &= -\frac{5}{13} + \frac{15}{26}i \end{aligned}$$

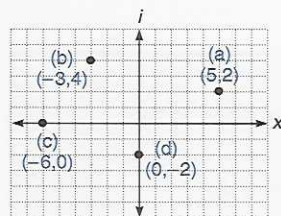


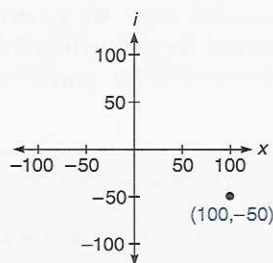
Figure 7-1

Example 7-1 F

Complex numbers can be graphed as ordered pairs in a rectangular coordinate system by letting horizontal distances represent the real part and vertical distances represent the imaginary part of each complex number. Figure 7-1 shows the graph of (a) $5 + 2i$, (b) $-3 + 4i$, (c) -6 , and (d) $-2i$. A graph of complex numbers, such as those in figure 7-1, is called an **Argand diagram**. Example 7-1 F illustrates one application.

In alternating current theory in electronics, complex numbers are used to represent *impedance* Z , a measure of the way in which a circuit retards the flow of current through it. The real part of an impedance is called the *resistance* R , and the imaginary part is called the *reactance* X . Thus, $Z = R + Xi$. The units for Z , R , and X are ohms. Graph circuit impedance Z if $R = 100$ ohms and $X = -50$ ohms.

$Z = R + Xi = 100 - 50i$. The graph is shown in the figure.



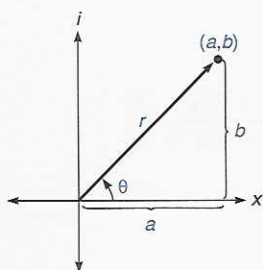


Figure 7-2

Polar form of a complex number

We can use the graph of a complex number as a guide to develop another way to represent a complex number. Given the complex number $a + bi$, let r represent the distance from the origin to the point (a, b) and let θ represent the angle in standard position determined by the ray containing the origin and (a, b) . See figure 7-2. Using the definitions of section 2-2 we obtain $\cos \theta = \frac{a}{r}$ and $\sin \theta = \frac{b}{r}$. Thus, $a = r \cos \theta$ and $b = r \sin \theta$. This means we can rewrite $a + bi$ as $r \cos \theta + (r \sin \theta)i$, or $r(\cos \theta + i \sin \theta)$. The expression $\cos \theta + i \sin \theta$ occurs so often it is abbreviated as **cis** θ , so $a + bi = r \text{ cis } \theta$. This form is called the **polar form** of a complex number.¹ Also, note that $r = \sqrt{a^2 + b^2}$ and that $\tan \theta = \frac{b}{a}$.

Polar form of a complex number

If $z = a + bi$ is a complex number that determines an angle θ , then

$$z = r \text{ cis } \theta$$

is its polar form, where **cis** θ means $\cos \theta + i \sin \theta$, and

$$r = \sqrt{a^2 + b^2}$$

The value r is called the **modulus** of z , which is also written $|z|$.

- Note**
1. The value of θ is not unique. All coterminal values produce the same rectangular form. Thus, $2 \text{ cis } 10^\circ$ is equivalent to $2 \text{ cis } 370^\circ$.
 2. The modulus is the distance from the origin to the complex coordinate (a, b) .

Polar-rectangular conversions

As with vectors (section 6-3), we generally give a value of θ so that $-180^\circ < \theta \leq 180^\circ$. A method for converting from rectangular form to polar form is a paraphrase of the method for converting vectors from rectangular to polar form.

Given a complex number $z = a + bi = r \text{ cis } \theta$. Then,

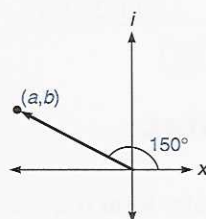
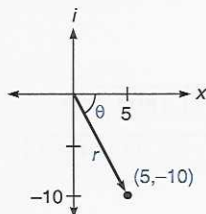
$$r = \sqrt{a^2 + b^2}, \quad \tan \theta = \tan^{-1} \frac{b}{a} \text{ (if } a \neq 0\text{), and}$$

$$\theta = \begin{cases} \theta' & \text{if } a > 0 \\ \theta' - 180^\circ & \text{if } \theta' > 0 \\ \theta' + 180^\circ & \text{if } \theta' < 0 \end{cases} \quad \text{if } a < 0$$

Note If $a = 0$ then θ is 90° if $b > 0$, and -90° if $b < 0$. A sketch will make the choice clear.

¹In 1893 Irving Stringham first used the notation $\text{cis } \beta = \cos \beta + i \sin \beta$.

Example 7-1 G



Convert between polar and rectangular form.

1. $5 - 10i$

The graph is shown in the figure.

$$r = |5 - 10i| = \sqrt{5^2 + (-10)^2} = \sqrt{125} = 5\sqrt{5}$$

$$\theta' = \tan^{-1} \frac{b}{a} = \tan^{-1} \left(\frac{-10}{5} \right) = \tan^{-1}(-2), \text{ so } \theta' \approx -63.4^\circ$$

$$\theta = \theta' \approx -63.4^\circ \quad a > 0$$

Thus, the polar form of $5 - 10i$ is $5\sqrt{5} \operatorname{cis}(-63.4^\circ)$ or about $(11.1, -63.4^\circ)$.

2. $5 \operatorname{cis} 150^\circ$

$$5 \operatorname{cis} 150^\circ = 5(\cos 150^\circ + i \sin 150^\circ)$$

$$= 5 \left(-\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2} \right)$$

$$= -\frac{5\sqrt{3}}{2} + \frac{5}{2}i$$

Thus, $5 \operatorname{cis} 150^\circ = -\frac{5\sqrt{3}}{2} + \frac{5}{2}i$ or about $-4.3 + 2.5i$.

Polar-rectangular conversions on a calculator

As stated in the previous section, most calculators are programmed to perform polar/rectangular conversions. This conversion works equally well for vectors (section 6-3) and for complex numbers. The conversion is done with keys marked “R → P” or “→ P” and “P → R” or “→ R” or something equivalent. The results are stored in locations referred to as x and y . Typical keystrokes are illustrated here. (The TI-81 steps are shown below.)

Part 1 of example 7-1 G would be done as follows.

$$z = 5 - 10i$$

$$5 \quad \boxed{\text{R} \rightarrow \text{P}} \quad 10 \quad \boxed{+/-} \quad \boxed{=}$$

Display: 11.18033989

$$\boxed{x \leftrightarrow y}$$

Display: -63.43494882

Thus, $z \approx (11.1, -63.4^\circ)$.

Part 2 of example 7-1 G would be done in the following way.

$$z = 5 \operatorname{cis} 150^\circ$$

$$5 \quad \boxed{\text{P} \rightarrow \text{R}} \quad 150 \quad \boxed{=}$$

Display: -4.330127019

$$\boxed{x \leftrightarrow y}$$

Display: 2.5

Thus, $z \approx -4.3 + 2.5i$.

The TI-81 uses the values X , Y , R , and θ . (Y , R , θ are ALPHA 1, X, and 3, respectively.) It also uses the two MATH functions “R → P” (rectangular to polar) and “P → R” (polar to rectangular).

MODE **Deg** **ENTER** Make sure the calculator is in degree mode.

Example 7-1 G, part 1:

MATH 1 5 **ALPHA** . **(-)** 10 **)**

ENTER

Display: 11.18033989

ALPHA 3 **ENTER**

Display: -63.43494882

Example 7-1 G, part 2:

MATH 2 5 **ALPHA** . 150 **)**

ENTER

Display: -4.330127019

ALPHA 1 **ENTER**

Display: 2.5

Multiplication and division of complex numbers in polar form

Multiplication and division of complex numbers in rectangular form is quite complicated. The following theorems show that these procedures are quite simple when the complex numbers are in polar form.

Complex multiplication—polar form

$$(r_1 \operatorname{cis} \theta_1)(r_2 \operatorname{cis} \theta_2) = r_1 r_2 \operatorname{cis} (\theta_1 + \theta_2)$$

Complex division—polar form

$$\frac{r_1 \operatorname{cis} \theta_1}{r_2 \operatorname{cis} \theta_2} = \frac{r_1}{r_2} \operatorname{cis} (\theta_1 - \theta_2), r_2 \neq 0$$

Concept

To *multiply*, multiply the moduli and add the angles. To *divide*, divide the moduli and subtract the angles.

We can see that the first theorem is true by converting the complex numbers into rectangular form and performing the multiplication as defined earlier in this section.

$$\begin{aligned} & (r_1 \operatorname{cis} \theta_1)(r_2 \operatorname{cis} \theta_2) \\ &= (r_1 \cos \theta_1 + i r_1 \sin \theta_1)(r_2 \cos \theta_2 + i r_2 \sin \theta_2) \\ &= r_1 r_2 \cos \theta_1 \cos \theta_2 + i r_1 r_2 \cos \theta_1 \sin \theta_2 + i r_1 r_2 \sin \theta_1 \cos \theta_2 + i^2 r_1 r_2 \sin \theta_1 \sin \theta_2 \\ &= r_1 r_2 \cos \theta_1 \cos \theta_2 - r_1 r_2 \sin \theta_1 \sin \theta_2 + i r_1 r_2 \cos \theta_1 \sin \theta_2 + i r_1 r_2 \sin \theta_1 \cos \theta_2 \\ & \quad \text{Recall that } i^2 = -1 \\ &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\ & \quad \text{Now use the identities for } \cos(\alpha + \beta) \text{ and } \sin(\alpha + \beta) \text{ from section 5-2.} \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \\ &= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \end{aligned}$$

The proof of the process for division in polar form is left as an exercise.

In electronics, Ohm's law states $V = IZ$, where V means voltage, I means current, and Z means impedance. The units are volts, amperes, and ohms, respectively. Often complex numbers are used to describe the values of volts, amperes, and ohms. The fact that the angles add in forming the product models the physical situation in an electronics circuit. This use is illustrated in example 7-1 H, part 2.

■ Example 7-1 H

Multiply or divide the complex numbers.

- $(2 \text{ cis } 110^\circ)(10 \text{ cis } 300^\circ)$
 $= 2 \cdot 10 \text{ cis}(110^\circ + 300^\circ)$
 $= 20 \text{ cis } 410^\circ$
 $= 20 \text{ cis } 50^\circ$ 410° and 50° are coterminal
- In a certain electronic circuit current $I = 4 \text{ cis } 30^\circ$ amperes and impedance $Z = 2 \text{ cis } 15^\circ$. Use Ohm's law, $V = IZ$, to compute voltage V .

$$V = IZ$$

$$= (4 \text{ cis } 30^\circ)(2 \text{ cis } 15^\circ)$$

$$= 8 \text{ cis } 45^\circ \text{ (volts)}$$
- $\frac{15 \text{ cis } 30^\circ}{18 \text{ cis } 80^\circ}$
 $= \frac{15}{18} \text{cis}(30^\circ - 80^\circ) = \frac{5}{6} \text{cis}(-50^\circ)$

De Moivre's theorem

Consider the following computations for successive powers of a complex number $r \text{ cis } \theta$.

$$\begin{aligned}(r \text{ cis } \theta)^1 &= r \text{ cis } \theta \\(r \text{ cis } \theta)^2 &= (r \text{ cis } \theta)(r \text{ cis } \theta) = r^2 \text{ cis } 2\theta \\(r \text{ cis } \theta)^3 &= (r \text{ cis } \theta)(r \text{ cis } \theta)^2 = (r \text{ cis } \theta)(r^2 \text{ cis } 2\theta) = r^3 \text{ cis } 3\theta \\(r \text{ cis } \theta)^4 &= (r \text{ cis } \theta)(r \text{ cis } \theta)^3 = (r \text{ cis } \theta)(r^3 \text{ cis } 3\theta) = r^4 \text{ cis } 4\theta\end{aligned}$$

It is logical to assume that the pattern above continues. This is true, and the result is called **De Moivre's theorem**. It actually turns out that the exponent can be any real number.

De Moivre's theorem

$$(r \text{ cis } \theta)^n = r^n \text{cis } n\theta \text{ for any real number } n.$$

This theorem is illustrated in example 7-1 I.

■ Example 7-1 I

Use De Moivre's theorem to compute the following.

- $(5 \text{ cis } 137^\circ)^3$; leave the answer in polar form.
 $= 5^3 \text{ cis}(3 \cdot 137^\circ)$
 $= 125 \text{ cis } 411^\circ$
 $= 125 \text{ cis } 51^\circ$

2. $(1 + 0.8i)^6$; leave the answer in rectangular form; round to the nearest tenth.

We could of course multiply $(1 + 0.8i)$ by itself five times to obtain the result. The amount of work is prohibitive. If we put the number in polar form we can use De Moivre's theorem.

$$|1 + 0.8i| = \sqrt{1^2 + 0.8^2} \approx 1.281; \theta = \tan^{-1} \frac{0.8}{1} \approx 38.66^\circ$$

Then $1 + 0.8i \approx 1.281 \text{ cis } 38.66^\circ$. Then

$$\begin{aligned} (1 + 0.8i)^6 &\approx (1.281 \text{ cis } 38.66^\circ)^6 \\ &= 1.281^6 \text{ cis}(6 \cdot 38.66^\circ) \\ &\approx 4.42 \text{ cis } 231.96^\circ \\ &= 4.42 \cos 231.96^\circ + 4.42 \sin 231.96^\circ i \\ &\approx -2.7 - 3.5i \end{aligned}$$

Thus, $(1 + 0.8i)^6 \approx -2.7 - 3.5i$, to the nearest tenth. ■

De Moivre's theorem for roots

In the complex number system, every number except 0 has n n th roots; that is, two square roots, three cube roots, four fourth roots, etc. These can be expressed by De Moivre's theorem by replacing n by $\frac{1}{n}$; recall that $x^{\frac{1}{2}} = \sqrt{x}$, $x^{\frac{1}{3}} = \sqrt[3]{x}$, $x^{\frac{1}{4}} = \sqrt[4]{x}$, etc.

De Moivre's theorem for roots

The n n th roots of $r \text{ cis } \theta$ are of the form

$$r^{\frac{1}{n}} \text{ cis} \left(\frac{\theta}{n} + \frac{k \cdot 360^\circ}{n} \right), 0 \leq k < n$$

where k and n are positive integers.

We can show that any number of the form above is an n th root of $r \text{ cis } \theta$ by raising it to the n th power.

$$\begin{aligned} \left[r^{\frac{1}{n}} \text{ cis} \left(\frac{\theta}{n} + \frac{k \cdot 360^\circ}{n} \right) \right]^n &= \left(r^{\frac{1}{n}} \right)^n \text{ cis} \left[n \left(\frac{\theta}{n} + \frac{k \cdot 360^\circ}{n} \right) \right] \\ &= r \text{ cis}(\theta + k \cdot 360^\circ) \\ &= r \text{ cis } \theta \quad \theta + k \cdot 360^\circ \text{ is coterminal to } \theta \end{aligned}$$

If $k \geq n$ we get a repetition of a previous root. The proof of this is left as an exercise. The proof of the fact that the roots are all distinct and that there are no other roots is beyond the scope of this text.

■ Example 7-1 J

Find the roots.

1. Find the three cube roots of 1.

$$1 = 1 \operatorname{cis} 0^\circ$$

Evaluate $1^{\frac{1}{3}} \operatorname{cis} \left(\frac{0^\circ}{3} + \frac{k \cdot 360^\circ}{3} \right) = \operatorname{cis}(k \cdot 120^\circ)$ for $k = 0, 1, 2$.

$$k = 0: \operatorname{cis} 0^\circ = \cos 0^\circ + i \sin 0^\circ = 1 + 0i = 1$$

$$k = 1: \operatorname{cis} 120^\circ = \cos 120^\circ + i \sin 120^\circ = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$k = 2: \operatorname{cis} 240^\circ = \cos 240^\circ + i \sin 240^\circ = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Thus, the three cube roots of 1 are $1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

2. Find decimal approximations (to tenths) of the four fourth roots of $10 - 2\sqrt{39}i$.

$$10 - 2\sqrt{39}i \approx 16 \operatorname{cis} 308.68^\circ$$

$$16^{\frac{1}{4}} \operatorname{cis} \left(\frac{308.68^\circ}{4} + \frac{k \cdot 360^\circ}{4} \right) \approx 2 \operatorname{cis}(77.17 + k \cdot 90^\circ)$$

Evaluate $2 \operatorname{cis}(77.17 + k \cdot 90^\circ)$ for $k = 0, 1, 2, 3$.

$$k = 0: 2 \operatorname{cis} 77.17^\circ = 2(\cos 77.17^\circ + i \sin 77.17^\circ) = 0.4 + 2.0i$$

$$k = 1: 2 \operatorname{cis} 167.17^\circ = 2(\cos 167.17^\circ + i \sin 167.17^\circ) = -2.0 + 0.4i$$

$$k = 2: 2 \operatorname{cis} 257.17^\circ = 2(\cos 257.17^\circ + i \sin 257.17^\circ) = -0.4 - 2.0i$$

$$k = 3: 2 \operatorname{cis} 347.17^\circ = 2(\cos 347.17^\circ + i \sin 347.17^\circ) = 2.0 - 0.4i$$

Thus, the four fourth roots of $10 - 2\sqrt{39}i$ are approximately $0.4 + 2.0i, -2.0 + 0.4i, -0.4 - 2.0i$, and $2.0 - 0.4i$.

3. In electronics apparent power P can be determined by $P = I^2 Z$. If we solve this for current I we get $I = \pm \sqrt{\frac{P}{Z}}$. Use $I = \sqrt{\frac{P}{Z}}$ to determine I to the nearest tenth if $P = 10 - 2i$ and $Z = 1 + 3i$.

$$P = 10 - 2i \approx 10.2 \operatorname{cis} 348.69^\circ$$

$$Z = 1 + 3i \approx 3.16 \operatorname{cis} 71.57^\circ$$

$$\frac{P}{Z} \approx \frac{10.2 \operatorname{cis} 348.69^\circ}{3.16 \operatorname{cis} 71.57^\circ} \approx 3.23 \operatorname{cis} 277.12^\circ$$

We evaluate the expression $3.23^{\frac{1}{2}} \operatorname{cis} \left(\frac{277.12^\circ}{2} + \frac{k \cdot 360^\circ}{2} \right)$ for $k = 0, 1$.

$$k = 0: \sqrt{3.23} \operatorname{cis} 138.56^\circ \approx -1.3 + 1.2i$$

$$k = 1: \sqrt{3.23} \operatorname{cis} 318.56^\circ \approx 1.3 - 1.2i$$

Thus, the current I is about $-1.3 + 1.2i$ or $1.3 - 1.2i$ (amperes). ■

Mastery points

Can you

- Identify the real and imaginary parts of a complex number?
- Form the conjugate of a complex number?
- Evaluate complex expressions involving addition, subtraction, multiplication, and division?
- Graph a complex number when given in rectangular or polar form?
- Convert between the rectangular and polar forms of complex numbers?
- Multiply and divide complex numbers in polar form?
- State and use De Moivre's theorem for integral powers?
- State and use De Moivre's theorem for roots?

Exercise 7-1

Identify the real and imaginary parts of the following complex numbers.

- | | | | |
|-------------|--------------|---------------|-------------|
| 1. $4 - 5i$ | 2. $3 + 11i$ | 3. $-4 + i$ | 4. $3i$ |
| 5. 12 | 6. $-i$ | 7. $-10 + 2i$ | 8. $-9 - i$ |

For each problem (a) find the complex conjugate and (b) graph the number and its conjugate in the same graph.

- | | | | |
|-------------|---------------|----------------|--------------|
| 9. $4 - 5i$ | 10. $3 + 11i$ | 11. $-4 + i$ | 12. $3i$ |
| 13. 12 | 14. $-i$ | 15. $-10 + 2i$ | 16. $-9 - i$ |

Simplify each expression by performing the indicated operations. Do all operations in rectangular form (do not convert to polar form).

- | | |
|--|-------------------------------------|
| 17. $(5 + 4i) + (-3 + 2i)$ | 18. $(-11 + 3i) - (-1 + 4i)$ |
| 19. $(3 + 2i) + (-6i) - (2 + 3i) - 10$ | 20. $(7 + 4i) - (-13 + 4i)$ |
| 21. $13 - 7i + 3i - 4$ | 22. $-8 - (4 + 3i) + 2i - (-6 - i)$ |
| 23. $(15 + 4i)(3 + 12i)$ | 24. $(-2 + 5i)(-3 - 5i)$ |
| 25. $-5i(6 - 4i)$ | 27. $(2 - 3i)^3$ |
| 26. $(4 - 3i)^2$ | 28. $i(2i)(3i)(-i)$ |
| 29. $\frac{5 + 4i}{5 + 2i}$ | 31. $\frac{3 + 6i}{-2 - i}$ |
| 30. $\frac{-2 + i}{2 + 4i}$ | 32. $\frac{5 + i}{5i}$ |

Write the polar form of each complex number. Round the results to the nearest tenth.

- | | | | | | |
|--------------|---------------------|---------------|---------------------|---------------|---------------|
| 33. $5 - 2i$ | 34. $\sqrt{2} + 3i$ | 35. $-1 + 3i$ | 36. $\sqrt{3} - 2i$ | 37. $-3 + 4i$ | 38. $13 - 9i$ |
|--------------|---------------------|---------------|---------------------|---------------|---------------|

Write the polar form of each complex number. Leave the result in exact form.

- | | | | |
|--------------------|----------------------------|--------------|----------------------|
| 39. $\sqrt{3} + i$ | 40. $1 - \sqrt{3}i$ | 41. $3 + 3i$ | 42. $-1 + \sqrt{3}i$ |
| 43. $-1 - i$ | 44. $\sqrt{5} - \sqrt{5}i$ | 45. $5i$ | 46. $-3i$ |

Write the rectangular form of the following numbers. Round the results to the nearest tenth.

47. $3 \text{ cis } 15^\circ$ 48. $5 \text{ cis } 20^\circ$ 49. $4.5 \text{ cis } 35^\circ$ 50. $10 \text{ cis } 40^\circ$
 51. $\sqrt{2} \text{ cis } 315^\circ$ 52. $200 \text{ cis } 8^\circ$ 53. $13.6 \text{ cis } (-25^\circ)$ 54. $12 \text{ cis } (-6^\circ)$

Write the rectangular form of the following numbers. Leave the result in exact form.

55. $\sqrt{3} \text{ cis } 30^\circ$ 56. $4 \text{ cis } 210^\circ$ 57. $10 \text{ cis } 300^\circ$ 58. $6 \text{ cis } 135^\circ$
 59. $\sqrt{10} \text{ cis } 180^\circ$ 60. $2 \text{ cis } 90^\circ$ 61. $\sqrt{8} \text{ cis } 315^\circ$ 62. $5 \text{ cis } 240^\circ$

Multiply or divide the following complex numbers. Leave the result in polar form.

63. $(5 \text{ cis } 30^\circ)(3 \text{ cis } 45^\circ)$ 64. $(2 \text{ cis } 18^\circ)(4.5 \text{ cis } 100^\circ)$ 65. $(5.4 \text{ cis } 300^\circ)(2 \text{ cis } 300^\circ)$
 66. $(0.5 \text{ cis } 230^\circ)(80 \text{ cis } 200^\circ)$ 67. $\frac{20 \text{ cis } 100^\circ}{5 \text{ cis } 20^\circ}$ 68. $\frac{100 \text{ cis } 45^\circ}{200 \text{ cis } 15^\circ}$
 69. $\frac{40 \text{ cis } 80^\circ}{18 \text{ cis } 160^\circ}$ 70. $\frac{90 \text{ cis } 300^\circ}{50 \text{ cis } 100^\circ}$

Use De Moivre's theorem to compute the power indicated. Leave the answer in the form in which the problem is stated (polar or rectangular).

71. $(8 \text{ cis } 100^\circ)^3$ 72. $(5 \text{ cis } 10^\circ)^4$ 73. $(3 \text{ cis } 200^\circ)^3$ 74. $(2 \text{ cis } 300^\circ)^5$
 75. $(0.5 - 1.2i)^8$ (round to nearest tenth) 76. $(0.8 + 0.6i)^{10}$ (round to nearest tenth)

77. Find the 3 cube roots of 8 in exact form.
 78. Find the 4 fourth roots of -1 in exact form.
 79. Find the 4 fourth roots of 81 in exact form.
 80. Find the 6 sixth roots of -64 in exact form.
 81. Find the 3 cube roots of $75 - 100i$ to the nearest tenth.
 82. Find the 4 fourth roots of $\sqrt{3} + 3i$ to the nearest tenth.
 83. In electronics, one version of Ohm's law says that $I = \frac{V}{Z}$, where I is current, V is voltage, and Z is impedance. Find I in a circuit in which V is $125 \text{ cis } 25^\circ$ and Z is $50 \text{ cis } 45^\circ$.
 84. Find I in a circuit in which $V = 200 \text{ cis } 40^\circ$ and $Z = 4 \text{ cis } 50^\circ$. See problem 83.
 85. Find V in a circuit where $I = 10 \text{ cis } 15^\circ$ and $Z = 5 \text{ cis } 30^\circ$. See problem 83.
 86. Find Z in a circuit where $I = 40 \text{ cis } 200^\circ$ and $V = 10 \text{ cis } 125^\circ$. See problem 83.

87. In a parallel electronics circuit with two legs, total impedance Z_T is $\frac{Z_1 Z_2}{Z_1 + Z_2}$. Find Z_T in a circuit in which $Z_1 = 2 + i$ and $Z_2 = 3 - 5i$. Leave the answer in polar form.
 88. Find Z_T in a parallel circuit in which $Z_1 = 12 + 3i$ and $Z_2 = 4 - 2i$. Leave the answer in polar form. See problem 87.
 89. Use $I = \sqrt{\frac{P}{Z}}$ to determine I if $P = 5 + 2i$ and $Z = 1 - 4i$. Leave the result in rectangular form, to the nearest hundredth. Use the first square root ($k = 0$ in De Moivre's theorem).
 90. Use $I = \sqrt{\frac{P}{Z}}$ to determine I if $P = -2 + 2i$ and $Z = 2 - i$. Leave the result in rectangular form, to the nearest hundredth. Use the first square root ($k = 0$ in De Moivre's theorem).
 91. Is the following an identity: $a \text{ cis } (-\theta) = -a \text{ cis } \theta$? Show why or why not.

Multiplication by i can be interpreted as a 90° rotation. If z represents the complex number given in each of the following problems, compute and graph (a) z , (b) iz , (c) $i^2 z$, (d) $i^3 z$.

92. $4 + 2i$ 93. $-3 + i$ 94. $5i$ 95. 6 96. $1 - i$ 97. $-1 - i$

98. Let $z_1 = -2 + 2i$, $z_2 = 1 - \sqrt{3}i$. Form the product in two ways: (a) by multiplication in rectangular form and (b) by changing each value into polar form (in exact form) and performing the multiplication in polar form. Then (c) convert the answer to (b) back to rectangular form and verify that the answers to (a) and (b) are the same.

99.  A numerically controlled machine is programmed to rotate a laser beam according to mathematical rules. The laser initially points to the point $1 + i$.

- Find the rectangular form of a complex number z such that the angle of the product $z(1 + i)$ is 30° greater than the angle of $1 + i$, without changing the modulus of $1 + i$.
- Give the rectangular form of the point to which the laser points after this rotation. Round the answer to two decimal places.
- Give the rectangular form of the point to which the laser points after eight such rotations, starting at the point $1 + i$. Round the answer to two decimal places.

100. In this section we stated that $\frac{r_1 \operatorname{cis} \theta_1}{r_2 \operatorname{cis} \theta_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$.

Prove that this is true. Use the proof in the text that $(r_1 \operatorname{cis} \theta_1)(r_2 \operatorname{cis} \theta_2) = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$ as a guide.

101.  **Warning:** This problem requires a *great* deal of algebraic manipulation. De Moivre's theorem for roots states that the n th roots of $r \operatorname{cis} \theta$ are of the form

$$(r \operatorname{cis} \theta)^{\frac{1}{n}} = r^{\frac{1}{n}} \operatorname{cis} \left(\frac{\theta}{n} + \frac{k \cdot 360^\circ}{n} \right), \quad 0 \leq k < n,$$

where k and n are positive integers. Show that if $k \geq n$, then the expression is a repetition of another root. That is, it is the same as the expression for some value of $k < n$. Do this in the following manner.

First, if $k \geq n$, then $\frac{k}{n} = a + \frac{b}{n}$, where $b < n$. (Think of the example $22 \geq 5$, so $\frac{22}{5} = 4 + \frac{2}{5}$.) This means $k = an + b$, $b < n$.

Next, show that the following steps are true:

$$\begin{aligned} & r^{\frac{1}{n}} \operatorname{cis} \left(\frac{\theta}{n} + \frac{k \cdot 360^\circ}{n} \right) \\ &= r^{\frac{1}{n}} \operatorname{cis} \left[\frac{\theta}{n} + \frac{(an + b) \cdot 360^\circ}{n} \right] \quad \text{Why?} \\ &= r^{\frac{1}{n}} \operatorname{cis} \left[\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n} \right) + a \cdot 360^\circ \right]. \end{aligned}$$

Show the algebra for this step

Finally, now expand this last expression into terms of sine and cosine (i.e., expand “ $\operatorname{cis} \alpha$ ”), and then apply the identities for the cosine and sine of a sum (section 5-2.).

7-2 Polar coordinates

Some natural phenomena, such as the motion of the planets, the contour of a cam on an automobile camshaft, the path traveled by a client on many of the rides in an amusement park, or the field strength around a radio transmitter, can be described most simply by describing the motion in terms of distance from some point as a line moves in a circle. The polar coordinate system is a coordinate system that is better suited to describing these phenomena than are rectangular coordinates. James Bernoulli is often credited with the creation of polar coordinates in 1691, although Isaac Newton used them earlier. A polar coordinate system is a series of concentric circles and an angle reference line (see figure 7-3). The common center of the circles is called the **pole**.

Basic definitions

Polar coordinates

The polar coordinates of a point is an ordered pair of the form (r, θ) , where r is the *radius*, and θ is an angle, often measured in radians.

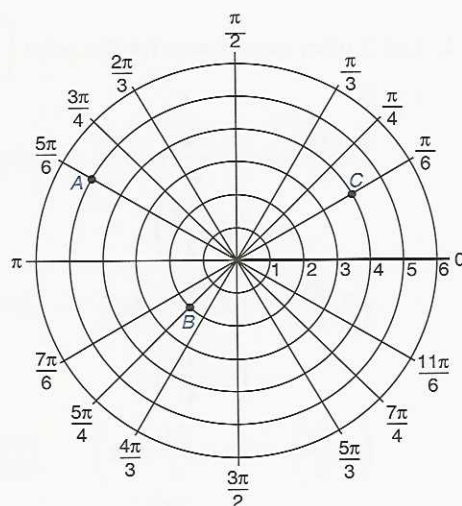


Figure 7-3

A point in polar form is located by finding the radius line corresponding to the angle θ , often stated in radians, and moving r units from the center along this line. Figure 7-3 shows the graphs² of the points $\left(5, \frac{5\pi}{6}\right)$ (A), $\left(2, \frac{5\pi}{4}\right)$ (B), and $\left(4, \frac{\pi}{6}\right)$ (C). In each of these cases $r > 0$.

If two points have equal radii and coterminal angles they will have the same graph. For this reason we define such points to be equivalent. If $r < 0$ we interpret this to mean a change of direction by π radians (the opposite direction).

Equivalence of points

1. $(r, \alpha) = (r, \beta)$ if α and β are coterminal angles.
2. $(-r, \theta) = (r, \theta + \pi)$.

Note $(-r, \theta) = (r, \theta - \pi)$ is also true.

This definition means that $(1, 0)$, $(1, 2\pi)$, $(1, 4\pi)$, $(1, -2\pi)$, $(1, -4\pi)$, $(-1, \pi)$, $(-1, 3\pi)$, $(-1, -\pi)$, etc. all describe the same point! This can occasionally lead to some confusion.

Example 7-2 A illustrates the basic definitions.

²Polar coordinate paper is widely available.

■ Example 7-2 A

1. List 3 other coordinates for the point $\left(2, \frac{3\pi}{4}\right)$, with at least one so that $r < 0$.

$$\begin{aligned}\left(2, \frac{3\pi}{4}\right) &= \left(2, \frac{3\pi}{4} + 2\pi\right) && \text{Adding } 2\pi \text{ gives a coterminal angle} \\ &= \left(2, \frac{11\pi}{4}\right)\end{aligned}$$

$$\begin{aligned}\left(2, \frac{3\pi}{4}\right) &= \left(2, \frac{3\pi}{4} + 4\pi\right) && \text{Adding } 4\pi \text{ gives a coterminal angle} \\ &= \left(2, \frac{19\pi}{4}\right)\end{aligned}$$

$$\begin{aligned}\left(2, \frac{3\pi}{4}\right) &= \left(-2, \frac{3\pi}{4} + \pi\right) && \text{Adding or subtracting } \pi \text{ gives a coterminal} \\ &= \left(-2, \frac{7\pi}{4}\right) && \text{angle in which } r \text{ changes sign}\end{aligned}$$

2. Plot the point $\left(-5, \frac{11\pi}{6}\right)$.

$$\left(-5, \frac{11\pi}{6}\right) = \left(5, \frac{11\pi}{6} - \pi\right) = \left(5, \frac{5\pi}{6}\right), \text{ which is plotted at } A \text{ in figure 7-3.}$$

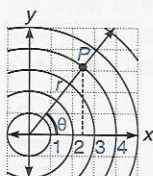


Figure 7-4

Polar-rectangular coordinate conversions

There is a way to relate polar and rectangular coordinates. Figure 7-4 shows polar and rectangular coordinates superimposed. From the definitions of section 2-2 we know that if $P = (x, y) = (r, \theta)$, $r > 0$, then $\cos \theta = \frac{x}{r}$ and

$$\sin \theta = \frac{y}{r}.$$

Thus, to convert from polar to rectangular coordinates we have only to use

$$\begin{aligned}x &= r \cos \theta \\ y &= r \sin \theta\end{aligned}$$

In fact, it will be an exercise to show that we can use the same relations when $r < 0$. Example 7-2 B illustrates a polar to rectangular conversion.

■ Example 7-2 B

Convert the polar coordinates $\left(2, \frac{\pi}{3}\right)$ to rectangular coordinates.

$$x = r \cos \theta = 2 \cos \frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1$$

$$y = r \sin \theta = 2 \sin \frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

Thus, the rectangular coordinates are $(1, \sqrt{3})$.

To convert from rectangular to polar coordinates we use the fact that $r^2 = x^2 + y^2$ and $\tan \theta = \frac{y}{x}$ if $x \neq 0$. This also means $\theta' = \tan^{-1} \frac{y}{x}$.

As with vectors and complex numbers (sections 6-3 and 7-1) we always leave the angle θ so that it is the smallest possible absolute value. In this case, using radian measure, we always choose θ so that $-\pi < \theta \leq \pi$.

A rule for finding θ is essentially the same as that for finding θ_V for vectors, and θ for complex numbers.

Given polar coordinates for point P , $P = (x, y) = (r, \theta)$,

$$r = \sqrt{x^2 + y^2}, \quad \tan \theta' = \tan^{-1} \frac{y}{x} \text{ if } x \neq 0, \text{ and}$$

$$\theta = \begin{cases} \theta' & \text{if } x > 0 \\ \theta' - \pi & \text{if } \theta' > 0 \\ \theta' + \pi & \text{if } \theta' < 0 \end{cases} \quad \text{if } x < 0$$

Note If $x = 0$ then θ is π if $y > 0$, and $-\pi$ if $y < 0$. This is clear from a sketch.

■ Example 7-2 C

Convert the rectangular coordinates into polar coordinates.

1. $(-2\sqrt{3}, 2)$

$$r^2 = (-2\sqrt{3})^2 + 2^2 = 16, \quad r = 4$$

$$\theta' = \tan^{-1} \frac{2}{-2\sqrt{3}} = \tan^{-1} \left(-\frac{\sqrt{3}}{3} \right) = -\tan^{-1} \frac{\sqrt{3}}{3} \quad \begin{array}{l} \tan^{-1} \text{ is an odd function,} \\ \text{so } \tan^{-1}(-x) = -\tan^{-1}x \end{array}$$

$$\text{so } \theta' = -\frac{\pi}{6}. \quad x < 0, \theta' < 0, \text{ so } \theta = \theta' + \pi = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}.$$

Therefore, the required polar coordinates are $\left(4, \frac{5\pi}{6}\right)$.

2. $(-2, -5)$

$$r^2 = (-2)^2 + (-5)^2 = 29, \quad r = \sqrt{29} \approx 5.39$$

$$\theta' = \tan^{-1} \frac{-5}{-2} \approx 1.19 \text{ (radians)}$$

$x < 0, \theta' > 0$, so $\theta = \theta' - \pi = \tan^{-1} \frac{5}{2} - \pi$ (exactly), or approximately $1.19 - \pi \approx -1.95$. Thus, the polar coordinates are $(\sqrt{29}, \tan^{-1} 2.5 - \pi)$ exactly, or approximately $(5.39, -1.95)$. ■

Conversions with a calculator

Engineering/scientific calculators are programmed to perform the conversions of example 7-2 C, using the same method and keys shown in section 6-3 for vectors and in section 7-1 for complex numbers. These calculators have keys marked $R \rightarrow P$ and $P \rightarrow R$, or something equivalent. The results are stored in

locations referred to as x and y . Typical keystrokes are illustrated here. For these examples we want the *calculator in radian mode*. The steps for the TI-81 are shown below.

Example 7-2 B would be done as follows:

$$\left(2, \frac{\pi}{3}\right) \text{ (polar)}$$

2 $\boxed{\text{P} \rightarrow \text{R}}$ $\boxed{(\pi \div 3)}$ $\boxed{=}$

Display: $\boxed{1}$

$\boxed{x \leftrightarrow y}$

Display: $\boxed{1.732050808}$

$$\text{Thus, } \left(2, \frac{\pi}{3}\right) \approx (1, 1.73).$$

Example 7-2 C, part 2, would be done in the following way:

$(-2, -5)$ (rectangular)

2 $\boxed{\pm}$ $\boxed{\text{R} \rightarrow \text{P}}$ 5 $\boxed{+/-}$ $\boxed{=}$

Display: $\boxed{5.385164807}$

$\boxed{x \leftrightarrow y}$

Display: $\boxed{-1.951302704}$

Thus, $(-2, -5)$ (rectangular) $\approx (5.39, -1.95)$ (polar).

The TI-81 uses the values X , Y , R , and θ . (Y , R , θ are $\boxed{\text{ALPHA}}$ 1, $\boxed{\times}$, and 3, respectively.) It also uses the two MATH functions “ $\text{R} \nabla \text{P}$ ” (rectangular to polar) and “ $\text{P} \nabla \text{R}$ ” (polar to rectangular).

Example 7-2 B:

$\boxed{\text{MODE}}$ Rad $\boxed{\text{ENTER}}$

Make sure the calculator is in radian mode.

$\boxed{\text{MATH}}$ 2 2 $\boxed{\text{ALPHA}}$ $\boxed{\cdot}$ $\boxed{(\pi \div 3)}$ $\boxed{)}$

$\boxed{\text{ENTER}}$

Display: $\boxed{1}$

$\boxed{\text{ALPHA}}$ 1 $\boxed{\text{ENTER}}$

Display: $\boxed{1.732050808}$

Example 7-2 C, part 2:

$\boxed{\text{MATH}}$ 1 $\boxed{(-)}$ 2 $\boxed{\text{ALPHA}}$ $\boxed{\cdot}$ $\boxed{(-)}$ 5

$\boxed{)}$ $\boxed{\text{ENTER}}$

Display: $\boxed{5.385164807}$

$\boxed{\text{ALPHA}}$ 3 $\boxed{\text{ENTER}}$

Display: $\boxed{-1.951302704}$

Conversion of equations between rectangular and polar form

Analytic geometry is the modeling of geometry in the language of algebra. Thus, a nonvertical line in analytic geometry is an equation of the form $y = mx + b$, and a circle with center at the origin and radius r is an equation of the form $x^2 + y^2 = r^2$.

Equations can also be written using polar coordinates. Examples are

$$r = 3 \cos \theta$$

$$r^2 = \sec \theta$$

$$r = 2$$

$$\cos \theta = 1$$

It is often useful to discover the polar coordinate version of a rectangular coordinate equation. We say that *a rectangular equation and a polar equation are equivalent if they describe the same set of points*, assuming the appropriate rectangular/polar conversions of the points themselves.

Conversion of equations from rectangular to polar form

To convert an equation in rectangular form into an equivalent equation in polar form, use the relations used to convert a point from rectangular to polar form:

$$x = r \cos \theta, y = r \sin \theta, \text{ and } r^2 = x^2 + y^2.$$

When possible we customarily write a polar equation in which r is described as a function of θ . That is, to the extent possible, we put all terms with r in one member of the equation, and all other terms in the other member.

Example 7-2 D illustrates conversions from rectangular to polar form.

■ Example 7-2 D

Convert each rectangular equation into polar form.

1. The line $y = 3x - 2$.

$$y = 3x - 2$$

$$r \sin \theta = 3(r \cos \theta) - 2$$

$$2 = 3r \cos \theta - r \sin \theta$$

$$2 = r(3 \cos \theta - \sin \theta)$$

$$r = \frac{2}{3 \cos \theta - \sin \theta}$$

$$x = r \cos \theta, y = r \sin \theta$$

It is customary to solve a polar equation for r if possible

2. The line $y = -2x$.

$$y = -2x$$

$$r \sin \theta = -2r \cos \theta$$

$$\sin \theta = -2 \cos \theta$$

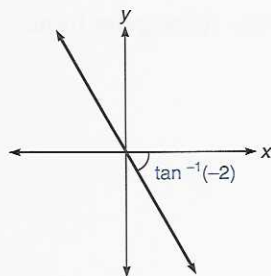
$$\frac{\sin \theta}{\cos \theta} = -2$$

$$\tan \theta = -2$$

$$x = r \cos \theta, y = r \sin \theta$$

Divide both members by r ; this assumes $r \neq 0$ (see below)

Divide both members by $\cos \theta$; this assumes $\cos \theta \neq 0$ (see below)



This equation is equivalent to the equation $y = -2x$. To see this, first observe that it does not mention r . This means r can be any value. The values of θ for which $\tan \theta = -2$ are in quadrants II and IV, where the tangent function takes on negative values. All points (r, θ) for which $\tan \theta = -2$ and r takes on any value are shown in the figure. This is the line $y = -2x$.

It was all right to assume $r \neq 0$ above because the resulting solution includes the pole as a solution (this is where $r = 0$).

It was also valid to assume $\cos \theta \neq 0$, for two reasons. One is that we arrive at an equation that satisfies the requirements and thus we do not need to consider the case where $\cos \theta = 0$. Second, when $\cos \theta = 0$, $\sin \theta$ is ± 1 . These values do not solve the equation $\sin \theta = -2 \cos \theta$, so that $\cos \theta = 0$ cannot occur in this situation.

3. The circle
- $x^2 + y^2 = 1$
- .

$$\begin{aligned}x^2 + y^2 &= 1 \\r^2 &= 1 & x^2 + y^2 = r^2 \\r &= \pm 1\end{aligned}$$

Either $r = 1$ or $r = -1$ describes a circle with radius one, since θ is not restricted but may take on any value. Thus, either equation is valid.

4. The hyperbola
- $x^2 - y^2 = 2$
- . (See footnote
- ³
- .)

$$\begin{aligned}x^2 - y^2 &= 2 \\(r \cos \theta)^2 - (r \sin \theta)^2 &= 2 \\r^2 \cos^2 \theta - r^2 \sin^2 \theta &= 2 \\r^2(\cos^2 \theta - \sin^2 \theta) &= 2 \\r^2 \cos 2\theta &= 2 & \cos 2\theta = \cos^2 \theta - \sin^2 \theta \\r^2 &= \frac{2}{\cos 2\theta} \\r^2 &= 2 \sec 2\theta\end{aligned}$$

Thus, either $r^2 \cos 2\theta = 2$ or $r^2 = 2 \sec 2\theta$ are valid equations of the given rectangular equation. ■

Conversion of equations from polar to rectangular form

To convert from polar to rectangular coordinates we use the relations

$$\sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r}, \quad r^2 = x^2 + y^2$$

Example 7-2 E illustrates converting polar equations into rectangular form.

■ Example 7-2 E

Convert the polar equation into rectangular form.

- 1.
- $r = 5 \sec \theta$

$$\begin{aligned}r &= \frac{5}{\cos \theta} \\r \cos \theta &= 5 \\r \cdot \frac{x}{r} &= 5 & \cos \theta = \frac{x}{r} \\x &= 5\end{aligned}$$

³Hyperbolas are not covered in this text. Any equation of the form $ax^2 - by^2 = c$, $a, b, c \neq 0$, and a and b have the same sign, is a hyperbola.

2. $r^2 = \cos 2\theta$

We do not have any relation for $\cos 2\theta$, so we use an identity to replace it.

$$r^2 = \cos^2 \theta - \sin^2 \theta$$

$$r^2 = \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\cos \theta = \frac{x}{r}, \sin \theta = \frac{y}{r}$$

$$r^2 = \frac{x^2}{r^2} - \frac{y^2}{r^2}$$

$$r^4 = x^2 - y^2$$

$$(x^2 + y^2)^2 = x^2 - y^2$$

$$r^4 = (r^2)^2 = (x^2 + y^2)^2$$

3. $r = 1 + \cos \theta$

$$r = 1 + \frac{x}{r}$$

$$r^2 = r + x$$

$$x^2 + y^2 = r + x$$

$$x^2 + y^2 - x = r$$

$$(x^2 + y^2 - x)^2 = r^2$$

$$(x^2 + y^2 - x)^2 = x^2 + y^2$$

$$x^4 - 2x^3 + y^4 + 2x^2y^2 - 2xy^2 - y^2 = 0$$

Multiply each member by r

Square both members to obtain r^2

Now replace r^2 by $x^2 + y^2$

Expand the left member and combine terms

Note Observe that r can be replaced by squaring both members as necessary to obtain r^2 , or any even power of r .

In example 7-2 E we used the identities $\sin \theta = \frac{y}{r}$ and $\cos \theta = \frac{x}{r}$. This is only valid if $r \neq 0$. The value of r is zero only if the pole is a solution to the given polar equation. Thus, when the pole is a solution to the polar equation we must verify that the point $(0,0)$ is a solution to the resulting rectangular equation.

In example 7-2 E, part 1, the pole is not a solution, since if $r = 0$, $0 = 5 \sec \theta = \frac{5}{\cos \theta}$. The equation $0 = \frac{5}{\cos \theta}$ has no solution. (The rectangular equation $x = 5$ does not pass through the origin either.)

In example 7-2 E, part 2, r can take on the value 0: $0 = \cos 2\theta$ has solutions. Thus, the pole is part of this equation. Observe that the point $(0,0)$ is also a solution to the equation $(x^2 + y^2)^2 = x^2 - y^2$.

Mastery points**Can you**

- Graph points in the polar coordinate system?
- Give alternate polar coordinates for a given point?
- Convert between polar and rectangular coordinates?
- Convert equations between polar and rectangular form?

Exercise 7-2

Graph the following points in polar coordinates.

- | | | | | | |
|--------------------------------------|------------------------------------|------------------------------------|---------------------------------------|--------------------------------------|--------------------------------------|
| 1. $(3, 0)$ | 2. $(4, \pi)$ | 3. $\left(2, \frac{\pi}{6}\right)$ | 4. $\left(1, \frac{\pi}{3}\right)$ | 5. $\left(2, \frac{3\pi}{4}\right)$ | 6. $\left(3, \frac{7\pi}{6}\right)$ |
| 7. $\left(6, \frac{11\pi}{6}\right)$ | 8. $\left(5, \frac{\pi}{2}\right)$ | 9. $(-2, \pi)$ | 10. $\left(-4, \frac{5\pi}{3}\right)$ | 11. $\left(-1, \frac{\pi}{3}\right)$ | 12. $\left(-5, \frac{\pi}{6}\right)$ |
| 13. $(4, 2)$ | 14. $(3, 5)$ | 15. $(-4, 6)$ | 16. $\left(-4, \frac{\pi}{2}\right)$ | 17. $\left(5, \frac{4\pi}{3}\right)$ | 18. $\left(4, \frac{5\pi}{6}\right)$ |

List three other coordinates for the point, with two points having $r > 0$ and one point having $r < 0$.

- | | | | | | |
|-------------------------------------|-------------------------------------|---------------------------------------|--------------------------------------|--------------|---------------------------------------|
| 19. $\left(2, \frac{\pi}{6}\right)$ | 20. $\left(1, \frac{\pi}{3}\right)$ | 21. $\left(6, \frac{11\pi}{6}\right)$ | 22. $\left(-5, \frac{\pi}{6}\right)$ | 23. $(2, 2)$ | 24. $\left(4, \frac{17\pi}{6}\right)$ |
|-------------------------------------|-------------------------------------|---------------------------------------|--------------------------------------|--------------|---------------------------------------|

Convert the following polar coordinates into rectangular coordinates. Leave the result in exact form.

- | | | | | | |
|-------------------------------------|-------------------------------------|--------------------------------------|---------------------------------------|--------------------------------------|--------------------------------------|
| 25. $\left(4, \frac{\pi}{2}\right)$ | 26. $\left(2, \frac{\pi}{3}\right)$ | 27. $\left(5, \frac{5\pi}{6}\right)$ | 28. $\left(1, \frac{11\pi}{6}\right)$ | 29. $\left(4, \frac{4\pi}{3}\right)$ | 30. $\left(2, \frac{5\pi}{3}\right)$ |
|-------------------------------------|-------------------------------------|--------------------------------------|---------------------------------------|--------------------------------------|--------------------------------------|

Convert the following polar coordinates into rectangular coordinates to two-decimal place accuracy.

- | | | | | | |
|--------------|----------------|-----------------|--------------|--------------|--------------|
| 31. $(2, 1)$ | 32. $(5, 1.2)$ | 33. $(3, 0.82)$ | 34. $(3, 5)$ | 35. $(4, 4)$ | 36. $(1, 6)$ |
|--------------|----------------|-----------------|--------------|--------------|--------------|

Convert the following rectangular coordinates into polar coordinates. Leave the result in exact form.

- | | | | | | |
|------------------------|---------------|---------------|----------------------|----------------|--------------|
| 37. $(-2\sqrt{3}, -2)$ | 38. $(3, -3)$ | 39. $(-2, 0)$ | 40. $(-1, \sqrt{3})$ | 41. $(-4, -4)$ | 42. $(0, 1)$ |
|------------------------|---------------|---------------|----------------------|----------------|--------------|

Convert the following rectangular coordinates into polar coordinates to two-decimal place accuracy.

- | | | | | | |
|--------------|---------------|---------------|----------------|--------------|---------------|
| 43. $(2, 3)$ | 44. $(-5, 2)$ | 45. $(1, -4)$ | 46. $(-4, -3)$ | 47. $(5, 4)$ | 48. $(-3, 5)$ |
|--------------|---------------|---------------|----------------|--------------|---------------|

Convert the following rectangular equations into polar equations.

- | | | | |
|----------------------------|---------------------|----------------------|-------------------|
| 49. $y = 4x$ | 50. $y = -2x$ | 51. $y = -3x + 2$ | 52. $y = 5x - 3$ |
| 53. $y = mx + b, b \neq 0$ | 54. $y = 2$ | 55. $y^2 - 2x^2 = 5$ | 56. $y^2 - x = 4$ |
| 57. $3x^2 + 2y^2 = 1$ | 58. $x^2 + y^2 = 3$ | | |

Convert the following polar equations into rectangular equations.

- | | | | |
|--------------------------|-------------------------|---------------------------------------|-------------------------------------|
| 59. $r = \sin \theta$ | 60. $2r = \cos \theta$ | 61. $r = 2 \sec \theta$ | 62. $r = 3 \csc \theta$ |
| 63. $r = 3 \sin 2\theta$ | 64. $r = 2 \cos \theta$ | 65. $r^2 = \sin 2\theta$ | 66. $r = \cos 2\theta$ |
| 67. $r^2 = \tan \theta$ | 68. $r \sin \theta = 5$ | 69. $r = \frac{3}{1 - 2 \sin \theta}$ | 70. $r = \frac{5}{4 - \cos \theta}$ |

71. Show that a polar equation of $2xy = 5$ is $r^2 = 5 \csc 2\theta$.
72. In the text we noted that $x = r \cos \theta$ if $r > 0$. Show that this is also true for a point given in polar coordinates where $r < 0$. (Hint: Consider a point $P = (r, \theta)$, where $r < 0$. Then $P = (-r, \theta + \pi)$, where $-r > 0$. Therefore, since $-r > 0$, $x = -r \cos(\theta + \pi)$ is true. Proceed from here.)
73. In the text we noted that $y = r \sin \theta$ if $r > 0$. Show that this is also true for a point given in polar coordinates where $r < 0$. See the hint in problem 72.
74. The shape of a cam that drives a certain sewing machine needle is described by the polar equation $r = 3 - 2 \cos \theta$. Convert this equation into rectangular form.
75. The path that an industrial robot must follow to paint a pattern on a part being manufactured is described by the curve $r = 1 - 2 \sin \theta$. Convert this equation into rectangular form.
76. The pattern of strongest radiation of a certain bidirectional radio antenna is described by the curve $r = 1 + \sin 2\theta$. Convert this equation into rectangular form.
77. The pattern of strongest radiation of a certain radio antenna is described by the equation $r = 1 + 2 \sin 2\theta$. This pattern is said to have side lobes. Convert this equation into rectangular form.
- In the October 1983 issue of *Scientific American*, Jearl Walker described several rides, the Scrambler and the Calypso, at the Geauga Lake Amusement Park near Cleveland, Ohio.
78. Assume the path taken by the Scrambler is described by the polar equation $r = 2 \cos 3\theta$. Convert this equation into rectangular form. It will be necessary to rewrite $\cos 3\theta$ in terms of $\cos \theta$. See problem 79 in section 5-3.
79. Assume the path of the Calypso is described by the equation $r = 1 - 3 \cos \theta$. Convert this equation into rectangular form.

7-3 Graphs of polar equations

There are several ways to graph polar equations. The graphing calculator can be used, or one can sketch a graph by hand. We illustrate how to obtain polar graphs with the TI-81 graphing calculator. To achieve a sketch of a graph by hand we employ three methods:

1. plotting points,
2. putting a polar equation into rectangular form, and
3. using the corresponding rectangular graph as a guide.

The examples show how to create polar graphs both by hand and by graphing calculator.

Graphing polar equations with the graphing calculator

The TI-81 uses a mode called Parametric Mode to graph polar equations. Parametric mode uses two functions to produce rectangular coordinates. The first produces the x -coordinates, and the second produces the y -coordinates. We illustrate how to use the process by graphing the polar equation $r = \sin^2 \theta$.

The polar equation $r = \sin^2 \theta$ generates a collection of polar coordinates (r, θ) . In this case, r is $\sin^2 \theta$, so the polar coordinates generated are $(\sin^2 \theta, \theta)$. To convert a polar coordinate to a rectangular coordinate we map (r, θ) to $(x, y) = (r \cos \theta, r \sin \theta)$ (section 7-2), so for this function we map $(\sin^2 \theta, \theta)$ to the point $(x, y) = (\sin^2 \theta \cdot \cos \theta, \sin^2 \theta \cdot \sin \theta)$.

As stated above, in parametric mode a point (x,y) is calculated from two separate equations that depend on a third variable, usually called t (the TI-81 uses T). Thus, we graph the set of rectangular coordinates $(\sin^2 t \cdot \cos t, \sin^2 t \cdot \sin t)$, which of course is the same as $(\sin^2 t \cdot \cos t, \sin^3 t)$. We usually let t take on the values from 0 to 2π to obtain the graph, but this may need to be adjusted depending on the functions in question.

As illustrated above, to graph a polar function of the form $r = f(\theta)$, put the calculator in parametric mode and graph

$$X_{1T} = f(T) \cdot \cos T$$

$$Y_{1T} = f(T) \cdot \sin T$$

To create the graph of the polar equation $r = \sin^2 \theta$ do the following:

MODE

Select Param instead of Function. Do this by positioning the blinking square over Param and using **ENTER**. Also select Polar instead of Rect. This has no effect except when using the trace feature. Then the values T , R and θ are shown instead of T , X and Y .

Y=

The display now shows the variables X_{1T} and Y_{1T} (as well as others). Enter the equations shown above. Note that in parametric mode the **X T** key is used to enter T . The display should look like $:X_{1T} = (\sin T)^2 \cdot \cos T$
 $:Y_{1T} = (\sin T)^3$.

RANGE

Remember, the exponent 3 is **MATH** 3.

The range display is different in parametric mode. Set the following values: $Tmin=0$

$$Tmax = 6.28 \quad (2\pi)$$

$$Tstep = .105 \quad \left(\frac{\pi}{30}\right)$$

$$Xmin = -1$$

$$Xmax = 1$$

$$Xscl = .5$$

$$Ymin = -1$$

$$Ymax = 1$$

$$Yscl = .5$$

ZOOM 5

We want the display to be square.

The graph is drawn as shown in figure 7-5.

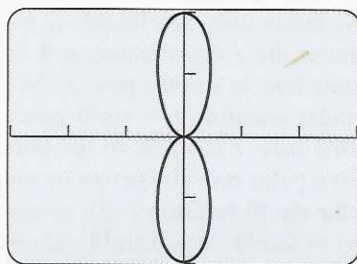


Figure 7-5

We show the range settings in the rest of this section in the order shown above. Thus, for the graph above we show

$$X_{1T} = (\sin T)^2 \cos T, Y_{1T} = (\sin T)^3,$$

RANGE 0,6.28,0.105,-1,1,.5,-1,1,.5, **ZOOM** 5

Graphing polar equations by plotting points

Of course any graph can be obtained by plotting enough points. Example 7-3 A shows a case where this might be done without plotting too many points.

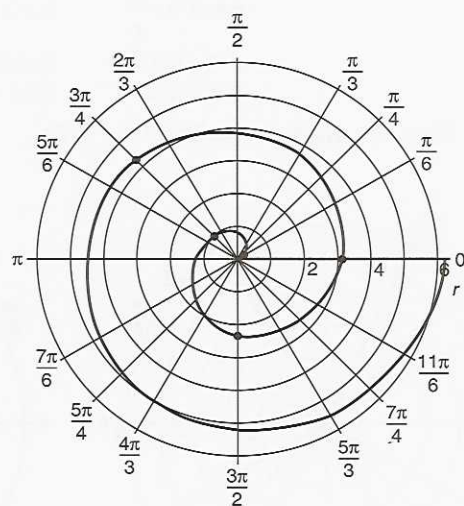
■ Example 7-3 A

Graph the polar equation $r = \frac{\theta}{2}$, $\theta \geq 0$.

We observe that r increases at half the rate of θ . A table of values is given in table 7-1, and these values are plotted in the figure.

θ	r
$\frac{\pi}{4}$	$\frac{\pi}{8} \approx 0.4$
$\frac{3\pi}{4}$	$\frac{3\pi}{8} \approx 1.2$
$\frac{5\pi}{4}$	$\frac{5\pi}{8} \approx 2.4$
$\frac{7\pi}{4}$	$\frac{7\pi}{8} \approx 3.1$
$\frac{9\pi}{4}$	$\frac{9\pi}{8} \approx 4.3$

Table 7-1



The graph shows points (r, θ) for $0 \leq \theta \leq 4\pi$, but the graph, a spiral, continues on forever.

$$X_{1T} = T/2 \cos T, Y_{1T} = T/2 \sin T,$$

RANGE 0,20,0.105,-6,6,1,-6,6,1, **ZOOM** 5

The next two methods presented can save the time required to calculate the many points that are often required to graph an equation by plotting points.

Graphing polar equations by putting in rectangular form

■ Example 7-3 B

Graph the polar equation $\theta = 2$.

An angle of 2 (radians) is shown in part (a) of the figure. Any point $(r, 2)$, $r > 0$, lies on the terminal side of this angle.

Recall that $(-r, \theta) = (r, \theta + \pi)$. Therefore $(-r, 2) = (r, 2 + \pi)$. Thus, when r is negative, its graph is along the line containing the terminal side of the angle $\theta = 2 + \pi$. Thus, the graph of $\theta = 2$ is the line shown in part (b) of the figure.

Another way to see the shape of this graph is to *rewrite the equation in rectangular form*.

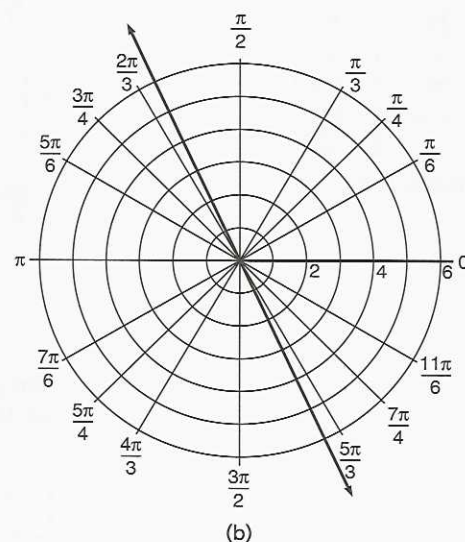
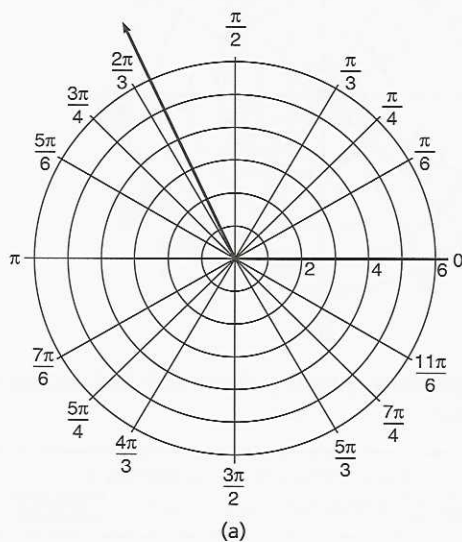
$$\tan \theta = \frac{y}{x}$$

$$\tan 2 = \frac{y}{x} \quad \theta = 2$$

$$y = (\tan 2)x \quad \text{Straight line, slope} = \tan 2$$

$$y \approx -2.2x \quad \tan 2 \approx -2.2$$

Thus, the graph is the line $y \approx -2.2x$.



The equation $\theta = 2$ does not describe r as a function of θ , since r is not even mentioned! This means that r can take on any value. Thus, any point in the graph is of the form $(r, 2)$. Parametrically we let $r = T$ and $\theta = 2$.

$$X_{1T} = T \cos 2, Y_{1T} = T \sin 2,$$

RANGE $-7, 7, 0.105, -6, 6, 1, -6, 6, 1$ **ZOOM** 5

Graphing polar equations using the rectangular form as a guide

The rectangular form of an equation can provide guidance for drawing the polar form of an equation. This is especially true when the rectangular form is periodic. There are two principles involved in this process. To discuss this we define the term “positive lobe.” A *positive lobe* is a portion of a graph in *rectangular* coordinates that starts and ends on the x -axis, and is continuous (it has no breaks, such as a vertical asymptote). Figure 7-6 (a) shows two positive lobes, at A and at B.

Every positive lobe corresponds to a lobe in a polar graph. A lobe (in a polar graph) is a closed figure that starts and ends at the pole. Figure 7-6 shows the correspondence between two positive lobes A and B in a rectangular graph (a), and the corresponding lobes in a polar graph (b).

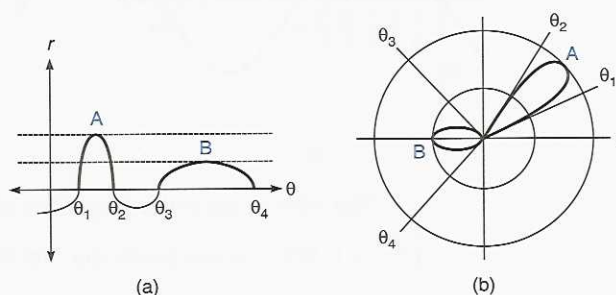


Figure 7-6

Another important point uses the fact that $(-r, \theta)$ has the same graph as $(r, \theta \pm \pi)$. This means that negative lobes in a rectangular graph may be redrawn as positive lobes by shifting their graphs $\pm\pi$ units, before graphing in polar coordinates. Although this method sometimes seems complicated, with a little practice it is much easier than having to plot many points. Example 7-3 C illustrates this method.

Example 7-3 C

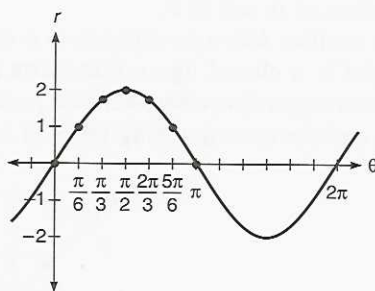
Graph the polar equation, using the rectangular form as a guide.

1. $r = 2 \sin \theta$

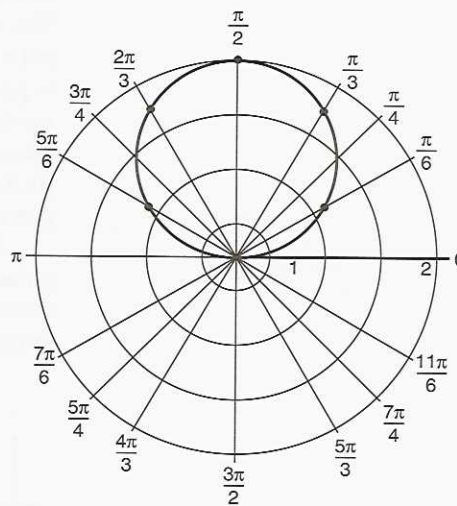
Part (a) of the figure is the graph of $r = 2 \sin \theta$, $0 \leq \theta \leq 2\pi$, in rectangular coordinates (as done in chapter 3). We observe that there is a positive lobe from $0 \leq \theta \leq \pi$. This lobe is graphed in part (b) of the figure. Table 7-2 shows values plotted in both graphs. The negative lobe from π to 2π gives the same graph in polar form as the positive lobe. This is because if we shifted the negative lobe π units to the left and made it positive, it would be the same as the positive lobe.

θ	$2 \sin \theta$
0	0
$\frac{\pi}{6}$	1
$\frac{\pi}{3}$	1.7
$\frac{\pi}{2}$	2
$\frac{2\pi}{3}$	1.7
$\frac{5\pi}{6}$	1
π	0

Table 7-2



(a)



(b)

The lobe in the polar graph is a circle with center at the polar point $\left(\frac{\pi}{2}, 1\right)$. We will not prove the fact that the graph is actually a circle.

$$X_{IT} = 2 \sin T \cos T, \quad Y_{IT} = 2 \sin T \sin T,$$

RANGE 0,3.14,0.105,-2,2,1,-2,2,1, ZOOM 5

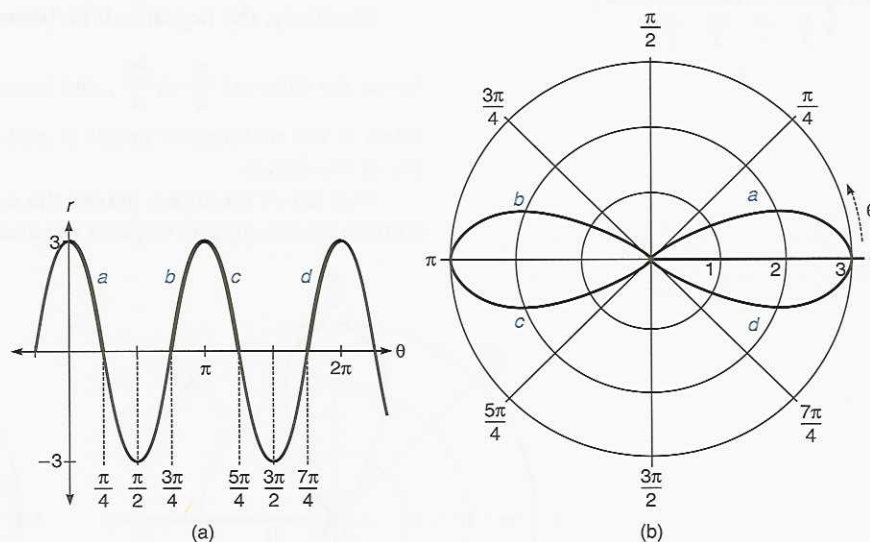
2. $r = 3 \cos 2\theta$

First graph in rectangular coordinates (as in section 3-3).

$$0 \leq 2\theta \leq 2\pi$$

$$0 \leq \theta \leq \pi \quad \text{Divide each member by 2}$$

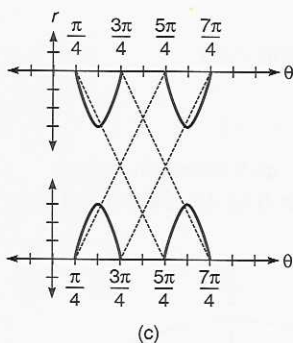
We get one basic cosine cycle, with amplitude 3, as θ takes on values from 0 to π . There are thus two basic cycles from 0 to 2π . Part (a) of the figure shows this graph.



Part (b) of the figure is obtained from part (a) in the following manner. Part (a) shows that as θ goes from 0 to $\frac{\pi}{4}$, r goes from 3 down to 0. In part (b), $r = 3$ when $\theta = 0$. As θ increases to $\frac{\pi}{4}$, r moves down to 0. This produces the half of a lobe labeled *a*.

Part (a) also shows that as θ goes from $\frac{3\pi}{4}$ to π , r goes from 0 to 3. This produces the half lobe *b* in part (b) of the figure. As θ continues from π to $\frac{5\pi}{4}$ (part (a)), r goes from 3 back down to 0. This produces the effect at *c* in part (b).

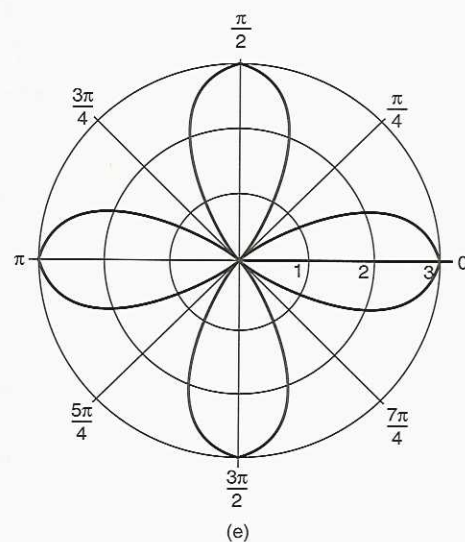
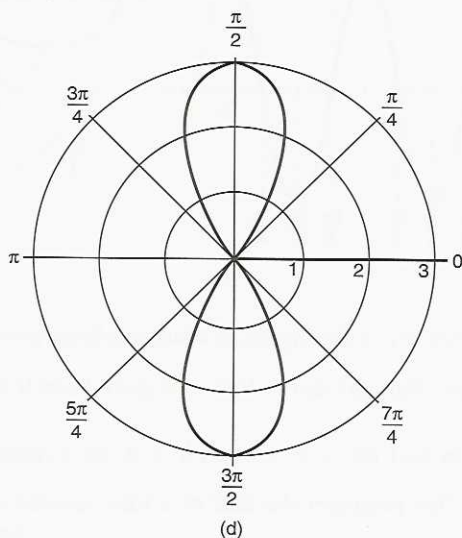
The half lobe at *d* (part (b)) is produced from what r is doing at *d* in part (a).



Part (c) of the figure shows one way to handle negative lobes in the rectangular graph. Shifting θ by π or $-\pi$ units causes r to change its sign; thus, we can shift the negative lobe between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, by adding π , to the interval $\frac{5\pi}{4}$ to $\frac{7\pi}{4}$, and in the process the negative lobe becomes positive.

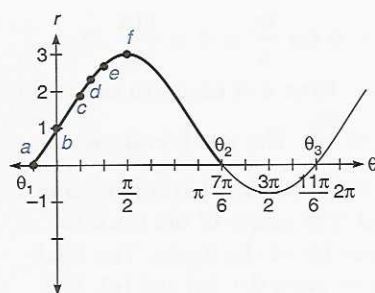
Similarly, the negative lobe between $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ is shifted $-\pi$ units to lie on the interval $\frac{\pi}{4}$ to $\frac{3\pi}{4}$, and becomes a positive lobe. These positive lobes in the rectangular graph in part (c) produce the lobes shown in part (d) of the figure.

Part (e) of the figure shows the complete graph produced by combining the graphs of parts (b) and (d).



$$X_{1T} = 3 * \cos 2T * \cos T, \quad Y_{1T} = 3 * \cos 2T * \sin T,$$

RANGE 0,6,3,0.105,-3,3,1,-3,3,1, ZOOM 5



(a)

3. $r = 1 + 2 \sin \theta$

The graph of $r = 1 + 2 \sin \theta$ in rectangular coordinates is shown in part (a) of the figure. It is the graph of $y = 2 \sin \theta$, shifted up by one unit.

Note that r goes from 0 up to 3 and back to 0 between angles θ_1 and θ_2 . It would help to find these two angles. They occur where $r = 0$, so we solve for θ where $r = 0$.

$$r = 1 + 2 \sin \theta$$

$$0 = 1 + 2 \sin \theta$$

$$-1 = 2 \sin \theta$$

$$-\frac{1}{2} = \sin \theta$$

$$\theta' = \sin^{-1} \frac{1}{2}$$

$$\theta' = \frac{\pi}{6} (30^\circ)$$

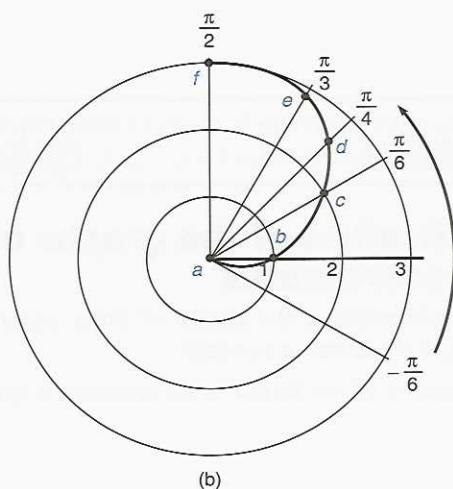
θ is an angle in quadrant III or IV, since $\sin \theta < 0$. Thus, θ is

$$\pi + \frac{\pi}{6} = \frac{7\pi}{6} \text{ and } 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}. \text{ By subtracting the period, } 2\pi,$$

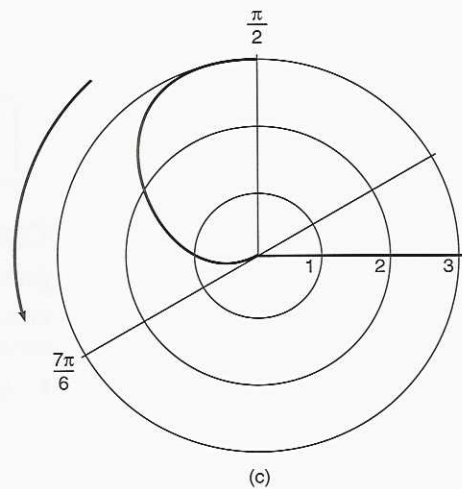
from $\frac{11\pi}{6}$ we can see that θ_1 is $-\frac{\pi}{6}$.

Now, referring to part (a) of the figure, we see that as θ goes from $-\frac{\pi}{6}$ to $\frac{\pi}{2}$, r goes from 0 to 3. This is shown in part (b) of the figure.

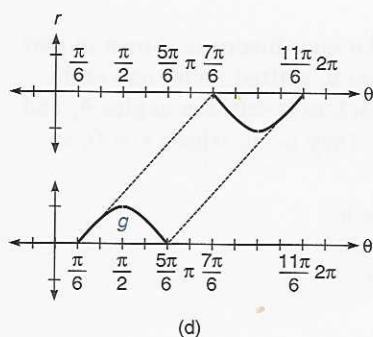
Similarly, as θ goes from $\frac{\pi}{2}$ to $\frac{7\pi}{6}$, r goes from 3 back down to 0. This is shown in part (c).



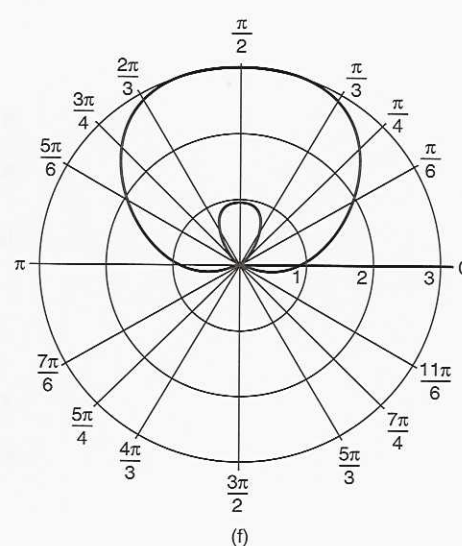
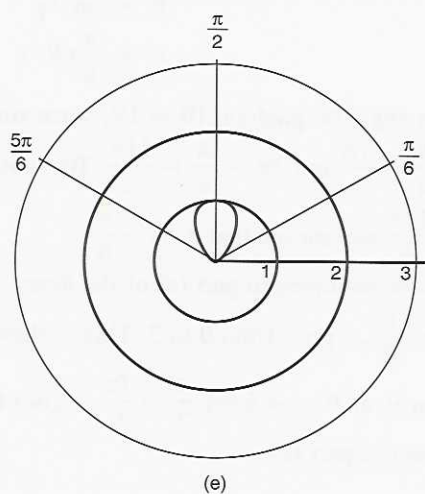
(b)



(c)



In part (a) of the figure we see that $r < 0$ for $\frac{7\pi}{6} \leq \theta \leq \frac{11\pi}{6}$. By adding or subtracting π , r becomes positive. Here it is easier to subtract π from each of these values, giving $\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$. The result is shown in rectangular coordinates in part (d), where we see that r varies from 0 to 1 and back to 0 as θ moves over this interval. The graph of the positive lobe g , in polar coordinates, is shown in part (e) of the figure. The final graph of $r = 1 + 2 \sin \theta$ is a combination of parts (b), (c) and (e). It is shown in part (f) of the figure.



$$X_{1T} = (1 + 2\sin T) \cos T, Y_{1T} = (1 + 2\sin T) \sin T,$$

RANGE 0,6.3,0.105,-3,3,1,-3,3,1, ZOOM 5

Classification of the graphs of equations in polar coordinates

Some classification of the graphs of polar equations has been done. In the following, k represents a constant.

1. An equation of the form $r = k\theta$ produces a *spiral* (example 7-3 A).

2. An equation of the form $r \sin \theta = k$ or $r \cos \theta = k$ produces a *horizontal or vertical straight line* for its graph. When graphing these, it is easiest to put the equation in rectangular form. For example, $r \sin \theta = 2$ can be transformed, using $\sin \theta = \frac{y}{r}$.

$$r \sin \theta = 2$$

$$r \cdot \frac{y}{r} = 2$$

$$y = 2$$

This is a horizontal straight line.

3. Equations of the form $\theta = k$ also produce *straight lines* (example 7-3 B).
4. Equations of the form $r = k$ produce *circles with center at the pole*.
5. Equations of the form $r = k \cos \theta$ or $r = k \sin \theta$ also produce *circles, which pass through the pole, but have centers elsewhere* (example 7-3 C, part 1).
6. The graph of an equation of the form $r = k \cos n\theta$ or $r = k \sin n\theta$ is called a *rose*. It has n leaves if n is odd, and $2n$ leaves if n is even. Example 7-3 C, part 2, is a four-leafed rose.
7. Equations of the form $r = a + b \cos \theta$ or $r = a + b \sin \theta$ produce a figure called the *limaçon*. Example 7-3 C, part 3, is an example. If $|a| = |b|$, the graph is heart shaped and is called a *cardioid*.

Mastery points

Can you

- Graph a polar equation by plotting points?
- Graph a polar equation by using its rectangular graph as a guide?

Exercise 7-3

Graph the polar equation.

- | | | | |
|------------------------------|----------------------------------|--|--|
| 1. $r = 6$ | 2. $r = 2$ | 3. $r = -3$ | 4. $r = -1$ |
| 5. $\theta = 1$ | 6. $\theta = 3$ | 7. $\theta = \frac{\pi}{3}$ | 8. $\theta = -1$ |
| 9. $r \sin \theta = 6$ | 10. $r \cos \theta = 1$ | 11. $r \cos \theta - 2 = 0$ | 12. $r \sin \theta = -3$ |
| 13. $r = 3 \sin \theta$ | 14. $r = 2 \cos \theta$ | 15. $r = 4 \cos \theta$ | 16. $r = \sin \theta$ |
| 17. $r = 3 \sin 2\theta$ | 18. $r = 2 \cos 3\theta$ | 19. $r = 3 \cos 4\theta$ | 20. $r = \sin 3\theta$ |
| 21. $r = 1 + \sin \theta$ | 22. $r = 1 - \sin \theta$ | 23. $r = 1 - 2 \cos \theta$ | 24. $r = 2 - 3 \sin \theta$ |
| 25. $r = 2 - 2 \sin 2\theta$ | 26. $r = 1 - \cos 2\theta$ | 27. $r = 1 + \cos 3\theta$ | 28. $r = 2 + \sin 2\theta$ |
| 29. $r = \theta, \theta < 0$ | 30. $r = 1 + \theta, \theta > 0$ | 31. $r = \frac{\theta}{4}, \theta > 0$ | 32. $r = 1 + \frac{\theta}{2}, \theta > 0$ |

33. The shape of a cam that drives a certain sewing machine needle is described by the polar equation $r = 3 - 2 \cos \theta$. Draw the cam by graphing this equation.
34. The path an industrial robot must follow to paint a pattern on a part being manufactured is described by the curve $r = 1 - 2 \sin \theta$. Graph this equation.

35. The pattern of strongest radiation of a certain bidirectional radio antenna is described by the curve $r = 1 + \sin 2\theta$. Graph this pattern.
36. The pattern of strongest radiation of a certain radio antenna is described by the equation $r = 1 + 2 \sin 2\theta$. This pattern is said to have side lobes. Graph the pattern.

As noted in section 7-2, in the October 1983 issue of *Scientific American*, Jearl Walker described several rides at the Geauga Lake Amusement Park near Cleveland, Ohio. The Scrambler is a ride whose motion could be described as a three-leafed rose, and the motion of the Calypso could be described as a limaçon.

37. Graph the path taken by the Scrambler, assuming that its motion is described by the polar equation $r = 2 \cos 3\theta$.
38. Graph the path of the Calypso, assuming that its motion is described by the equation $r = 1 - 3 \cos \theta$.

Chapter 7 summary

- **Imaginary unit** $i = \sqrt{-1}$
- **Complex number (rectangular form)** A number of the form $a + bi$, a and b both real numbers.
- **Complex conjugate** The complex conjugate of $a + bi$ is $a - bi$.
- **Equality of complex numbers** $a + bi = c + di$ if and only if $a = c$ and $b = d$.
- **Polar form of a complex number** If $z = a + bi$ is a complex number that determines an angle θ , then $r \operatorname{cis} \theta$ is its polar form, where $\operatorname{cis} \theta$ means $\cos \theta + i \sin \theta$. The value r is called the modulus of z , which is also written $|z|$, and

$$r = \sqrt{a^2 + b^2}, \quad \tan \theta' = \tan^{-1} \frac{b}{a}, \text{ and}$$

$$\theta = \begin{cases} \theta' & \text{if } a > 0 \\ \theta' - 180^\circ & \text{if } \theta' > 0 \\ \theta' + 180^\circ & \text{if } \theta' < 0 \end{cases} \quad \text{if } a < 0$$

- **Complex multiplication—polar form**
 $(r_1 \operatorname{cis} \theta_1)(r_2 \operatorname{cis} \theta_2) = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$.
- **Complex division—polar form**
 $\frac{r_1 \operatorname{cis} \theta_1}{r_2 \operatorname{cis} \theta_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$, $r_2 \neq 0$.
- **De Moivre's theorem** $(r \operatorname{cis} \theta)^n = r^n \operatorname{cis} n\theta$ for any real number n .

- **De Moivre's theorem for roots** The n n th roots of $r \operatorname{cis} \theta$ are of the form $(r \operatorname{cis} \theta)^{\frac{1}{n}} = r^{\frac{1}{n}} \operatorname{cis} \left(\frac{\theta}{n} + \frac{k \cdot 360^\circ}{n} \right)$, $0 \leq k < n$, where k and n are positive integers.
- The polar coordinates of a point is an ordered pair of the form (r, θ) , where r is the radius and θ is an angle.
- **Equivalence of points in polar coordinates**
 1. $(r, \alpha) = (r, \beta)$ if α and β are coterminal angles.
 2. $(-r, \theta) = (r, \theta \pm \pi)$.

- **Relation between polar and rectangular coordinates** If $P = (x, y) = (r, \theta)$, $r > 0$, then $\cos \theta = \frac{x}{r}$ and $\sin \theta = \frac{y}{r}$. Thus, $x = r \cos \theta$ and $y = r \sin \theta$. Also,

$$r = \sqrt{x^2 + y^2}, \quad \tan \theta' = \tan^{-1} \frac{y}{x}, \text{ and}$$

$$\theta = \begin{cases} \theta' & \text{if } x > 0 \\ \theta' - \pi & \text{if } \theta' > 0 \\ \theta' + \pi & \text{if } \theta' < 0 \end{cases} \quad \text{if } x < 0$$

- To convert a rectangular equation into a polar equation use the relations $x = r \cos \theta$, $y = r \sin \theta$, and $r^2 = x^2 + y^2$.
- To convert a polar equation into a rectangular equation use the relations $\sin \theta = \frac{y}{r}$, $\cos \theta = \frac{x}{r}$, and $r^2 = x^2 + y^2$.

Chapter 7 review

[7-1] Identify the real and imaginary parts of the following complex numbers.

1. $3 - i$ 2. $3i$ 3. 12 4. $-i$

Simplify each expression by performing the indicated operations.

5. $(2 - 4i) + (-3 + 2i)$ 6. $(-7 + 4i) - (-13 + 3i)$
 7. $(5 - 4i)(3 + 12i)$ 8. $(-2 + 5i)(3 - 6i)$
 9. $4i(2 - 3i)$ 10. $(4 - i)(2 - 3i)(5 + 2i)$
 11. $\frac{10 - 4i}{5 + 2i}$ 12. $\frac{-2 + i}{2 - 4i}$ 13. $\frac{8 + 6i}{14i}$

If $z_1 = 2 + i$ and $z_2 = 3 - 2i$, evaluate the following expressions.

14. $2z_1 - z_2$ 15. $z_1(3 - z_2)$
 16. $\frac{z_1 + z_2}{z_1 - z_2}$ 17. $\frac{z_1(z_1 - z_2)}{2z_2}$

In each of the following problems, (a) find the complex conjugate of the given number and (b) graph both the number and its complex conjugate in the same graph.

18. $-4 + 3i$ 19. $-4 + i$ 20. $3i$ 21. 17

Write the polar forms of the following numbers (to the nearest tenth).

22. $3 - 2i$ 23. $\sqrt{3} + 3i$ 24. $-1 - 2i$

Write the polar forms of the following complex numbers. Leave the result in exact form.

25. $\sqrt{3} - i$ 26. $3 + 3i$ 27. $-4i$

Write the rectangular forms of the following complex numbers (to the nearest tenth).

28. $3 \text{ cis } 35^\circ$ 29. $3 \text{ cis } 243^\circ$

Write the rectangular forms of the following complex numbers. Leave the result in exact form.

30. $3 \text{ cis } 240^\circ$ 31. $10 \text{ cis } 330^\circ$

Multiply the following complex numbers.

32. $(2 \text{ cis } 25^\circ)(3 \text{ cis } 45^\circ)$
 33. $(2 \text{ cis } 18^\circ)(6.5 \text{ cis } 122^\circ)$

Divide the following complex numbers.

34. $\frac{40 \text{ cis } 120^\circ}{5 \text{ cis } 20^\circ}$ 35. $\frac{50 \text{ cis } 45^\circ}{100 \text{ cis } 9^\circ}$

36. Compute the cube of $2 \text{ cis } 130^\circ$.

37. Compute the fourth power of $2 \text{ cis } 150^\circ$.

38. Compute an approximation to $(0.8 + 0.6i)^8$ (to the nearest tenth).

39. Find the 4 fourth roots of 16 in exact form.

40. Find the 3 cube roots of -27 in exact form.

41. In electronics one version of Ohm's law says that

$I = \frac{V}{Z}$, where I is current, V is voltage, and Z is impedance. Find I in a circuit in which $V = 130 \text{ cis } 25^\circ$ and Z is $30 \text{ cis } 75^\circ$.

[7-2] Graph the following points in polar coordinates.

42. $(2, 0)$ 43. $(2, \pi)$ 44. $\left(3, \frac{5\pi}{6}\right)$
 45. $\left(-6, \frac{11\pi}{6}\right)$ 46. $\left(1, \frac{\pi}{2}\right)$ 47. $(-3, \pi)$
 48. $(2, 1)$ 49. $(3, 4)$ 50. $(-4, 1)$

Convert the following polar coordinates into rectangular coordinates. Leave the result in exact form.

51. $\left(3, \frac{11\pi}{6}\right)$ 52. $\left(4, \frac{2\pi}{3}\right)$ 53. $\left(-2, \frac{5\pi}{3}\right)$

Convert the following polar coordinates into rectangular coordinates to two-decimal-place accuracy.

54. $(2, 1)$ 55. $(5, 2)$ 56. $(1, 0.5)$

Convert the following rectangular coordinates into polar coordinates to two-decimal-place accuracy.

57. $(2, 1)$ 58. $(-5, 3)$ 59. $(-1, -4)$

Convert the following rectangular equations into polar equations.

60. $y = -3x$ 61. $y = 4x + 2$
 62. $2y^2 - x^2 = 5$ 63. $y^2 - 3x = 0$

Convert the following polar equations into rectangular equations.

64. $r = \sin \theta$ 65. $r = 2 \sec \theta$
 66. $r^2 = \sin 2\theta$ 67. $r^2 = \tan \theta$
 68. $r \sin \theta = 2$ 69. $r = \frac{3}{2 - \sin \theta}$

[7-3] Graph the polar equations.

70. $r = -2$ 71. $\theta = \frac{5\pi}{6}$
 72. $2r \sin \theta = 8$ 73. $r = 3 \cos \theta$
 74. $r = \sin 2\theta$ 75. $r = 3 \cos 3\theta$
 76. $r = 1 + 2 \cos \theta$ 77. $r = 2 - \sin \theta$

Chapter 7 test

Simplify each expression by performing the indicated operations.

1. $(12 - 4i) - (-3 + 2i)$ 2. $(1 - 4i)(2 + 8i)$
3. $\frac{2 - 5i}{2 - 3i}$
4. If $z = 1 + 2i$, evaluate $3z^2 - 2z + 5$.
5. Write the polar form of the number $4 - 5i$ (to the nearest tenth).
6. Write the rectangular form of the number $2 \operatorname{cis} 120^\circ$. Leave the result in exact form.
7. Multiply $(2 \operatorname{cis} 325^\circ)(7 \operatorname{cis} 145^\circ)$. Leave the result in polar form.
8. Divide $\frac{6 \operatorname{cis} 140^\circ}{2 \operatorname{cis} 20^\circ}$. Leave the result in polar form.
9. Compute the cube of $3 \operatorname{cis} 150^\circ$.
10. Find the 4 fourth roots of -16 in exact form.

Graph the following points in polar coordinates.

11. $\left(2, \frac{11\pi}{6}\right)$ 12. $\left(2, \frac{\pi}{3}\right)$ 13. $(-2, 1)$

Convert the following polar coordinates into rectangular coordinates to two-decimal-place accuracy.

14. $(3, 0.8)$ 15. $(-5, 2)$

Convert the following rectangular coordinates into polar coordinates. Leave the result in exact form.

16. $(-\sqrt{3}, -1)$ 17. $(-5, 5)$

Convert the following rectangular equations into polar equations.

18. $y = -3x + 5$ 19. $2y^2 - x = 5$

Convert the following polar equations into rectangular equations.

20. $r = 2 \csc \theta$ 21. $r^2 = \cos 2\theta$

Graph the polar equations.

22. $r = -\theta, \theta > 0$ 23. $r = 2 \csc \theta$
24. $r = 3 \cos 2\theta$

Appendix A

Graphing—Addition of Ordinates

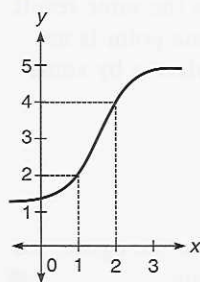


Figure A-1

Although the advent of desktop computers and calculators with graphics capabilities has given many users of trigonometry the ability to graph a function without putting pencil to paper, there are often instances where it is useful to be able to obtain a quick sketch of a function by hand. The method we examine in this section, **addition of ordinates**, can be very helpful in this regard. Also, studying this method leads to a fuller understanding of functions, and this understanding will aid even the person using a computer's graphing capabilities. We assume a thorough knowledge of section 3-3.

We learned in section 2-1 that a function is a set of ordered pairs in which no first element repeats. When we graph a function in a rectangular coordinate system, we use two perpendicular axes. We use horizontal distances to represent domain elements, or values of the argument of the function. Vertical distances are used to represent range elements, or values of the function for a given value of the argument. In this way, each ordered pair represents a domain element and the corresponding range element. In an ordered pair, the first element is sometimes called the **abscissa**; the second element is then called the **ordinate**. For example, consider the graph of a function f , shown in figure A-1. Although we do not know the "formula" for f (i.e., an expression that defines f), or even whether such a formula exists, we can still see that $f(1) = 2$ and $f(2) = 4$. We know that $f(1) = 2$, because if we go a horizontal distance of 1, corresponding to domain element 1, or an abscissa of 1, we see that the associated vertical distance is 2, representing a range element, or ordinate, of 2.

In general, if we go a horizontal distance x , the corresponding vertical distance represents $f(x)$.

Since we use the y -axis to plot values of $f(x)$, we often write $y = f(x)$, or replace $f(x)$ by y in a formula.

The graph of a linear function is relatively simple. A linear function is a function of the form $f(x) = mx + b$. (We often simply write $y = mx + b$.) The graph of such a function is a straight line. This means that if we can find two points on the line (i.e., that satisfy the function) we know the graph is a straight line which must pass through them.

Thus, to graph a linear function, we find two points that lie on the line and draw a straight line through them. Usually these two points are the x - and y -intercepts.

Example A

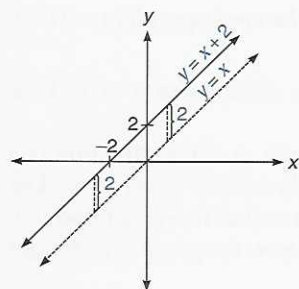
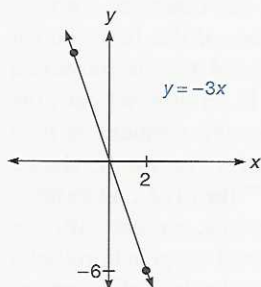
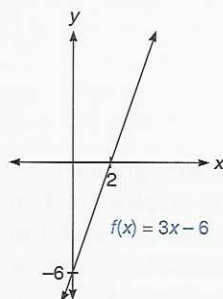


Figure A-2

1. Graph the linear function $f(x) = 3x - 6$.

It is easier to do the necessary algebra if we replace $f(x)$ by y , giving $y = 3x - 6$.

To find the x -intercept we replace y by 0, since for any point on the x -axis, y is 0.

$$\begin{aligned} y &= 3x - 6 \\ 0 &= 3x - 6 \\ 2 &= x \end{aligned}$$

Therefore, the x -intercept is $(2, 0)$.

To find the y -intercept we replace x by 0.

$$\begin{aligned} y &= 3x - 6 \\ y &= 3(0) - 6 \\ y &= -6 \end{aligned}$$

Therefore, the y -intercept is $(0, -6)$.

In the figure we plot these two points and draw the straight line through them.

2. Graph the linear function $f(x) = -3x$.

First we write $y = -3x$. Replacing x by 0 or y by 0 gives the same result, $(0, 0)$. The origin is, therefore, the only intercept. Since one point is not enough to determine a straight line, we find another. Replace x by some value other than 0, say 2.

$$\begin{aligned} y &= -3x \\ y &= -3(2) \\ y &= -6 \end{aligned}$$

Therefore, a second point that lies on the line is $(2, -6)$. In the figure, we plot both points and draw the line that passes through them. ■

Now, consider the graph of $f(x) = x + 2$. First, we shall relabel this as $y = x + 2$. Its graph is shown in figure A-2. If we graph the equation $y = x$, shown in dashed lines in figure A-2, we see that every point in the graph of $y = x + 2$ is two units above every point in the graph of $y = x$. Another way to view this is that the graph of $y = x + 2$ is the sum of the graphs of $y = x$ and $y = 2$. This summing is done vertically. (We are adding the ordinates.) That is, to graph $y = x + 2$ for a given value of x , we take the vertical distance in the graph of $y = x$ and add the vertical distance in $y = 2$. This process is shown for $x = 3$ and for $x = -5$ in figure A-3. Note that we treat each distance as a directed distance. If the function is positive, we move up; if negative, we move down.

For example, at the point $x = 3$, we move up three units and then two more units. The length 3 represents the value of the equation $y = x$ when x is 3. The length 2 represents the value of the equation $y = 2$ when x is 3.

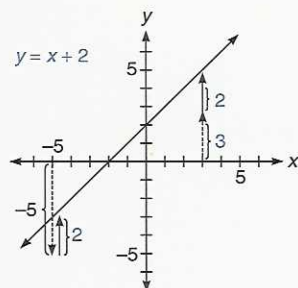


Figure A-3

At the point $x = -5$, we add vertical lengths -5 and 2 . The length -5 (5 in the downward direction) represents the value of the equation $y = x$ when x is -5 , and the length 2 represents the value of the equation $y = 2$ when x is -5 .

Figure A-4 illustrates the process of addition of ordinates in general. We assume that we have a function h described by an expression that is the sum of the expressions for two functions f and g . That is, $h(x) = f(x) + g(x)$. For the purpose of illustration, $f(x)$ might be the expression $0.25 + \sin x$, and $g(x)$ might be the expression $\cos(x + 0.5)$. Then, $h(x)$ would be

$$\begin{aligned} h(x) &= f(x) + g(x) \\ &= (0.25 + \sin x) + [\cos(x + 0.5)] \\ &= \sin x + \cos(x + 0.5) + 0.25 \end{aligned}$$

In figure A-4 we see the graphs of the two functions, f and g , and the addition of ordinates at x_1 , x_2 , x_3 , x_4 , and x_5 .

Figure A-4(a) shows the process at x_1 . $f(x_1)$ and $g(x_1)$ are both positive, so the value of $h(x_1)$ is represented by the combined heights of f and g at this point.

Figure A-4(b) shows the case where $f(x_2) = g(x_2)$. Since these values are equal, the combined value $f(x_2) + g(x_2)$ is twice the height of either f or g at this point.

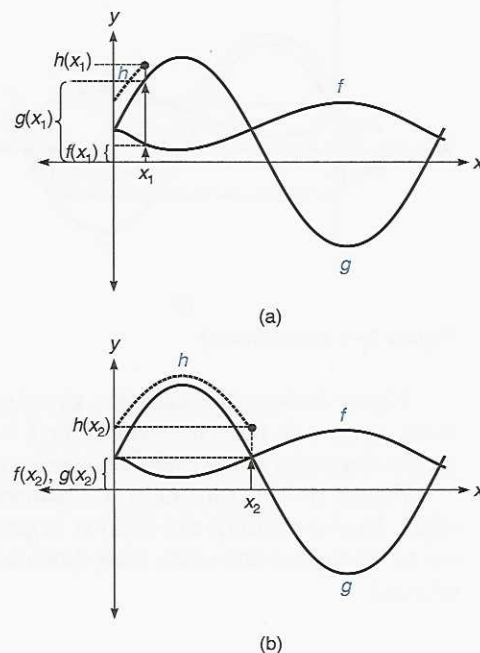
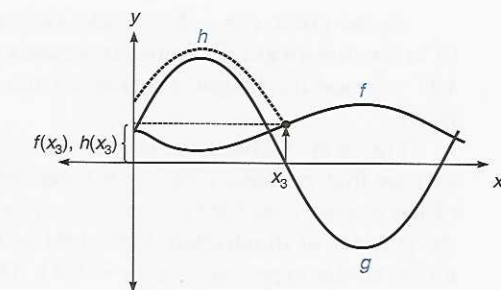
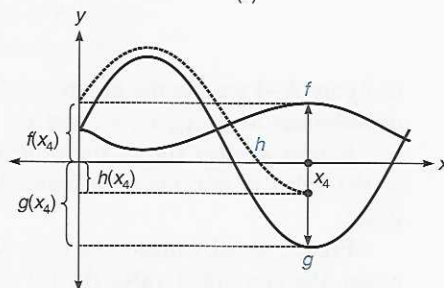


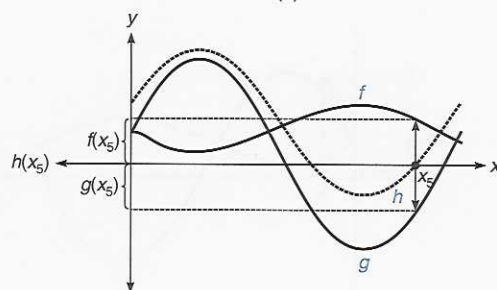
Figure A-4



(c)



(d)



(e)

Figure A-4 (continued)

Figure A-4(c) illustrates the situation where one function is 0 at a point. Here, $g(x_3) = 0$; thus, the sum of $f(x_3) + g(x_3)$ is just $f(x_3)$. In other words, at x_3 , the functions h and f take on equal values.

Figure A-4(d) illustrates a situation where the functions have opposite signs. $f(x_4)$ is positive and $g(x_4)$ is negative. Thus, to form the value of $h(x_4)$ we go up to $f(x_4)$ and come back down a distance corresponding to the height of $g(x_4)$.

Figure A-4(e) is like the situation at x_4 , except that $f(x_5)$ and $g(x_5)$ have equal absolute values. Thus, their sum is 0, so that $h(x_5) = 0$.

This discussion can provide *guidelines for finding the graph of a sum of two functions using addition of ordinates*.

1. Wherever the two functions cross, the result is twice the height of the functions (as in figure A-4 (b)).
2. Wherever one function is zero the result is the same as the other function (figure A-4 (c)).
3. Wherever the functions have opposite signs but equal absolute values the result is 0 (figure A-4 (e)).

Using these guidelines we can often obtain quite a few points in the result. We must then graph enough intermediate points (as in figure A-4 (a) and (d)) to get a good idea of what the result looks like.

The function $f(x) = x + \sin x$ provides a concrete example. First we rewrite $y = x + \sin x$ for convenience. We view this as the sum of the graphs of $y = x$ and $y = \sin x$. See figure A-5, where the graph of each of these functions is shown.

To obtain the graph of $y = x + \sin x$, we proceed in the following way. We select an abscissa (moving a given horizontal distance along the x -axis) and then add the ordinate of $y = x$ and $y = \sin x$. This is shown for several abscissas in figure A-6. Observe that wherever the function $y = \sin x$ is zero, the result is the other function, $y = x$. This is guideline 2, above. Many other points in the graph of $y = x + \sin x$ are also shown in figure A-6, on the dotted line. We graph only enough points to see what the finished graph looks like. Actually, with a little experience, we often require only a few points to see the result. The graph of $y = x + \sin x$ is shown in figure A-7. We can view this as the function $y = \sin x$ “riding on” the function $y = x$.

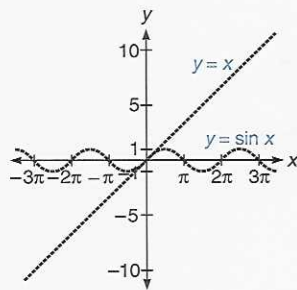


Figure A-5

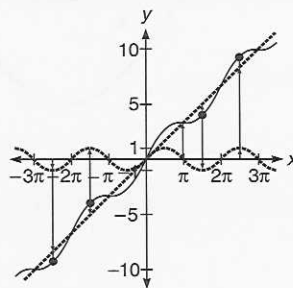


Figure A-6

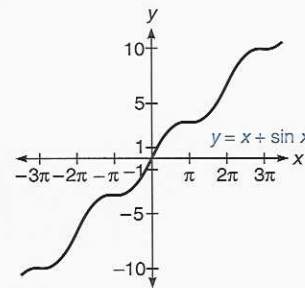
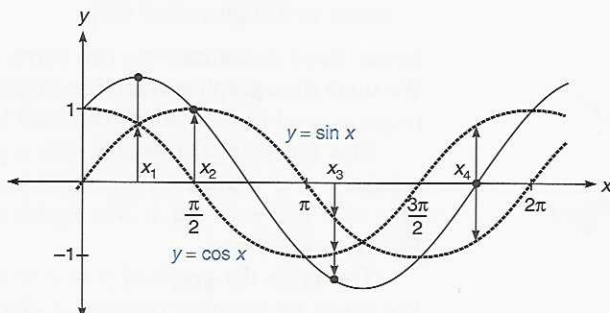


Figure A-7

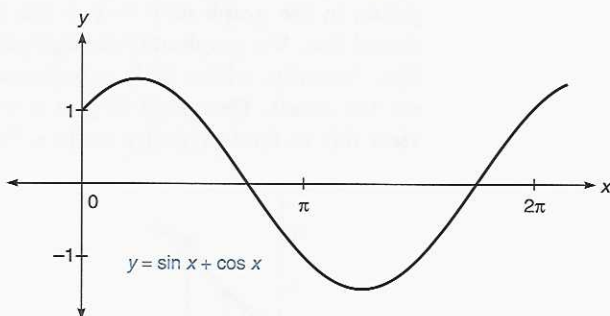
■ Example B

1. Graph the equation $y = \sin x + \cos x$ for $0 \leq x \leq 2\pi$.

The figure shows the graphs of $y = \sin x$ and $y = \cos x$ in dashed lines, along with the additions of ordinates for several values of x . In particular, at x_1 the functions have equal values and so the result is twice the height of either. At x_2 one function is zero, so the result is the same as the nonzero function. At x_3 both functions are negative, but there is nothing special about either (i.e., none of the three guidelines apply). We must add the signed distances here. At x_4 the functions have equal absolute values but opposite signs. Thus, the result is 0 at this point.



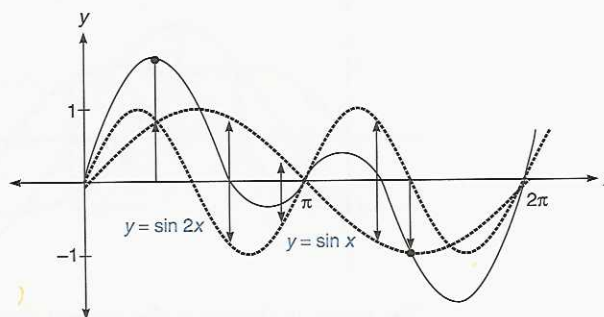
The result is shown at many other points also, on the solid line. The graph of $y = \sin x + \cos x$ is shown in the next figure.



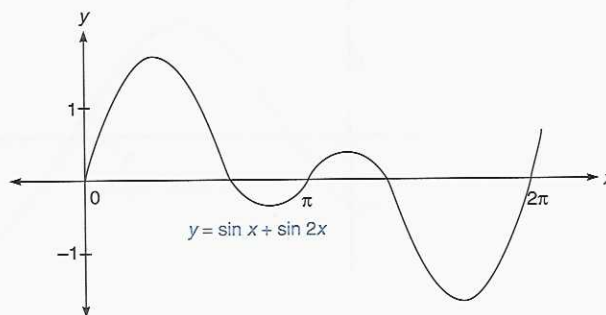
Note It can be shown that $\sin x + \cos x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$.

2. Graph the equation $y = \sin x + \sin 2x$ for $0 \leq x \leq 2\pi$.

The figure shows the graphs of $y = \sin x$ and $y = \sin 2x$ in dashed lines, along with the additions of ordinates for several values of x , and the result at many other points (solid line).



The result is shown in the next figure.



If the expression includes subtraction, we rewrite it in terms of addition.

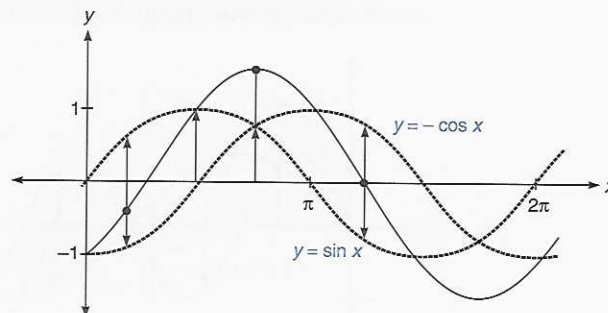
■ Example C

Graph the equation $y = \sin x - \cos x$ for $0 \leq x \leq 2\pi$.

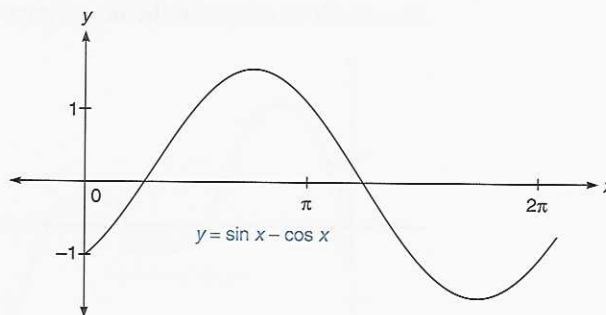
Since it is easier to picture addition of ordinates than to picture subtraction, we rewrite the equation as

$$y = \sin x + (-\cos x)$$

We now graph $y = \sin x$ and $y = -\cos x$. These are shown in the first figure with the additions of ordinates for several values of x and the result on the solid line.



The result is shown in the second figure.

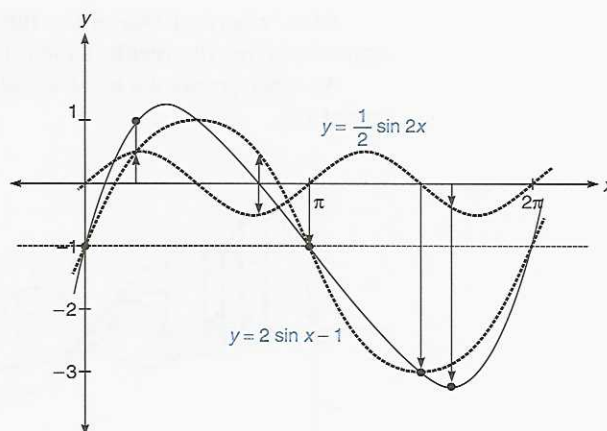


Note It can be shown that $\sin x - \cos x = \sqrt{2} \sin\left(x - \frac{\pi}{4}\right)$. ■

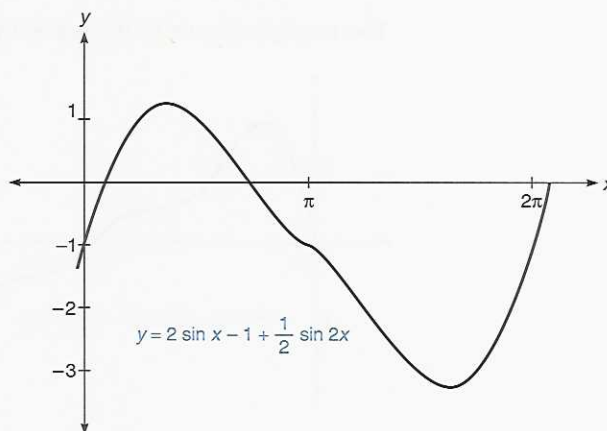
■ Example D

1. An electronic signal that is described by the equation $y = 2 \sin x - 1$ is serving as a carrier for another signal described by $y = \frac{1}{2} \sin 2x$. That is, the finished signal is described by $y = 2 \sin x - 1 + \frac{1}{2} \sin 2x$. Graph this signal for $0 \leq x \leq 2\pi$.

We will graph $y = 2 \sin x - 1$ and $y = \frac{1}{2} \sin 2x$ in dashed lines. Although we could graph $y = 2 \sin x - 1$ itself by the method of addition of ordinates, we should recall that this is just the graph of $y = 2 \sin x$ shifted down by one unit (section 3-3). The graphs of these two signal components are shown in the first figure, as well as the addition of several ordinates. The results are shown for many points on the solid line.



The result is shown in the second figure.



2. Assume a pure musical tone is being represented by $y = \sin x$. Its second harmonic is then $y = \frac{1}{2} \sin 2x$, and $y = \frac{1}{3} \sin 3x$ represents the third harmonic. Assume that all three tones are present and describe the result graphically.

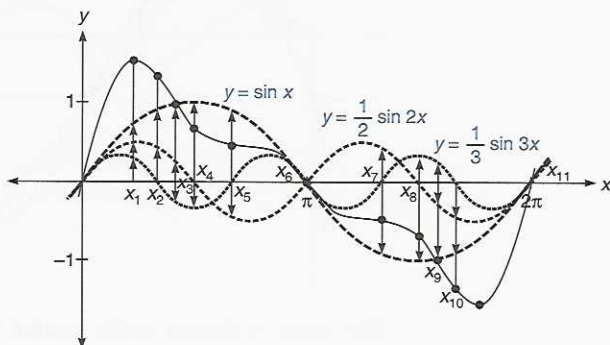
We graph the three functions separately and then add the ordinates of all three. The three graphs are shown in the first figure, as are the addition of ordinates for several values of x and the result for many points on the dotted line.

There are some points that can be of help in a complicated case like this.

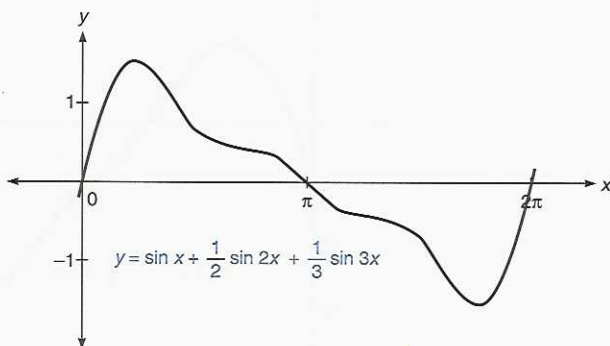
Wherever one of the functions is zero, the result is the sum of the other two functions. Thus, good selections for x are values where any of the three functions cross the x -axis (points $x_2, x_4, x_5, x_6, x_7, x_8, x_{10}$, and x_{11}).

Also, wherever two of the functions have equal absolute values but opposite signs, the result is the third function (points x_3 and x_9).

At other points we must simply compute the sum for three values (point x_1).



The result is shown in the second figure.



Mastery points

Can you

- Graph a function using addition of ordinates?

Exercise A

Graph the following linear functions.

1. $f(x) = 5x - 3$

4. $f(x) = \frac{1}{2}x + 2$

2. $f(x) = -2x + 7$

5. $f(x) = \frac{2}{3}x - 1$

3. $f(x) = -x + 3$

6. $f(x) = -2x$

Graph the following functions, using addition of ordinates, for $0 \leq x \leq 2\pi$.

7. $y = x + 2 \sin x$

8. $y = -x + \sin x$

9. $y = 2x + 2 \sin x$

10. $y = 2x + \sin 2x$

11. $y = \sin x + \sin 3x$

12. $y = \sin 2x + \cos x$

13. $y = \sin 2x + \cos 3x$

14. $y = 2x - \sin x$

15. $y = -x - \sin x$

16. $y = 2x - 2 \sin x$

17. $y = \sin x - \sin 3x$

18. $y = \sin x - \cos x$

19. $y = \sin 2x - \cos x$

20. The equation of time combines the effects of the inclination of the earth in its orbit about the sun, and the eccentricity of the orbit. It gives the difference between mean solar time and actual solar time throughout the year. The function that describes the effect of the eccentricity of the earth's orbit can be approximately described by $y = 7.5 \sin \frac{\pi}{6}x$, and the function that describes the effect of the inclination of the earth is approximately $y = 10 \sin \frac{\pi}{3}x$. The amplitudes tell the number of degrees of deviation between mean and actual solar time.

To find the combined effects, we graph the sum of the two functions, $y = 7.5 \sin \frac{\pi}{6}x + 10 \sin \frac{\pi}{3}x$. Graph this equation of time for $0 \leq x \leq 12$ (for 12 months).

21. The electric voltage supplied to United States homes has a frequency of 60 cycles per second (cps), or a period of $\frac{1}{60}$ of one second. It can be described by the function $y = 180 \sin 120\pi x$, where the amplitude is instantaneous voltage and x is in seconds.
- Graph this function. Use divisions of $\frac{1}{240}$ of one second, and graph two complete cycles.
 - Suppose that a wind-driven home generator is generating electricity, but due to a fault in its speed governor it is generating at 50 cps and is only producing 25 volts. Assuming that these sources are in phase at some point and that they are tied together (electrically speaking), the resulting voltage can be described by $y = 180 \sin 120\pi x + 25 \sin 100\pi x$ (until a fuse blows). Graph this resultant voltage for $0 \leq x \leq \frac{1}{25}$.

22. Most people have heard about the theory of biorhythms. First stated by Dr. Wilhelm Fliess in Germany at the beginning of the century, this theory maintains that at birth three cycles are started—physical, emotional, and intellectual. These have periods of 23, 28, and 33 days, respectively, and can be (presumably) described by sinusoidal waves. With a time axis in days, we compute the equation for the physical as follows:

$$0 \leq x \leq 23$$

$$\frac{2\pi}{23}(0) \leq \frac{2\pi}{23}x \leq \frac{2\pi}{23}(23)$$

$$0 \leq \frac{2\pi}{23}x \leq 2\pi$$

Thus, the equation for the physical state is $y = \sin \frac{2\pi}{23}x$.

- Find the equations for the emotional and intellectual cycles.
- Graph the function that is the sum of these three cycles using addition of ordinates. Graph it for the first 33 days of life. (All three functions start together, or are in phase, at birth.)
- Find the period of the resulting function, in both days and years.

Appendix B

Further Uses for Graphing Calculators and Computers

The use of a scientific calculator has been assumed throughout this text. We have also shown the use of graphing calculators. This appendix presents problems that are best done *only* with a graphing calculator.

Specific examples refer to the Texas Instruments TI-81 programmable, graphing calculator and a Casio fx-7000G programmable, graphing calculator. It is assumed that the user has read at least the introductory material in the handbook for the calculator.

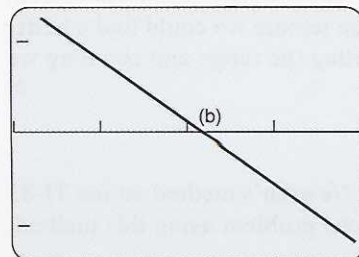
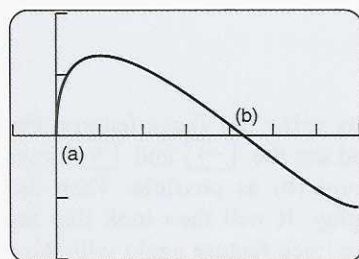
Finding zeros of functions

This topic was introduced in section 5-4. We revisit it here with examples that are difficult to solve algebraically. We also show how to accomplish these same tasks with the Casio and Texas Instruments calculators.

Estimating zeros from the graph

The approximate zeros of a function can be found by graphing the function. This is because a zero of a function is an x -intercept of the function's graph.

■ Example A



Find approximate zeros to the function $f(x) = 2 \sin(\sqrt{2}x) - \frac{1}{2}$ for $0 \leq x < 2\pi$ by graphing.

Texas Instruments TI-81 Calculator

The following steps would produce a graph similar to that shown in the upper screen in the figure. It is essential that the **MODE** key be used to set the calculator to radian (Rad) mode. As can be seen in the graph the function has

Steps	Explanation
Range	Enter the x - and y -axis limits.
0 ENTER	Xmin becomes 0.
7.5 ENTER	Xmax becomes 7.5.
1 ENTER	Xscl becomes 1.
(-) 2.5 ENTER	Ymin becomes -2.5.
2.5 ENTER	Ymax becomes 2.5.
1 ENTER	Yscl becomes 1.
2nd CLEAR	Quit

Enter the function into Y_1 .

Y= 2 **sin** **(** **2nd** **x²** **(** 2 **X/T** **)** **)**
- 0.5 **GRAPH**

zeros at approximate x -values of 0.1 and 4.1. It would be possible to obtain a better estimate by using the trace feature. For example, select **TRACE** and use the **◀** and **▶** keys to move the blinking dot as close to the point (b) as possible. Then use **ZOOM** 2 **ENTER** to expand the display. It will then look like the lower screen. Using the **TRACE** feature again will show that point (b) is between the values 4.16 and 4.20. The trace feature shows the current value of x and y in the display. By noting the values of x when the value of y changes sign we can find estimates of the value of x at the zero at (b). By repeating the zoom and trace features we could find a better and better estimate of the zero at (b). By resetting the range and zooming we could repeat the process for the zero at (a).

Casio fx-7000G Calculator

The following steps will produce a graph similar to that shown in the previous figure. As can be seen in the graph the function has zeros at approximate x -values of 0.1 and 4.1

Steps	Explanation
Range	Enter the x - and y -axis limits.
0 EXE	Xmin becomes 0.
7.5 EXE	Xmax becomes 7.5.
1 EXE	Xscl becomes 1.
(-) 2.5 EXE	Ymin becomes -2.5.
2.5 EXE	Ymax becomes 2.5.
1 EXE	Yscl becomes 1.

We are back at the execution level at this point. If not, use the **Range** key again.

Graph	"Y =" appears
2 sin (√ (2 ALPHA	
+)) - 0.5 EXE	

It is possible to obtain a better estimate by using the *Trace* feature. For example, select **SHIFT** **Graph** (trace) and use the **→** and **←** keys to move the blinking dot as close to the point (b) as possible. Then use **SHIFT** **×** (multiply) to expand the display. It will then look like the lower screen in the preceding figure. Using the trace feature again will show that point (b) is approximately 4.149. (The trace feature shows the current value of x in the display.) By repeating the trace feature we could find a better and better estimate of the zero at (b). By resetting the range and zooming we could repeat the process for the zero at (a). ■

The TI-81 and Newton's Method

Section 5-4 presented solving equations using Newton's method on the TI-81 calculator. The next example solves the previous problem using this method.

Example B

Find the zeros of $y = 2 \sin(\sqrt{2x}) - \frac{1}{2}$ for $0 \leq x < 2\pi$ (from example A) using Newton's method.

We assume the function is entered into Y_1 as described in example A, and use the trace function to position the cursor near the point (a) in the figure there. Then select **PRGM** 1 (assumes the program NEWTON is stored as the first program). The display shows Prgm1 in the display. Use **ENTER** to run the program and find the approximate value of x at the point (a) to be 0.0319236557 when the program is finished.

Note The zoom feature must be used several times before the root at (a) can be found by the program NEWTON. Not zooming produces an error.

This is because the program calculates a value $D = \frac{XMAX - XMIN}{100}$. This

value D is used to compute new values of x . Using $XMAX$ of 7.5 and $XMIN$ of 0 produces a value of D that causes the program to try to use a negative value of x . The value $\sqrt{2x}$ causes a problem when this happens. Rerun the program by resetting the range to its initial values, then selecting **GRAPH** and then **TRACE** again, placing the cursor near (b). This will show that the zero at (b) is approximately 4.172907423. ■

Newton's method, used in example B, cannot be conveniently programmed into the Casio calculator. This is because the Texas Instruments calculator has a built-in function called $NDeriv$ that is not available in the Casio. For the Casio, we illustrate another method for approximating zeros of functions—the method of bisection. This method is also suitable for programming on a computer. At the end of this appendix is a Pascal language program for this method.

The Casio fx-7000G and the method of bisection

Figure B-1 illustrates the method of bisection¹ for some hypothetical function. We let z mean zero, L mean lower, U mean upper, and A mean average. Suppose we know there is exactly one zero of the function, at z in the figure, between two values, L and U (part 1 of figure B-1). We can see that $f(L) < 0$ and $f(U) > 0$.

If we let $A = \frac{L + U}{2}$ be the average of L and U , A divides the interval

between L and U in half. We calculate $f(A)$ and discover that $f(A) < 0$. This means the zero z must be between A and U . This is because the function must go from a negative value at A to a positive value at U , thus taking on the value 0 somewhere in between. Thus, change L to mean the value at c , and repeat the process (part 2 of figure B-1).

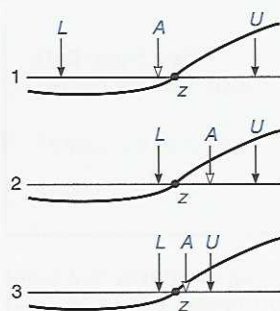


Figure B-1

¹To "bisect" means to cut in half.

We now consider the new smaller interval from L to U , and compute the midpoint of this interval, $A = \frac{L + U}{2}$. Since $f(A) > 0$ the zero is between L and A . We therefore let U represent this value of c (part 3 of figure B-1) and repeat the process, obtaining the new point A . We repeat this process until we find a value x so that $f(x) = 0$, or until the interval is so short that its midpoint is a good enough approximation to the zero.

To simplify the program we require $f(L) < 0$ and $f(U) > 0$, as in figure B-1. If this is not true, we interchange the values of L and U . This does not affect the procedure.

An algorithm for this procedure is described below. A program for a Casio fx-7000G calculator and a Pascal language program are given at the end of this appendix.

Finding a root of a function by bisection

Assumptions

1. The function has exactly one zero between two values L and U , $L < U$.
2. The function is continuous for all values from L to U inclusive.
3. $f(L) \neq 0$ and $f(U) \neq 0$.
4. $f(L)$ and $f(U)$ have different signs (the root is not of even multiplicity).

Algorithm

{Comments are enclosed in braces.}

```

Start:  Read in the values  $L$  and  $U$ .
        If  $f(L) > 0$  then                                {We want  $f(L) < 0$ ,  $f(U) > 0$ }
            interchange  $L$  and  $U$ .
Loop:   Let  $A = \frac{L + U}{2}$ .                                {A is the average of  $L$  and  $U$ }
        If  $f(A) = 0$  then
            go to Finish.                                {A is the zero and we are done}
        If  $|L - U| < \text{some\_predetermined\_value}$  then
            go to Finish.
        If  $f(A) < 0$  then                                {See figure B-1}
            Let  $L = A$ .                                    {Part 1 of the figure.}
        otherwise { $f(A) > 0$  so}
            let  $U = A$ .                                    {Part 2 of the figure.}
            go to Loop.
Finish: Print out the value of  $A$ .

```

The program for the Casio given at the end of this appendix requires two input values of x , which provide an upper (U) and lower (L) bound for an interval that contains a single zero of an equation. These values might easily be found by graphing the equation. The program is stored in, say program 8.

The function for which the zeros are to be calculated is stored as program 9. It must be an expression in the variable X and its last statement must be $\rightarrow Y$.

Example C illustrates using this program.

■ Example C

Find the zeros of $y = 2 \sin(\sqrt{2}x) - 0.5$ (from example A) using the method of bisection, programmed into program 8 on a Casio fx-7000G. (We assume the Casio program that implements the method of bisection has been entered into program 8.)

To enter the equation $y = 2 \sin(\sqrt{2}x) - 0.5$ into memory as program 9 proceed as follows.

Key strokes	Comments
MODE 2	Enter WRT (Program WRITE) mode.
Use the → key to select Program 9.	
Then EXE .	
2 sin (√ (2 ALPHA	
+)) - 0.5 →	
ALPHA -	
MODE 1	Go back to run mode.

To run the program to find the zero proceed as follows. Based on figure B-1 we use $L = 0$, $U = 1$.

Key strokes	Comments
Prog 8 EXE	Execute program 8.
?	The ? in the display means to enter a value.
0 EXE	Enter the value 0. This is L .
?	
1 EXE	Enter the value 1. This is U .

0.03192365567 appears when the program is finished.

The zero at (a) in figure B-1 is thus approximately 0.03192365567. Re-running the program with suitable values for L and U shows that the zero at (b) is approximately 4.172907423. ■

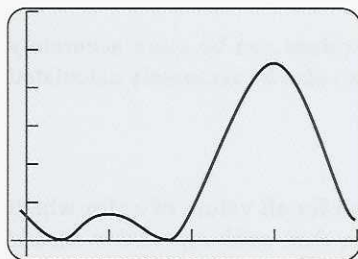


Figure B-2

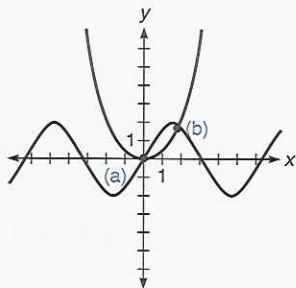
Note that the program in 8 can be used with any expression that is entered into program 9. The only requirement is that the expression end with “ $\rightarrow y$.”

Note If the graph of a function does not cross the x axis at a zero, the zero is said to have even multiplicity. For example, $f(x) = (\sin x - \frac{1}{2})^2$ has a zero at $\frac{\pi}{6}$, but neither the method of bisection nor Newton's method will find it. Figure B-2 shows the graph of this function.

Solving systems of two equations in two variables

A system of two equations in two variables is a collection of two equations, each of which is described in terms of the same two variables, usually x and y . To solve a system of two equations in two variables is to find all ordered pairs (x,y) that satisfy both equations. Graphically, these points are where the graphs intersect. Example D shows two methods to solve these systems with the graphing, programmable calculator.

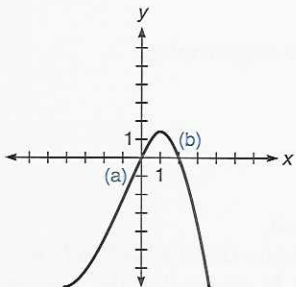
■ Example D



Solve the system of equations $y = 2 \sin x$ and $y = \frac{1}{2}x^2$.

Method 1: Graph both equations in the same coordinate system. The point(s) of intersection of the graphs are the solutions. In the figure, there are two such points. One is the origin (a) $(0,0)$; the second appears to be approximately $(1.9, 1.9)$ (b).

By expanding the graph near this second point we might get a better estimate. A TI-81 displays both the x - and y -values when tracing. On the Casio fx-7000G, use the $\boxed{X \leftrightarrow Y}$ feature to see both the x - and y -values.



Method 2: A second method, which will obtain just the x -value, is to graph the difference of the two x -expressions, in this case $y = (2 \sin x) - (\frac{1}{2}x^2)$.

The points in which we are interested are zeros of this new function. The graph is shown in the figure, where we estimate the x -value of the point at (b) to be 1.9. Using Newton's method or bisection we can quickly find that $x \approx 1.933753763$.

The y -values can be found by evaluating either of the two equations $y = 2 \sin x$ or $y = \frac{1}{2}x^2$ for y using $x = 1.933753763$.

Doing this gives $y \approx 1.869701808$. ■

The advantage of method 2 is that the x -values can be more accurately estimated, since they fall on the x -axis. They can also be accurately calculated using the methods cited earlier.

Verifying identities

An identity states that two expressions are equal for all values of x (for which each is defined) (chapter 5). If this is the case, then *each expression should have the same graph*.

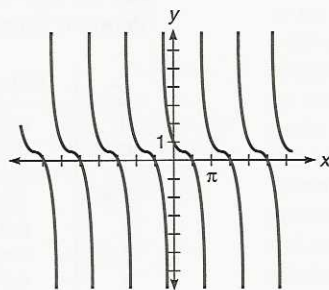
■ Example E

Show that $\frac{\cos x}{\cos x + \sin x} = \frac{\cot x}{1 + \cot x}$ is probably an identity by graphing each member of the equation.

(This is an identity from section 5-1.)

Graph both $y = \frac{\cos x}{\cos x + \sin x}$ and $y = \frac{\cot x}{1 + \cot x}$. The figure is the graph of both, which indicates that it is probably an identity.

Note Most calculators and computers do not have the cotangent function built in. Thus, each instance of $\cot x$ must be replaced by $\frac{1}{\tan x}$ to graph the expression.



The fact that two graphs look the same is not *proof* that two expressions are identical. A graph is finite and constructed by a machine. For example, on a Casio fx-7000G the graphs of $y = \sin(6x)$ and $y = \sin(100x)$ are the same! This phenomenon is explored in the exercises.

Exercises for Appendix B

- Graph the following equations for $0 \leq x \leq 4$.
- Using the graph, estimate the value of all zeros of the equation for $0 \leq x \leq 4$.
- Using the method of bisection or Newton's method find the value of each zero to at least six decimal places.

1. $y = \sin\left(\frac{x^2}{2}\right)$

2. $y = \sin(\sqrt{3}x) - \frac{1}{2}$

3. $y = \sin x - \sin 2x$

4. $y = \sin 3x + \sin 2x$

5. $y = 2 \sin x - \frac{x}{2}$

6. $y = \cos 3x + \sin 2x$

Solve the following systems of equations by (a) estimating the solutions from their graphs and then (b) using Newton's method or the method of bisection to find the value of each solution to at least six decimal places.

7. $y = \sin x; y = \frac{x}{2}$

8. $y = \sin 3x; y = \sqrt{x}$

9. $y = \cos \frac{x}{2}; y = \frac{x}{2}$

10. $y = x; y = \sqrt{x}$

Show that the following are probably identities by graphing the left and right members and observing that the graphs look the same. Use $-2\pi \leq x \leq 2\pi$ for the domain of the graph.

11. $\csc \theta + \cot \theta = \frac{1 + \cos \theta}{\sin \theta}$

12. $\tan \theta + \sec \theta = \frac{1 + \sin \theta}{\cos \theta}$

13. $\frac{\csc \theta}{\sec \theta + \tan \theta} = \frac{\cos \theta}{\sin \theta + \sin^2 \theta}$

14. $\frac{\sec \theta}{\csc \theta + \cot \theta} = \frac{\tan \theta}{1 + \cos \theta}$

15. $\frac{1}{\sec \theta - \cos \theta} = \cot \theta \csc \theta$

16. $\frac{\cot \theta}{\sec \theta} = \csc \theta - \sin \theta$

17. Problems 7 through 19 in appendix A are problems for which a calculator is well suited. Do some of these problems. The results obtained with a calculator should be the same as the answers obtained in the method discussed in appendix A, addition of ordinates.
18. Do problem 20 in appendix A.
19. Do problem 21 in appendix A.
20. Do problem 22 in appendix A.
21. *Fourier series* are functions defined as a sum of an infinite number of terms of the form $a \sin(bx)$ or $a \cos(bx)$. They are used extensively in electronics, music, linguistics, and other areas to analyze wave forms. It turns out that any periodic wave form can be described by an appropriate Fourier series. The four examples in this problem describe “sawtooth” and square-wave forms.
- Graph an approximation to each of the following Fourier series by entering at least the first five terms into the calculator. Use the values $-\pi \leq x \leq \pi$ for the graphs. (As more and more terms are used the graphs become more accurate.)
- $f(x) = \sin x + \frac{1}{2} \sin(2x) + \frac{1}{3} \sin(3x) + \frac{1}{4} \sin(4x) + \cdots$
 - $f(x) = \sin x + \frac{1}{3} \sin(3x) + \frac{1}{5} \sin(5x) + \frac{1}{7} \sin(7x) + \cdots$
 - $f(x) = \frac{1}{2} \sin(2x) + \frac{1}{4} \sin(4x) + \frac{1}{6} \sin(6x) + \frac{1}{8} \sin(8x) + \cdots$
 - $f(x) = \sin x - \frac{1}{2} \sin(2x) + \frac{1}{3} \sin(3x) - \frac{1}{4} \sin(4x) + \cdots$

22. As stated in this section, on a Casio graphing calculator the graphs of $y = \sin 6x$ and of $y = \sin 100x$ are the same. (At the range settings used for graphing the sine function: Xmin = -6.2831853, Xmax = 6.2831853.) The same phenomenon can occur with any graphing calculator or computer.

To see why this sort of thing can happen, one must realize that the plotting area of most graphing devices is broken down into a grid of points, called pixels,² which are either darkened in or left light. The number of these pixels is insufficient for some purposes.

By way of example, assume a situation where pixels represent 0.5 units. (Range settings for which this might occur in the TI-81 and the Casio fx-7000G are given below.) This situation is shown in figure B-3. To graph a function we compute values for the function for each value of the domain in increments of 0.5.

²Pixel means “picture element.”

Table B-1 constructs a set of values for the two functions $y = \sin(x)$ and $y = \sin[(4\pi + 1)x]$, in x -increments of 0.5. These ordered pairs are plotted in figure B-3. Observe that the ordered pairs at the values of x shown have the same y -values, although for other values of x the two functions are not the same. The calculator can only darken complete 0.5 by 0.5 unit squares. When this is done we have the graph in figure B-4. We can see that *either function would produce the graph shown in figure B-4.*

x	$\sin(x)$	$\sin[(4\pi + 1)x]$
0	0.00	0.00
0.5	0.48	0.48
1	0.84	0.84
1.5	1.00	1.00
2	0.91	0.91
2.5	0.60	0.60
3	0.14	0.14
3.5	-0.35	-0.35
4	-0.76	-0.76
4.5	-0.98	-0.98
5	-0.96	-0.96
5.5	-0.71	-0.71
6	-0.28	-0.28
6.5	0.22	0.22

Table B-1

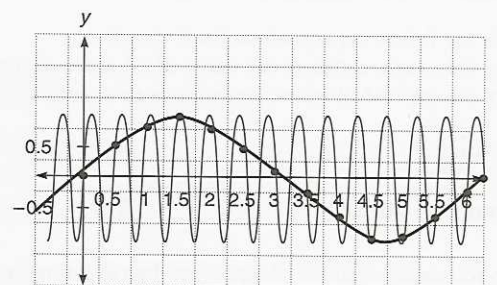


Figure B-3

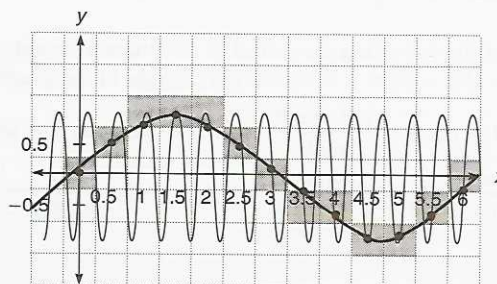


Figure B-4

To investigate this further, consider the following problems.

- a. Make a table of values for the following four functions, letting x take on the values 0, 0.5, 1, 1.5, etc., up to 6.0. This is just an extension of table B-1, where the values for the first two functions are already done.

$$\begin{aligned}y &= \sin(x) \\y &= \sin[(4\pi + 1)x] \\y &= \sin[(8\pi + 1)x] \\y &= \sin[(12\pi + 1)x]\end{aligned}$$

Try to make a generalization from the results.

Note The four functions will give identical values for a given x , but this does not mean the functions are identical! Just reconsider figure B-3 to see that the first two functions are not identical.

- b. Make a table of values for the following four functions, letting x take on the values 0, 0.25, 0.5, 0.75, etc., up to 3. This corresponds to a calculator display with pixels 0.25 units wide.

$$\begin{aligned}y &= \sin(x) \\y &= \sin[(8\pi + 1)x] \\y &= \sin[(16\pi + 1)x] \\y &= \sin[(24\pi + 1)x]\end{aligned}$$

Again, look for a pattern.

- c. Make a table of values for the following four functions, letting x take on the values 0, 0.1, 0.2, 0.3, etc., up to 1.2. This corresponds to a calculator display 0.1 units wide.

$$\begin{aligned}\sin(x) \\ \sin[(20\pi + 1)x] \\ \sin[(40\pi + 1)x] \\ \sin[(60\pi + 1)x]\end{aligned}$$

Try to find a pattern.

Note On a TI-81 the following settings create pixels of 0.1 units wide:

$$X_{\min} = -4.8, X_{\max} = 4.7, X_{\text{res}} = 1.$$

$$\text{There are 96 pixels on this calculator, and} \\ \frac{4.7 - (-4.8) + 0.1}{96} = 0.1.$$

$$\text{For pixels 0.25 wide, find values so that} \\ \frac{X_{\max} - X_{\min} + 0.25}{96} = 0.25. \text{ A similar}$$

calculation can be used to obtain pixels 0.5 units wide.

The Casio fx-7000G has 95 pixels, so the settings for creating pixels 0.1 units wide are $X_{\min} = -4.7$, $X_{\max} = 4.7$. Of course the division shown above then has the value 95 in the denominator.

- d. Consider the results of the previous parts of this problem, and try to predict some values β so that $\sin(x)$ and $\sin(\beta x)$ have the same graph on a device that has pixels 0.01 units wide. (That is, that the functions $y = \sin x$ and $y = \sin(\beta x)$ have the same values for $x = 0, 0.01, 0.02, 0.03$, etc.)
- e. Show why the generalizations of parts a, b, and c are true (use identities from section 5-2).

Program for the Casio fx-7000G calculator to calculate zeros of a function using the method of bisection. It is assumed the function to be solved is defined in program 9.

Generic program	Casio program	Casio key strokes/comments	
Prepare the calculator to accept a program.		MODE 2	Enter WRT (Program) mode.
		Use the ⇒ key to select Program 8.	Actually any program location will do.
		EXE	Ready to enter program 8.
Input the lower value L and the upper value U .	?→L: ?→U	SHIFT → → ALPHA √ : SHIFT → → ALPHA 1 EXE	

Generic program	Casio program	Casio key strokes/comments
Let $x = L$. Compute $f(L)$. Store the result in A.	L→X:Prog 9:Y→A	ALPHA $\sqrt{x}$ → ALPHA + : Prog 9 : ALPHA - → ALPHA x^{-1} EXE
If $f(L) < 0$ then go to 1	A<0→Goto 1	ALPHA x^{-1} SHIFT 3 0 SHIFT 7 SHIFT Prog 1 EXE
else switch the values in U and L.	U→T:L→U:T→L	ALPHA 1 → ALPHA ÷ : ALPHA $\sqrt{x}$ → ALPHA 1 : ALPHA ÷ → ALPHA $\sqrt{x}$ EXE
1:	Lbl 1	SHIFT ⇐ 1 EXE
Let $X = \frac{L+U}{2}$	(L+U)÷2→X	(ALPHA $\sqrt{x}$ + ALPHA 1) ÷ 2 → ALPHA + EXE
Compute $f(x)$; store result in Y.	Prog 9	Prog 9 EXE Assumes the equation to be solved will be in 9.
If $f(A) = 0$ then go to 3	Y=0→Goto 3	ALPHA - SHIFT 8 0 SHIFT 7 SHIFT Prog 3 EXE
If $ U - L \leq \text{error}$ then go to 3	Abs (U-L)≤1 E-11 →Goto 3	SHIFT x^y (ALPHA 1 - ALPHA $\sqrt{x}$) SHIFT 6 1 EXP (-) 1 1 SHIFT 7 SHIFT Prog 3 EXE
If $Y > 0$ go to 2	Y>0→Goto 2	ALPHA - SHIFT 2 0 SHIFT 7 SHIFT Prog 2 EXE
else let $L = x$ go to 1	X→L:Goto 1	ALPHA + → ALPHA $\sqrt{x}$: SHIFT Prog 1 EXE
2:	Lbl 2	SHIFT ⇐ 2 EXE
let $U = x$ go to 1	X→U:Goto 1	ALPHA + → ALPHA 1 : SHIFT Prog 1 EXE
3:	Lbl 3	SHIFT ⇐ 3 EXE
Display the final value, x.	X	ALPHA +
Stop the program.		MODE 1 Go back to run mode.

Pascal language program that implements the method of bisection for finding approximations to zeros of functions.

```

program      BISECTION (input,output);
const        ERROR = 1.0E-6;           { Practical smallest difference of
                                         two real data values in Pascal. }

var          Lower, Upper, Average : real;
              Done : boolean;
{ ***** The function for which we want to approximate zeros. ***** }
function     F (x : real) : real;
begin
              F := 2*sin(sqrt(2 * x)) - 0.5      { DEFINE FUNCTION HERE }
end { F };
{ ***** }
procedure    Read_in (var L, U : real);
begin
              Write('Enter Lower bound, then Upper bound: ');
              ReadLn(L, U);
end { Read_in };
procedure    Interchange (var L, U : real);
var temp : real;
begin
              temp := L;
              L := U;
              U := temp
end { Interchange };
begin      { BISECTION }
              Done := FALSE;
              Read_in(Lower, Upper);
              If F(Lower) > 0 then Interchange(Lower, Upper);
              While not Done do begin
                  Average := (Lower + Upper) / 2;
                  If (f(Average) = 0) or (abs(Upper-Lower) < ERROR) then
                      Done := TRUE
                  else { Average is not a zero and is not within ERROR units of a zero }
                      If f(Average) < 0 then Lower := Average
                      else { f(Average) > 0 } Upper := Average
                  end { while };
              WriteLn(Average);
end { BISECTION }.

```


Appendix C

Development of the Identity $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

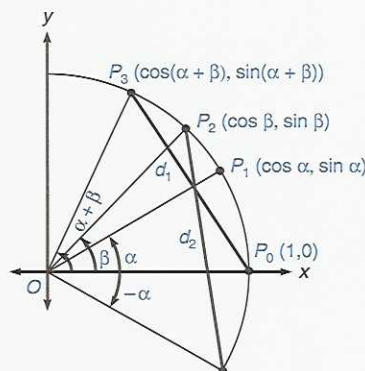


Figure C-1

A proof that this identity is true is beyond the scope of this text, but an argument for its correctness can be obtained in the following way.

Let α and β be two angles in standard position (see figure C.1). Let P_1 be the point where the terminal side of α intersects the unit circle, and P_2 be the point where angle β intersects the unit circle. Let P_3 be the point where the angle $\alpha + \beta$ (the sum of the angles α and β) intersects the circle. Let P_0 be the point (1,0). Finally, let P_4 be the point where the terminal side of angle $-\alpha$ intersects the unit circle.

On the unit circle the x - and y -coordinates of a point are the cosine and sine values for the appropriate angle. Thus, the point P_1 has coordinates $(\cos \alpha, \sin \alpha)$. The coordinates for the other points are shown in the figure.

Angle $\alpha + \beta$, or angle P_0OP_3 in standard position, has the same measure as angle P_4OP_2 . It is a geometric property that central angles of a circle having equal measure will have chords of equal length. Thus, the chords P_3P_0 and P_2P_4 have the same length. The length of a line segment with end points (x_1, y_1) and (x_2, y_2) is given by the distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We apply this to the chords mentioned above.

Let d_1 = length of P_3P_0 , and d_2 = length of P_2P_4 .

$$d_1 = d_2$$

$$\begin{aligned} & \sqrt{(\cos(\alpha + \beta) - 1)^2 + (\sin(\alpha + \beta) - 0)^2} \\ &= \sqrt{(\cos \beta - \cos \alpha)^2 + (\sin \beta - (-\sin \alpha))^2} \end{aligned}$$

We now square both sides.

$$\begin{aligned} [\cos(\alpha + \beta) - 1]^2 + [\sin(\alpha + \beta) - 0]^2 &= (\cos \beta - \cos \alpha)^2 + \\ & \quad [\sin \beta - (-\sin \alpha)]^2 \end{aligned}$$

Performing the indicated operations we obtain

$$\begin{aligned} & \cos^2(\alpha + \beta) - 2 \cos(\alpha + \beta) + 1 + \sin^2(\alpha + \beta) \\ &= \cos^2 \beta - 2 \cos \alpha \cos \beta + \cos^2 \alpha + \sin^2 \beta + 2 \sin \alpha \sin \beta + \sin^2 \alpha \end{aligned}$$

Then

$$\begin{aligned} & [\cos^2(\alpha + \beta) + \sin^2(\alpha + \beta)] - 2 \cos(\alpha + \beta) + 1 \\ &= (\cos^2 \beta + \sin^2 \beta) + (\cos^2 \alpha + \sin^2 \alpha) + 2 \sin \alpha \sin \beta - 2 \cos \alpha \cos \beta \end{aligned}$$

Using the fundamental identity $\sin^2 \theta + \cos^2 \theta = 1$, we obtain

$$\begin{aligned} 1 - 2 \cos(\alpha + \beta) + 1 &= 1 + 1 + 2 \sin \alpha \sin \beta - 2 \cos \alpha \cos \beta \\ -2 \cos(\alpha + \beta) &= 2 \sin \alpha \sin \beta - 2 \cos \alpha \cos \beta \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \end{aligned}$$

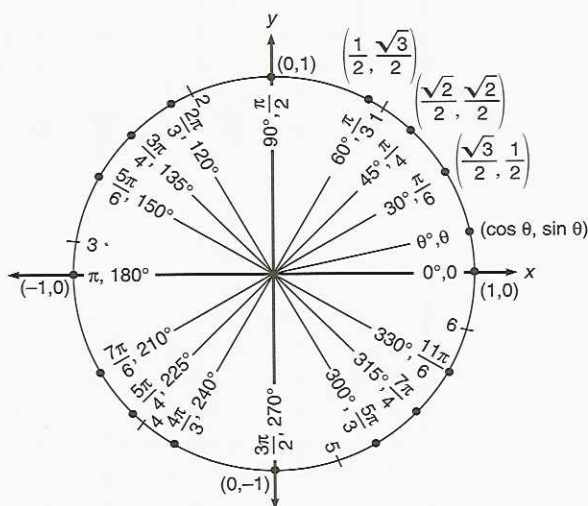
Divide each member by -2

Appendix D

Useful Templates

This appendix includes items that it might prove useful to reproduce in quantity and have handy. Note that rectangular coordinate graph paper and polar coordinate paper are both widely available in book stores.

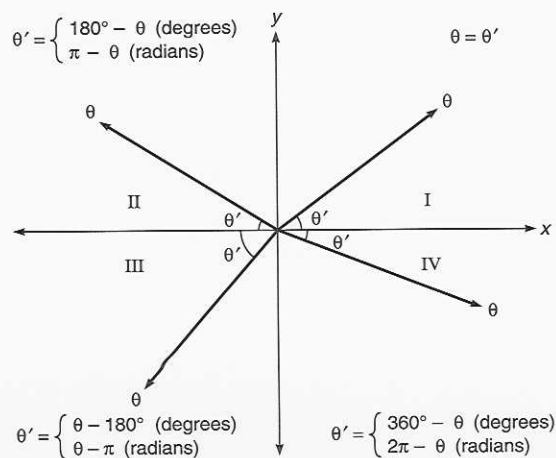
The Unit Circle



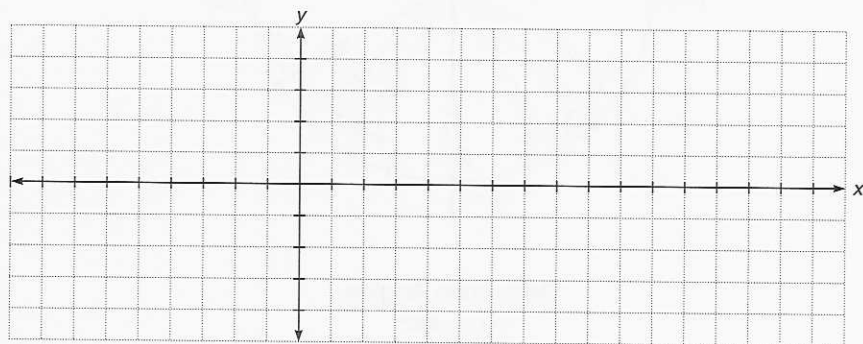
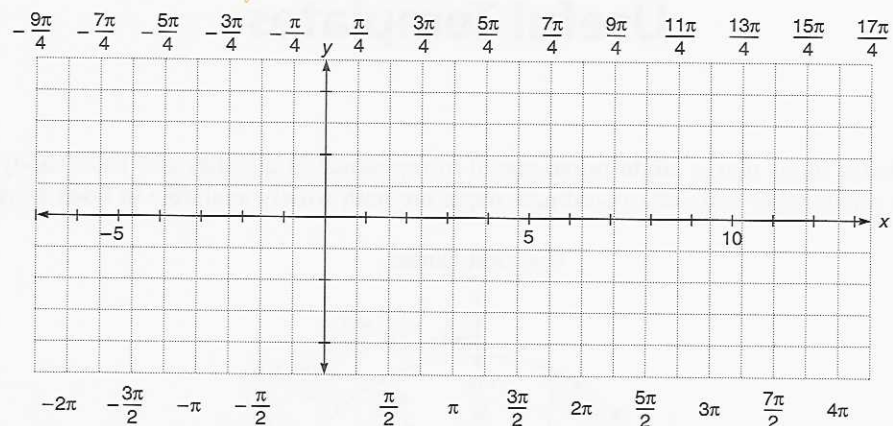
Reference Angles

$$0 < \theta < 360^\circ$$

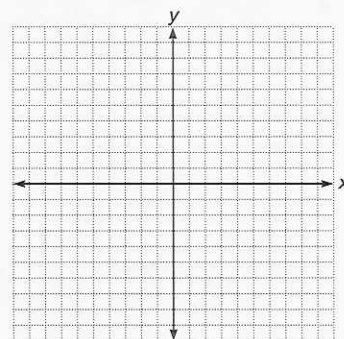
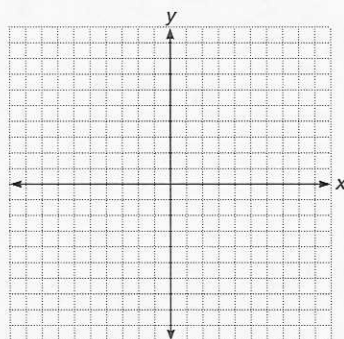
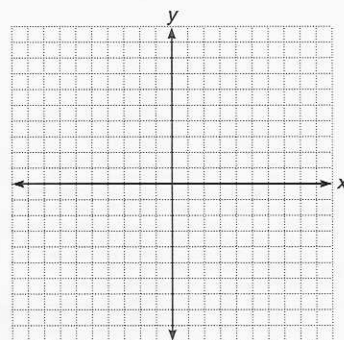
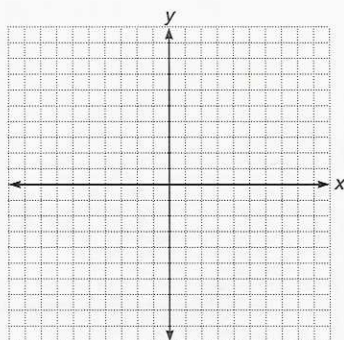
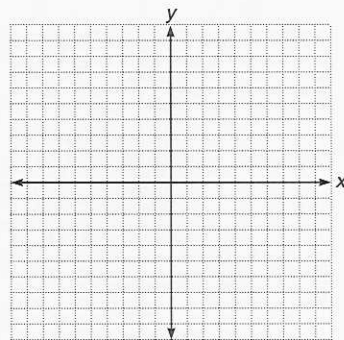
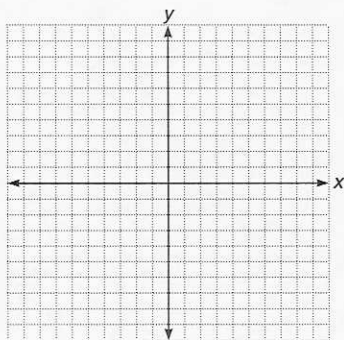
$$0 < \theta < 2\pi$$



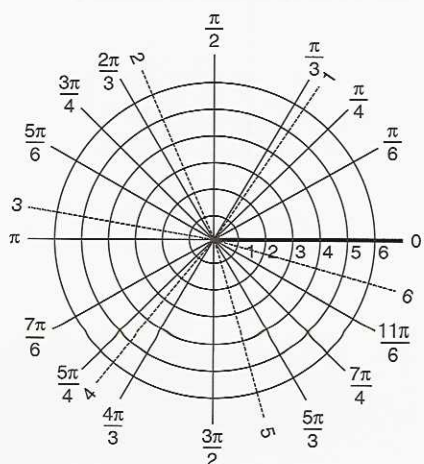
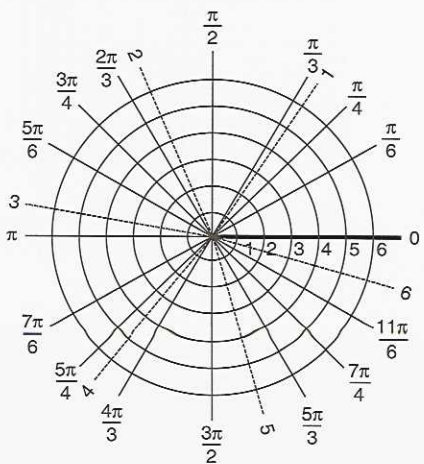
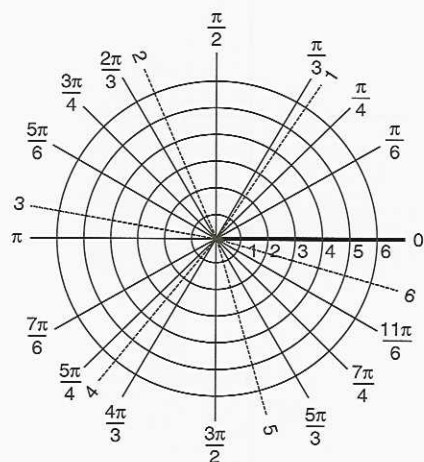
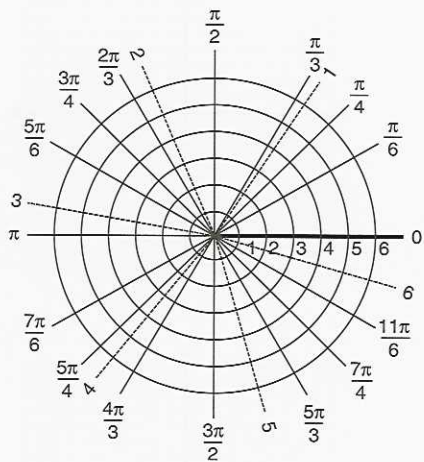
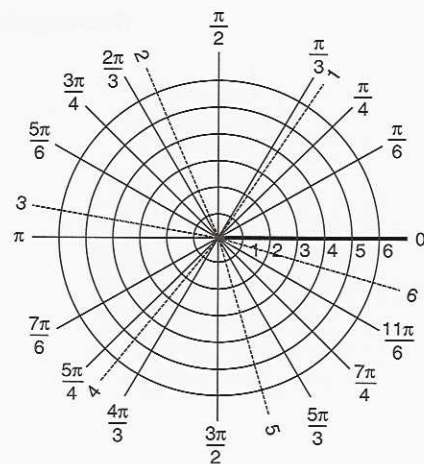
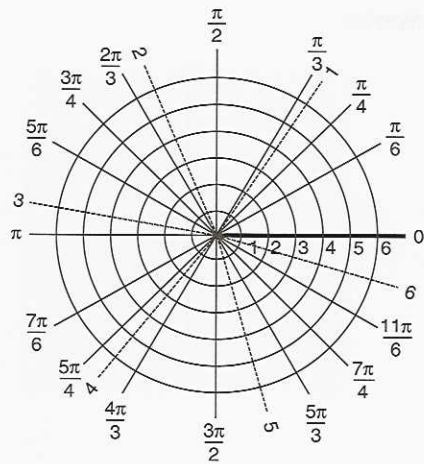
Templates Useful for Graphing Many Trigonometric Functions



Rectangular Coordinates



Polar Coordinates



Appendix E

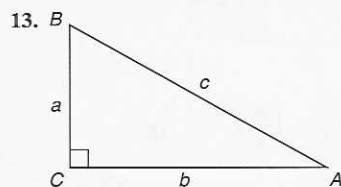
Answers and Solutions

Chapter 1

Exercise 1-1

Answers to odd-numbered problems

1. 13.417° , acute 3. 0.2° , acute
5. 25.555° , acute 7. 165.783° , obtuse
9. 33.099° , acute 11. 159.983° , obtuse



15. 48.2° 17. $71^\circ 48'$ 19. 106°
21. right triangle, hypotenuse = 0.5
23. right triangle, hypotenuse = $2\sqrt{5}$
25. right triangle, hypotenuse = 2.57
27. $c = 15$ 29. $a = 6$
31. $b = 6\sqrt{5} \approx 13$ 33. $c = \sqrt{14} \approx 4$
35. $c = 50\sqrt{13} \approx 180$ 37. $c = 4$
39. $b = \sqrt{185.31} \approx 13.6$
41. $c = 7\sqrt{2} \approx 9.9$ 43. $a = 3\sqrt{47} \approx 21$
45. $c = \sqrt{2} \approx 1$ 47. 61 ft 49. 90.1 ft
51. 20.07 ohms 53. 3,770 ohms
55. Not accurate, measurements do not satisfy Pythagorean theorem
57. 23 minutes 59. no, 1.1 ft
61. 107 knots 63. 17 knots
65. 27 units

Solutions to trial exercise problems

$$\begin{aligned} 6. \quad 87^\circ 2' 13'' &= \left(87 + \frac{2}{60} + \frac{13}{3,600} \right)^\circ \\ &= (87 + 0.0333 + 0.0036)^\circ \\ &= 87.037^\circ \end{aligned}$$

This angle is acute since it is less than 90° .

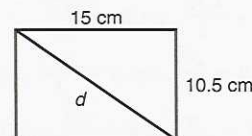
$$\begin{array}{r} 18. \quad \begin{array}{r} 28^\circ \quad 17' \\ + 28^\circ \quad 52' \\ \hline 56^\circ \quad 69' \\ 56^\circ \quad 60' + 9' \\ \hline 57^\circ \quad 9' \end{array} \quad \begin{array}{r} 20. \quad 90^\circ \\ - 72^\circ 19' 51'' \\ \hline 89^\circ 60' \\ - 72^\circ 19' 51'' \\ \hline 89^\circ 59' 60'' \\ - 72^\circ 19' 51'' \\ \hline 17^\circ 40' 9'' \end{array} \\ \begin{array}{r} 180^\circ \\ - 57^\circ \\ \hline 122^\circ \quad 60' \\ - 57^\circ \quad 9' \\ \hline 122^\circ \quad 51' \end{array} \end{array}$$

The angle is $122^\circ 51'$.

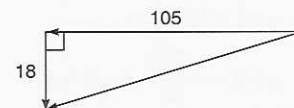
23. $(2\sqrt{5})^2 = 2^2(\sqrt{5})^2 = 4(5) = 20$;
 $2^2 + 4^2 = 4 + 16 = 20$.
Since $(2\sqrt{5})^2 = 2^2 + 4^2$, the triangle is a right triangle; the hypotenuse is the longest side, with length $2\sqrt{5}$.
29. $a^2 + b^2 = c^2$, so $a^2 + 8^2 = 10^2$
 $a^2 + 64 = 100$
 $a^2 = 36$
 $a = 6$
41. $c^2 = a^2 + b^2$
 $c^2 = (3\sqrt{2})^2 + (4\sqrt{5})^2$
 $= 9(2) + 16(5) = 98$
 $c = \sqrt{98} = \sqrt{2 \cdot 49} = 7\sqrt{2}$

54. The sum of the angles is $110^\circ 0' + 33^\circ 28' + 28^\circ 32' = 172^\circ$. Since the sum of the angles of a triangle is 180° , we can see that there is quite a bit of inaccuracy in these measurements. Although in practice we would not expect the three measurements to add exactly to 180° (they are approximations, as are any measurements), an error of 8° out of 180° (more than 4% error) probably indicates an error in a measurement.

57. The diagonal is $\sqrt{15^2 + 10.5^2} \approx 18.31$ inches. (See the diagram.) Divide 18.31 inches by 0.8 inch per minute to get 23 minutes.



61. The headings of the aircraft and the wind form the wind triangle shown.

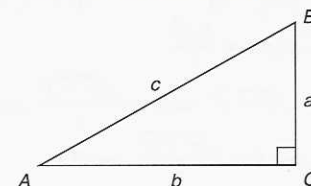


Ground speed = $\sqrt{105^2 + 18^2} \approx 107$ knots

Exercise 1-2

Answers to odd-numbered problems

1.



$$\begin{aligned} \sin A &= \frac{a}{c}, \quad \cos A = \frac{b}{c}, \quad \tan A = \frac{a}{b}, \\ \cot A &= \frac{b}{a}, \quad \sec A = \frac{c}{b}, \quad \csc A = \frac{c}{a}, \\ \sin B &= \frac{b}{c}, \quad \cos B = \frac{a}{c}, \quad \tan B = \frac{b}{a}, \\ \cot B &= \frac{a}{b}, \quad \sec B = \frac{c}{a}, \quad \csc B = \frac{c}{b} \end{aligned}$$

$$3. c = 5, \sin B = \frac{4}{5}, \cos B = \frac{3}{5}, \tan B = \frac{4}{3}, \csc B = \frac{5}{4}, \sec B = \frac{5}{3}, \cot B = \frac{3}{4}$$

$$5. c = \sqrt{10}, \sin B = \frac{3\sqrt{10}}{10}, \cos B = \frac{\sqrt{10}}{10}, \tan B = 3, \csc B = \frac{\sqrt{10}}{3}, \sec B = \frac{\sqrt{10}}{10}, \cot B = \frac{1}{3}$$

$$7. c = 4\sqrt{2}, \sin B = \frac{\sqrt{14}}{8}, \cos B = \frac{5\sqrt{2}}{8}, \tan B = \frac{\sqrt{7}}{5}, \csc B = \frac{4\sqrt{14}}{7}, \sec B = \frac{4\sqrt{2}}{5}, \cot B = \frac{5\sqrt{7}}{7}$$

$$9. b = 1, \sin B = \frac{\sqrt{5}}{5}, \cos B = \frac{2\sqrt{5}}{5}, \tan B = \frac{1}{2}, \csc B = \sqrt{5}, \sec B = \frac{\sqrt{5}}{2}, \cot B = 2$$

$$11. c = \sqrt{313}, \sin B = \frac{13\sqrt{313}}{313}, \cos B = \frac{12\sqrt{313}}{313}, \tan B = \frac{13}{12}, \csc B = \frac{\sqrt{313}}{13}, \sec B = \frac{\sqrt{313}}{12}, \cot B = \frac{12}{13}$$

$$13. b = 8, \sin A = \frac{3}{5}, \cos A = \frac{4}{5}, \tan A = \frac{3}{4}, \csc A = \frac{5}{3}, \sec A = \frac{5}{4}, \cot A = \frac{4}{3}$$

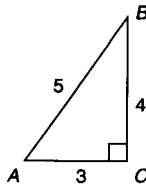
$$15. c = \sqrt{19}, \sin A = \frac{\sqrt{57}}{19}, \cos A = \frac{4\sqrt{19}}{19}, \tan A = \frac{\sqrt{3}}{4}, \csc A = \frac{\sqrt{57}}{3}, \sec A = \frac{\sqrt{19}}{4}, \cot A = \frac{4\sqrt{3}}{3}$$

$$17. b = \sqrt{z^2 - x^2}, \sin B = \frac{\sqrt{z^2 - x^2}}{z}, \cos B = \frac{x}{z}, \tan B = \frac{\sqrt{z^2 - x^2}}{x}, \csc B = \frac{z}{\sqrt{z^2 - x^2}}, \sec B = \frac{z}{x}, \cot B = \frac{x}{\sqrt{z^2 - x^2}}$$

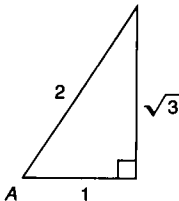
$$19. b = \sqrt{3}, \sin B = \frac{\sqrt{3}}{2}, \cos B = \frac{1}{2}, \tan B = \sqrt{3}, \csc B = \frac{2\sqrt{3}}{3}, \sec B = 2, \cot B = \frac{\sqrt{3}}{3}$$

$$21. c = \sqrt{106}, \sin B = \frac{5\sqrt{106}}{106}, \cos B = \frac{9\sqrt{106}}{106}, \tan B = \frac{5}{9}, \csc B = \frac{\sqrt{106}}{5}, \sec B = \frac{\sqrt{106}}{9}, \cot B = \frac{9}{5}$$

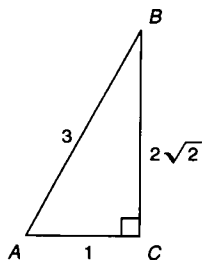
$$23. \cos A = \frac{3}{5}, \tan A = \frac{4}{3}, \cot A = \frac{3}{4}, \sec A = \frac{5}{3}, \csc A = \frac{5}{4}$$



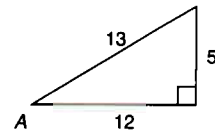
$$25. \sin A = \frac{\sqrt{3}}{2}, \tan A = \sqrt{3}, \cot A = \frac{\sqrt{3}}{3}, \sec A = 2, \csc A = \frac{2\sqrt{3}}{3}$$



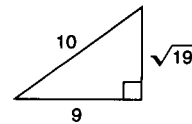
$$27. \sin A = \frac{2\sqrt{2}}{3}, \cos A = \frac{1}{3}, \tan A = 2\sqrt{2}, \cot A = \frac{\sqrt{2}}{4}, \csc A = \frac{3\sqrt{2}}{4}$$



$$29. \cos A = \frac{12}{13}, \tan A = \frac{5}{12}, \cot A = \frac{12}{5}, \sec A = \frac{13}{12}, \csc A = \frac{13}{5}$$

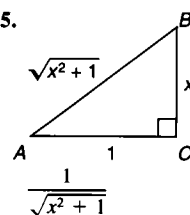


$$31. \sin A = \frac{\sqrt{19}}{10}, \tan A = \frac{\sqrt{19}}{9}, \cot A = \frac{9\sqrt{19}}{19}, \sec A = \frac{10}{9}, \csc A = \frac{10\sqrt{19}}{19}$$

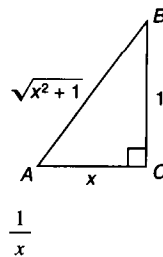


$$33. \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 = \frac{b^2}{c^2} + \frac{a^2}{c^2} = \frac{b^2 + a^2}{c^2} = \frac{c^2}{c^2} = 1$$

35.



37.



Solutions to trial exercise problems

15. $a = \sqrt{3}, b = 4$. Find ratios for angle A.

$$c^2 = a^2 + b^2 \\ c^2 = (\sqrt{3})^2 + 4^2 \\ c^2 = 3 + 16 = 19 \\ c = \sqrt{19}$$

$$\sin A = \frac{a}{c} = \frac{\sqrt{3}}{\sqrt{19}} = \frac{\sqrt{3}}{\sqrt{19}} \cdot \frac{\sqrt{19}}{\sqrt{19}} = \frac{\sqrt{57}}{19};$$

$$\cos A = \frac{b}{c} = \frac{4}{\sqrt{19}} = \frac{4}{\sqrt{19}} \cdot \frac{\sqrt{19}}{\sqrt{19}} = \frac{4\sqrt{19}}{19};$$

$$\tan A = \frac{a}{b} = \frac{\sqrt{3}}{4}; \csc A = \frac{1}{\sin A} = \frac{1}{\frac{\sqrt{3}}{\sqrt{19}}} = \frac{\sqrt{19}}{\sqrt{3}}$$

$$= \frac{\sqrt{19}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{57}}{3}; \sec A = \frac{1}{\cos A} = \frac{1}{\frac{4}{\sqrt{19}}} = \frac{\sqrt{19}}{4};$$

$$\cot A = \frac{1}{\tan A} = \frac{1}{\frac{\sqrt{3}}{4}} = \frac{4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$$

- 19.
- $a = 1$
- ,
- $c = 2$
- . Find ratios for angle
- B
- .

$$a^2 + b^2 = c^2, \text{ so}$$

$$1 + b^2 = 4$$

$$b^2 = 3$$

$$b = \sqrt{3}$$

$$\sin B = \frac{\text{opp}}{\text{hyp}} = \frac{b}{c} = \frac{\sqrt{3}}{2};$$

$$\cos B = \frac{\text{adj}}{\text{hyp}} = \frac{a}{c} = \frac{1}{2};$$

$$\tan B = \frac{\text{opp}}{\text{adj}} = \frac{b}{a} = \frac{\sqrt{3}}{1} = \sqrt{3};$$

$$\cot B = \frac{1}{\tan B} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3};$$

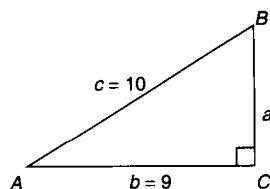
$$\sec B = \frac{1}{\cos B} = \frac{1}{\frac{1}{2}} = 2;$$

$$\csc B = \frac{1}{\sin B} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2\sqrt{3}}{3}$$

- 31.
- $\cos A = 0.9$

$$\cos A = \frac{9}{10} = \frac{b}{c}, \text{ so a triangle in}$$

which $b = 9$ and $c = 10$ would work.
This is shown in the diagram.



Using the Pythagorean theorem, we find a .

$$c^2 = a^2 + b^2$$

$$100 = a^2 + 81$$

$$\sqrt{19} = a$$

With this we can now compute the five remaining ratios.

$$\sin A = \frac{a}{c} = \frac{\sqrt{19}}{10};$$

$$\tan A = \frac{a}{b} = \frac{\sqrt{19}}{9};$$

$$\cot A = \frac{1}{\tan A} = \frac{9\sqrt{19}}{19};$$

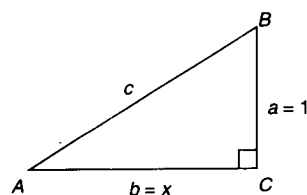
$$\csc A = \frac{1}{\sin A} = \frac{1}{\frac{\sqrt{19}}{10}} = \frac{10\sqrt{19}}{19};$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\frac{9}{10}} = \frac{10}{9}$$

- 37.
- $\tan B = x$
- ; if we think of
- x
- as
- $\frac{x}{1}$
- we

can use the triangle shown in the

diagram, since $\tan B = \frac{b}{a} = \frac{x}{1}$.



From the Pythagorean theorem we find c .

$$c^2 = a^2 + b^2$$

$$c^2 = 1^2 + x^2 \text{ so}$$

$$c = \sqrt{x^2 + 1}$$

Now we find $\tan A$:

$$\tan A = \frac{a}{b} = \frac{1}{x}. \text{ (Note that we did}$$

not actually need c .)

Exercise 1-3

Answers to odd-numbered problems

1. 0.5192 3. 0.2116 5. 1.8137
 7. 0.6465 9. 2.5048 11. 0.9793
 13. 0.9801 15. 0.0790 17. 0.7037
 19. 0.5534 21. 0.7447 23. 15.7801
 25. 971.40 27. 0.47 29. 742.5 watts

Solutions to trial exercise problems

$$9. \sec 66.47^\circ = \frac{1}{\cos 66.47^\circ}$$

$$66.47 \quad \boxed{\cos} \quad \boxed{1/x}$$

$$\boxed{\text{TI-81}} \quad \boxed{[(]} \quad \boxed{\text{COS}} \quad 66.47 \quad \boxed{) } \quad \boxed{[x^{-1}]} \quad \boxed{\text{ENTER}}$$

$$\approx 2.5048$$

$$\text{Display: } \boxed{2.50482689}$$

Make sure calculator is in degree mode.

- 11.
- $\sin 78.33^\circ$

$$78.33 \quad \boxed{\sin}$$

$$\boxed{\text{TI-81}} \quad \boxed{\text{SIN}} \quad 78.33 \quad \boxed{\text{ENTER}}$$

$$\approx 0.9793$$

$$\text{Display: } \boxed{0.9793288556}$$

Make sure calculator is in degree mode.

- 13.
- $\sin 78^\circ 33'$

$$\boxed{\text{A}} \quad 78 \quad \boxed{+} \quad 33 \quad \boxed{\div} \quad 60 \quad \boxed{=} \quad \boxed{\sin}$$

$$\boxed{\text{P}} \quad 78 \quad \boxed{\text{ENTER}} \quad 33 \quad \boxed{\text{ENTER}} \quad 60 \quad \boxed{\div} \quad \boxed{+} \quad \boxed{\sin}$$

$$\boxed{\text{TI-81}} \quad \boxed{\text{SIN}} \quad \boxed{[(]} \quad 78 \quad \boxed{+} \quad 33 \quad \boxed{\div} \quad 60 \quad \boxed{) } \quad \boxed{\text{ENTER}}$$

$$\approx 0.9801$$

$$\text{Display: } \boxed{0.9800983128}$$

Make sure calculator is in degree mode.

31. 11.8 inches

$$33. c^2 = 1^2 + 1^2$$

$$= 1 + 1$$

$$= 2$$

$$c = \sqrt{2}$$

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = \frac{1}{1} = 1$$

- 35.
- 53.2°
- 37.
- 62.0°
- 39.
- 31.6°

- 41.
- 31.7°
- 43.
- 63.4°
- 45.
- 78.00°

- 47.
- 68.63°
- 49.
- $A = 51.7^\circ$
- ,
- $b \approx 12.0$
- ,

$$c \approx 19.4 \quad 51. B = 76.3^\circ$$
, $b \approx 45.5$,

$$c \approx 46.9 \quad 53. B = 60.6^\circ$$
, $a \approx 0.379$,

$$c \approx 0.771 \quad 55. A = 12^\circ$$
, $a \approx 4.6$,

$$c \approx 22.3 \quad 57. B = 75^\circ$$
, $a \approx 2.6$, $b \approx 9.7$

$$59. A = 24.5^\circ$$
, $a \approx 51$, $b \approx 111$

$$61. c \approx 20.4$$
, $A \approx 40.0^\circ$, $B \approx 50.0^\circ$

$$63. c \approx 1.36$$
, $A \approx 9.3^\circ$, $B \approx 80.7^\circ$

$$65. b \approx 17.8$$
, $A \approx 44.9^\circ$, $B \approx 45.1^\circ$

$$67. a \approx 98.4$$
, $A \approx 62.5^\circ$, $B \approx 27.5^\circ$

$$69. b \approx 5$$
, $A \approx 67.4^\circ$, $B \approx 22.6^\circ$

$$71. a \approx 32.6$$
, $A \approx 21.2^\circ$, $B \approx 68.8^\circ$

$$73. 55.1 \text{ ohms} \quad 75. 34^\circ \quad 77. 18.6 \text{ volts}$$

$$79. 32,800 \text{ ft} \quad 81. 217 \text{ mm}$$

$$83. 2.3^\circ \quad 85. 265.8$$

$$87. A = 30^\circ$$
, $a = \frac{8\sqrt{3}}{3}$, $c = \frac{16\sqrt{3}}{3}$

$$25. R = \frac{LC}{2 \sin I}, LC = 611.1 \text{ meters}, I = 18^\circ 20'$$

$$R = \frac{611.1}{2 \sin 18^\circ 20'} \approx \frac{611.1}{2(0.3145)} \approx \frac{611.1}{0.6290} \approx 971.40$$

$$27. y = x \cos A \cos B - x^2 \cos A \sin B - x^3 \sin A$$

$$= 1.2 \cos 10^\circ \cos 15^\circ - 1.2^2 \cos 10^\circ \sin 15^\circ - 1.2^3 \sin 10^\circ$$

$$\approx 0.47 \text{ (to two decimal places)}$$

(A) 1.2 $\times$ 10 $\cos$ $\times$ 15 $\cos$ $-$

1.2 x^2 $\times$ 10 $\cos$ $\times$ 15 $\sin$ $-$

1.2 y^x 3 $\times$ 10 $\sin$ $=$

(P) 1.2 ENTER 10 $\cos$ $\times$ 15 $\cos$ $\times$

1.2 x^2 10 $\cos$ $\times$ 15 $\sin$ $\times$ $-$

1.2 ENTER 3 y^x 10 $\sin$ $\times$ $-$

(TI-81) 1.2 STO XIT ENTER 10 STO

MATH ENTER 15 STO MATRX

ENTER XIT $\cos$ ALPHA

MATH $\cos$ ALPHA MATRX

$-$ XIT x^2 $\cos$ ALPHA

MATH $\sin$ ALPHA MATRX $-$

XIT MATH 3 $\sin$ ALPHA MATH

ENTER

$$31. r = \frac{f}{2 \cos \theta} = \frac{21.4}{2 \cos 25^\circ} \approx \frac{21.4}{1.8126} \approx 11.8 \text{ inches}$$

$$47. \sec \theta = \frac{6.45}{2.35}$$

$$\cos \theta = \frac{2.35}{6.45}$$

$$\theta = \cos^{-1} \frac{2.35}{6.45}$$

$$\theta \approx 68.63^\circ \quad \text{Display: } 68.6329624$$

$$2.35 \div 6.45 = \cos^{-1}$$

(TI-81) $\cos^{-1}$ (2.35 $\div$ 6.45) ENTER

Make sure calculator is in degree mode.

Remember, $\cos^{-1}$ is SHIFT $\cos$ or 2nd $\cos$ on most calculators.

$$48. \sin \theta = \frac{1}{\sqrt{10.8}}$$

$$\theta = \sin^{-1} \frac{1}{\sqrt{10.8}}$$

$$\theta \approx 17.72^\circ \quad \text{Display: } 17.71547234$$

$$10.8 \sqrt{x} \text{ 1/x } \sin^{-1}$$

(TI-81) $\sin^{-1}$ ($\sqrt{x}$ 10.8) x^{-1}

ENTER

Make sure calculator is in degree mode.

Remember, $\sin^{-1}$ is SHIFT $\sin$ or 2nd $\sin$ on most calculators.

$$49. a = 15.2, B = 38.3^\circ$$

$$A = 90^\circ - 38.3^\circ = 51.7^\circ$$

Using angle B: $\sin B = \frac{b}{c}$, $\cos B = \frac{a}{c}$, $\tan B = \frac{b}{a}$

$$\sin 38.3^\circ = \frac{b}{c}, \cos 38.3^\circ = \frac{15.2}{c}, \tan 38.3^\circ = \frac{b}{15.2}$$

Use the cosine and tangent ratios:

$$\cos 38.3^\circ = \frac{15.2}{c}, \tan 38.3^\circ = \frac{b}{15.2}; c = \frac{15.2}{\cos 38.3^\circ},$$

$$15.2(\tan 38.3^\circ) = b; c \approx 19.4, 12.0 \approx b$$

$$59. c = 122, B = 65.5^\circ$$

$$A = 90^\circ - 65.5^\circ = 24.5^\circ$$

$$\sin 65.5^\circ = \frac{b}{122}; \cos 65.5^\circ = \frac{a}{122}; \tan 65.5^\circ = \frac{b}{a}$$

$$b = 122 \sin 65.5^\circ \approx 111; a = 122 \cos 65.5^\circ \approx 51$$

$$67. b = 51.3, c = 111.0$$

$$c^2 = a^2 + b^2; 111^2 = a^2 + 51.3^2; a \approx 98.4$$

$$\sin A = \frac{a}{111.0}; \cos A = \frac{51.3}{111.0}; \tan A = \frac{51.3}{a}$$

$$\text{Using } \cos A = \frac{51.3}{111.0}, \text{ we find } A \approx 62.5^\circ$$

$$B = 90^\circ - 62.5^\circ \approx 27.5^\circ$$

$$76. \text{Construct a line perpendicular to one side } b, \text{ as shown. Since } b = 8.25'', \text{ half that is } 4.125''.$$

$$\text{Since } \theta = \frac{360^\circ}{7}, \frac{\theta}{2} = \frac{1}{2} \cdot \frac{360^\circ}{7} = \frac{180^\circ}{7}.$$

$$\sin \frac{\theta}{2} = \frac{\text{opp}}{\text{hyp}} = \frac{4.125}{a}$$

$$\sin \frac{180^\circ}{7} = \frac{4.125}{a}$$

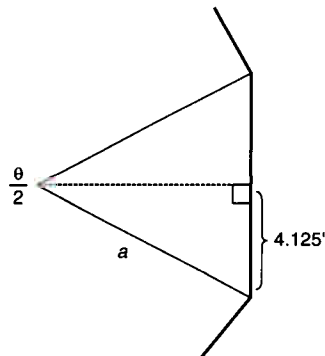
$$a = \frac{4.125}{\sin \frac{180^\circ}{7}} \approx 9.51$$

$$4.125 \div (\sin (180 \div 7)) =$$

$$\text{Display: } 9.507155093$$

(TI-81) 4.125 $\div$ $\sin$ (180 $\div$ 7)

ENTER



87. $B = 60^\circ, b = 8$

$$A = 90^\circ - 60^\circ = 30^\circ$$

$$\sin 60^\circ = \frac{8}{c}, \text{ so } c = \frac{8}{\sin 60^\circ} = \frac{8}{\frac{\sqrt{3}}{2}} = \frac{8}{1} \cdot \frac{2}{\sqrt{3}} = \frac{16\sqrt{3}}{3}$$

$$\tan 60^\circ = \frac{8}{a}, \text{ so } a = \frac{8}{\tan 60^\circ} = \frac{8}{\sqrt{3}} = \frac{8\sqrt{3}}{3}$$

Exercise 1–4

Answers to odd-numbered problems

1. $\tan \theta \cot \theta$

$$\frac{1}{\cot \theta} \cot \theta$$

3. $\cos \theta (1 - \sec \theta)$

$$\cos \theta \left(1 - \frac{1}{\cos \theta} \right)$$

$$\cos \theta - \frac{\cos \theta}{\cos \theta}$$

$$\cos \theta - 1$$

5. $\sec \theta (\cot \theta + \cos \theta - 1)$

$$\sec \theta \cot \theta + \sec \theta \cos \theta - \sec \theta$$

$$\frac{1}{\cos \theta} \frac{\cos \theta}{\sin \theta} + \frac{1}{\cos \theta} \cos \theta - \sec \theta$$

$$\frac{1}{\sin \theta} + 1 - \sec \theta$$

7. $\cos \alpha - \sin \alpha$

$$\frac{\cos \alpha}{\cos \alpha} - \frac{\sin \alpha}{\cos \alpha}$$

$$1 - \tan \alpha$$

9. $1 - \cos^2 \theta$

$$\sin^2 \theta + \cos^2 \theta - \cos^2 \theta$$

$$\sin^2 \theta$$

11. $\cos \beta (\sec \beta - \cos \beta)$

$$\cos \beta \sec \beta - \cos^2 \beta$$

$$\cos \beta \frac{1}{\cos \beta} - \cos^2 \beta$$

$$1 - \cos^2 \beta$$

$$\sin^2 \beta + \cos^2 \beta - \cos^2 \beta$$

$$\sin^2 \beta$$

13. $(\cos \theta + \sin \theta) (\cos \theta - \sin \theta) +$

$$2 \sin^2 \theta$$

$$\cos^2 \theta + \cos \theta \sin \theta - \cos \theta \sin \theta -$$

$$\sin^2 \theta + 2 \sin^2 \theta$$

$$\cos^2 \theta + \sin^2 \theta$$

$$1$$

15. a. $\sin^2 16^\circ 50' + \cos^2 16^\circ 50'$

$$\sin^2 16.83^\circ + \cos^2 16.83^\circ$$

$$0.0838 + 0.9162$$

$$1$$

b. $\sin^2 50^\circ + \cos^2 50^\circ$

$$0.5868 + 0.4132$$

$$1$$

17. $\tan 32^\circ 40' \approx 0.6412$

$$\frac{\sin 32^\circ 40'}{\cos 32^\circ 40'} \approx \frac{0.5398}{0.8418} \approx 0.6412$$

19. $\tan x (\cot x + \csc x) = 1 + \sec x$

$$\tan x \cdot \cot x + \tan x \cdot \csc x$$

$$\tan x \cdot \frac{1}{\tan x} + \frac{\sin x}{\cos x} \cdot \frac{1}{\sin x}$$

$$1 + \frac{1}{\cos x}$$

$$1 + \sec x$$

21. $\sin \beta (\cot \beta - \csc \beta + \sin \beta) = \cos \beta$

$$\sin \beta \cot \beta - \sin \beta \csc \beta + \sin \beta$$

$$\sin \beta \cdot \frac{\cos \beta}{\sin \beta} - \sin \beta \cdot \frac{1}{\sin \beta} + \sin \beta$$

$$\cos \beta - 1 + \sin^2 \beta$$

$$\cos \beta - (\cos^2 \beta + \sin^2 \beta) + \sin^2 \beta$$

$$\cos \beta - \cos^2 \beta$$

23. $\cos \alpha (\csc \alpha + \sec \alpha) = \cot \alpha + 1$

$$\cos \alpha \cdot \csc \alpha + \cos \alpha \cdot \sec \alpha$$

$$\cos \alpha \cdot \frac{1}{\sin \alpha} + \cos \alpha \cdot \frac{1}{\cos \alpha}$$

$$\frac{\cos \alpha}{\sin \alpha} + 1$$

$$\cot \alpha + 1$$

25. 60° 27. 60° 29. 11.5° 31.

$$77.5^\circ$$
 33. 33.1°

$$35. 10^\circ$$
 37. 30° 39. 6.5° 41. 15°

$$43. 25.3^\circ$$

$$45. \sin B = \frac{b}{c}, \cos B = \frac{a}{c}, \tan B = \frac{b}{a}$$

$$\frac{b}{a} = \tan B = \frac{\sin B}{\cos B} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{b}{a}$$

Solutions to trial exercise problems

8. $\frac{\sin \theta + \cos \theta - 2}{\cos \theta} = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\cos \theta} - \frac{2}{\cos \theta}$

$$= \tan \theta + 1 - 2 \sec \theta$$

26. $\sqrt{3} \tan x = 1$

$$\tan x = \frac{1}{\sqrt{3}}$$

$$\tan x = \frac{\sqrt{3}}{3}$$

$$x = \tan^{-1} \frac{\sqrt{3}}{3}$$

$$x = 30^\circ, \text{ since } \tan 30^\circ = \frac{\sqrt{3}}{3}$$

42. $3 \sin 2x = 0.75$

$$\sin 2x = 0.25$$

$$2x = \sin^{-1} 0.25$$

$$x = \frac{\sin^{-1} 0.25}{2}$$

$$x \approx 7.2^\circ$$

$$0.25 \quad \boxed{\sin^{-1}} \quad \boxed{\div} \quad 2 \quad \boxed{=}$$

$$\text{Display: } \boxed{7.238756093}$$

$$\boxed{\text{TI-81}} \quad \boxed{\text{SIN}^{-1}} \quad .25 \quad \boxed{\div} \quad 2 \quad \boxed{\text{ENTER}}$$

Chapter 1 review

1. 17.57° , acute 2. 84.15° , acute

3. 125.62° , obtuse 4. 39.76° , acute

5. $59^\circ 56' 52''$ 6. 61.3°

7. $4\sqrt{13} \approx 14.4$ 8. $\sqrt{595} \approx 24.4$

9. $\sqrt{105} \approx 10.2$ 10. 52 ft

11. 40.0 ohms 12. $c = \sqrt{58}$, $\sin A =$

$$\frac{3\sqrt{58}}{58} \approx 0.3939, \cos A = \frac{7\sqrt{58}}{58} \approx 0.9191,$$

$$\tan A = \frac{3}{7} \approx 0.4286, \csc A = \frac{\sqrt{58}}{3}$$

$$\approx 2.5386, \sec A = \frac{\sqrt{58}}{7} \approx 1.0880,$$

$$\cot A = \frac{7}{3} \approx 2.3333$$
 13. $b = 10\sqrt{2}$,

$$\cos B = \frac{1}{3} \approx 0.333, \sin B = \frac{2\sqrt{2}}{3} \approx 0.9428,$$

$$\cot B = \frac{\sqrt{2}}{4} \approx 0.3536, \sec B = 3, \csc B =$$

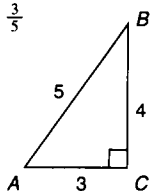
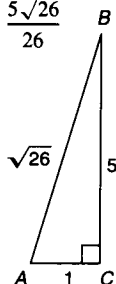
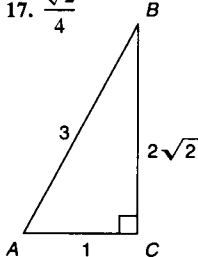
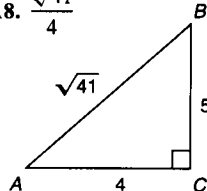
$$\frac{3\sqrt{2}}{4} \approx 1.0607, \tan B = 2\sqrt{2} \approx 2.8284$$

14. $c = \sqrt{14}$, $\sin A = \frac{\sqrt{14}}{7} \approx 0.5345, \cos A$

$$= \frac{\sqrt{35}}{7} \approx 0.8452, \tan A = \frac{\sqrt{10}}{5} \approx 0.6325,$$

$$\csc A = \frac{\sqrt{14}}{2} \approx 1.8708, \sec A = \frac{\sqrt{35}}{5} \approx$$

$$1.1832, \cot A = \frac{\sqrt{10}}{2} \approx 1.5811$$

15. $\frac{3}{5}$ 16. $\frac{5\sqrt{26}}{26}$ 17. $\frac{\sqrt{2}}{4}$ 18. $\frac{\sqrt{41}}{4}$ 

19. 0.2672 20. 2.6927 21. 0.3230

22. 0.6534 23. 0.2924 24. 1.1098

25. 1.6426 26. 13.2347 27. 0.6128

28. 1.4617 29. 2.0057 30. 17.1984

31. a. $\sin 53.20^\circ \approx 0.8007$, $\cos 53.20^\circ$ ≈ 0.5990 , $\tan 53.20^\circ \approx 1.3367$, $\csc 53.20^\circ$ ≈ 1.2489 , $\sec 53.20^\circ \approx 1.6694$, $\cot 53.20^\circ$ ≈ 0.7481 b. $\sin 53^\circ 20' \approx 0.8021$, $\cos 53^\circ 20' \approx 0.5972$, $\tan 53^\circ 20' \approx 1.3432$, $\csc 53^\circ 20' \approx 1.2467$, $\sec 53^\circ 20' \approx 1.6746$, $\cot 53^\circ 20' \approx 0.7445$ 32. 410.10133. 39.2° 34. 82.1° 35. 35.7° 36. 64.8° 37. 44.6° 38. 9.0° 39. $c \approx 28.1$, $b \approx 15.7$, $B \approx 33.9^\circ$ 40. $a \approx 20.0$, $A \approx 60.3^\circ$, $B \approx 29.7^\circ$ 41. $c \approx 9.44$, $A \approx 25.1^\circ$, $B \approx 64.9^\circ$ 42. $b \approx 58.9$, $a \approx 29.9$, $A \approx 26.9^\circ$

43. 4.39 km 44. 36.0 mm

45. $\sin^2 45^\circ + \cos^2 45^\circ$ 46. $\cot \theta \sec \theta$

$$\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 \quad \frac{\cos \theta}{\sin \theta} \left(\frac{1}{\cos \theta}\right)$$

$$\frac{2}{4} + \frac{2}{4} = 1$$

$$\frac{1}{\sin \theta}$$

$$\csc \theta$$

$$47. \csc \theta (\sin \theta - \tan \theta)$$

$$\csc \theta \sin \theta - \csc \theta \tan \theta$$

$$\frac{1}{\sin \theta} \sin \theta - \frac{1}{\sin \theta} \frac{\sin \theta}{\cos \theta}$$

$$1 - \frac{1}{\cos \theta}$$

$$1 - \sec \theta$$

$$48. \frac{\sin \theta - 1}{\sin \theta}$$

$$\frac{\sin \theta}{\sin \theta} - \frac{1}{\sin \theta}$$

$$1 - \csc \theta$$

$$49. \frac{\cos \alpha + 2 - \cot \alpha}{\cos \alpha}$$

$$\frac{\cos \alpha}{\cos \alpha} + \frac{2}{\cos \alpha} - \frac{\cot \alpha}{\cos \alpha}$$

$$1 + 2 \left(\frac{1}{\cos \alpha} \right) - \frac{\cos \alpha}{\cos \alpha}$$

$$1 + 2 \sec \alpha - \frac{1}{\sin \alpha}$$

$$1 + 2 \sec \alpha - \csc \alpha$$

$$50. (1 + \sin \theta)(1 - \sin \theta)$$

$$1 - \sin \theta + \sin \theta - \sin^2 \theta$$

$$1 - \sin^2 \theta$$

$$\cos^2 \theta + \sin^2 \theta - \sin^2 \theta$$

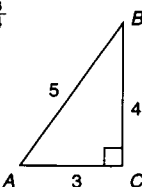
$$\cos^2 \theta$$

51. 20.9°

Chapter 1 test

1. 26.462° 2. 64.21° 3. $6\sqrt{3}$

4. 73.8 ft

5. $\frac{3}{4}$ 

6. 0.4305 7. 4.9313 8. 1.6107

9. 603.6 watts 10. 7.6° 11. 76.0° 12. $b \approx 9.7$, $c \approx 10.4$, $B \approx 68.6^\circ$ 13. $c \approx 9.5$, $A \approx 33.4^\circ$, $B \approx 56.6^\circ$

14. 15,540 ft

15. $\cos \theta (\sec \theta - \cos \theta)$ 16. 15.7°

$$\cos \theta \sec \theta - \cos^2 \theta$$

$$\cos \theta \frac{1}{\cos \theta} - \cos^2 \theta$$

$$1 - \cos^2 \theta$$

$$\sin^2 \theta$$

Chapter 2

Exercise 2-1

Answers to odd-numbered problems

1. a. yes b. $D = \{3, 4, 6, 7\}$, $R = \{5, 9, 10\}$ c. not one to one

3. a. yes

b. $D = \{-2, 3, 4\}$, $R = \{-2, 3, 4\}$ c. $h^{-1} = \{(-2, -2), (4, 3), (3, 4)\}$

5. not a function 7. a. yes

b. $D = \{1, 2, 3, 4\}$, $R = \{1, 2, 3, 4\}$ c. $f^{-1} = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

9. a. 9 b. 19 c. 11 d. not defined

e. not defined 11. a. $(-2, -13)$ b. $(0, -3)$ c. $(\sqrt{3}, 5\sqrt{3} - 3)$ d. $(\frac{1}{2}, -\frac{1}{2})$ e. $(5, 22)$ 13. a. $(-2, 0)$ b. $(0, -6)$ c. $(\sqrt{3}, -\sqrt{3} - 3)$ d. $(\frac{1}{2}, \frac{-25}{4})$ e. $(5, 14)$ 15. a. $(-2, 1)$ b. $(0, -1)$ c. $(\sqrt{3}, 2 + \sqrt{3})$ d. $(\frac{1}{2}, -\frac{1}{4})$ e. $(5, 29)$ 17. a. $(-2, 46)$ b. $(0, 2)$ c. $(\sqrt{3}, 26)$ d. $(\frac{1}{2}, \frac{31}{16})$ e. $(5, 1,852)$ 19. a. $(-2, -2)$ b. $(0, 0)$ c. $(\sqrt{3}, \frac{\sqrt{3}-1}{2})$ d. $(\frac{1}{2}, \frac{1}{7})$ e. $(5, \frac{5}{8})$ 21. a. 494°C b. 476°C c. 292°C

Solutions to trial exercise problems

8. $g = \{(-3, 5), (5, 8), (8, 13), (13, 21)\}$

a. Is a function because no first element repeats.

b. $\text{Domain}_g = \{-3, 5, 8, 13\}$; $\text{Range}_g = \{5, 8, 13, 21\}$ c. Is a one-to-one function because no second element repeats. The inverse is $\{(5, -3), (8, 5), (13, 8), (21, 13)\}$.

19. $h(x) = \frac{x}{x+3}$

a. $h(-2) = \frac{-2}{-2+3} = \frac{-2}{1} = -2; (-2, -2)$

b. $h(0) = \frac{0}{0+3} = \frac{0}{3} = 0; (0, 0)$

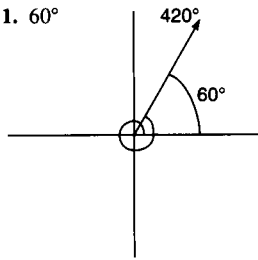
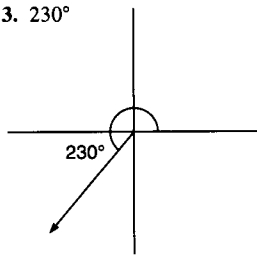
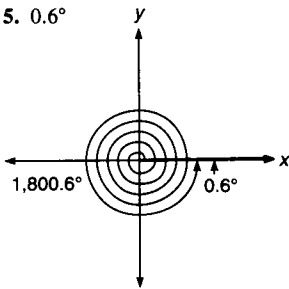
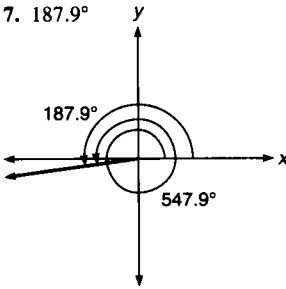
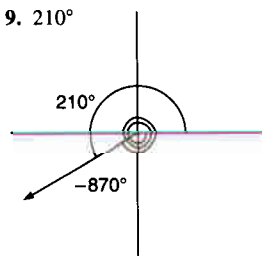
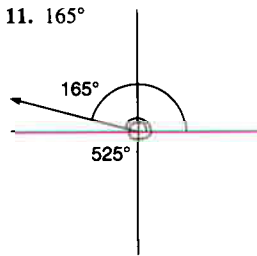
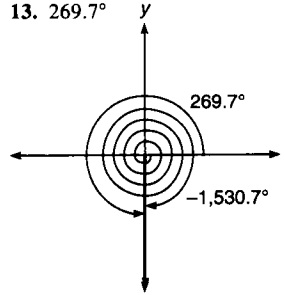
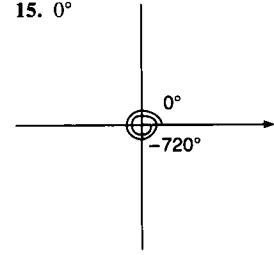
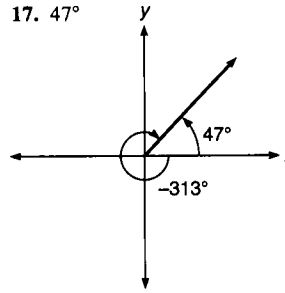
c. $h(\sqrt{3}) = \frac{\sqrt{3}}{\sqrt{3}+3} = \frac{\sqrt{3}}{\sqrt{3}+3} \cdot \frac{\sqrt{3}-3}{\sqrt{3}-3} = \frac{3-3\sqrt{3}}{3-9}$
 $= \frac{-3(-1+\sqrt{3})}{-6} = \frac{\sqrt{3}-1}{2}; \left(\sqrt{3}, \frac{\sqrt{3}-1}{2}\right)$

d. $h\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{\frac{1}{2}+3} = \frac{\frac{1}{2}}{\frac{7}{2}} = \frac{1}{2} \cdot \frac{2}{7} = \frac{1}{7}; \left(\frac{1}{2}, \frac{1}{7}\right)$

e. $h(5) = \frac{5}{5+3} = \frac{5}{8}; \left(5, \frac{5}{8}\right)$

Exercise 2-2

Answers to odd-numbered problems

 1. 60°

 3. 230°

 5. 0.6°

 7. 187.9°

 9. 210°

 11. 165°

 13. 269.7°

 15. 0°

 17. 47°

 19. 347° 21. 353.9°

23. $\sin \theta = \frac{2\sqrt{5}}{5}$, $\cos \theta = \frac{\sqrt{5}}{5}$, $\tan \theta = 2$, $\csc \theta = \frac{\sqrt{5}}{2}$,

$\sec \theta = \sqrt{5}$, $\cot \theta = \frac{1}{2}$ 25. $\sin \theta = \frac{8\sqrt{89}}{89}$, $\cos \theta = \frac{-5\sqrt{89}}{89}$,
 $\tan \theta = -\frac{8}{5}$, $\csc \theta = \frac{\sqrt{89}}{8}$, $\sec \theta = -\frac{\sqrt{89}}{5}$, $\cot \theta = \frac{-5}{8}$

27. $\sin \theta = \frac{-\sqrt{2}}{2}$, $\cos \theta = \frac{\sqrt{2}}{2}$, $\tan \theta = -1$, $\csc \theta = -\sqrt{2}$,

$\sec \theta = \sqrt{2}$, $\cot \theta = -1$ 29. $\sin \theta = \frac{4\sqrt{17}}{17}$, $\cos \theta = \frac{-\sqrt{17}}{17}$,
 $\tan \theta = -4$, $\csc \theta = \frac{\sqrt{17}}{4}$, $\sec \theta = -\sqrt{17}$, $\cot \theta = -\frac{1}{4}$

31. $\sin \theta = \frac{-3\sqrt{13}}{13}$, $\cos \theta = \frac{-2\sqrt{13}}{13}$, $\tan \theta = \frac{3}{2}$,
 $\csc \theta = \frac{-\sqrt{13}}{3}$, $\sec \theta = \frac{-\sqrt{13}}{2}$, $\cot \theta = \frac{2}{3}$

33. $\sin \theta = \frac{3\sqrt{38}}{19}$, $\cos \theta = \frac{-\sqrt{19}}{19}$, $\tan \theta = -3\sqrt{2}$,
 $\csc \theta = \frac{\sqrt{38}}{6}$, $\sec \theta = -\sqrt{19}$, $\cot \theta = \frac{-\sqrt{2}}{6}$ 35. $\sin \theta =$

$-\frac{\sqrt{10}}{5}$, $\cos \theta = \frac{-\sqrt{15}}{5}$, $\tan \theta = \frac{\sqrt{6}}{3}$, $\csc \theta = \frac{-\sqrt{10}}{2}$, $\sec \theta =$
 $-\frac{\sqrt{15}}{3}$, $\cot \theta = \frac{\sqrt{6}}{2}$ 37. $\sin \theta = \frac{-\sqrt{10}}{4}$, $\cos \theta = \frac{\sqrt{6}}{4}$, $\tan \theta =$

$\frac{-\sqrt{15}}{3}$, $\csc \theta = \frac{-2\sqrt{10}}{5}$, $\sec \theta = \frac{2\sqrt{6}}{3}$, $\cot \theta = \frac{-\sqrt{15}}{5}$

$$\begin{aligned}
 39. \sin \theta &= -\frac{\sqrt{5}}{5}, \cos \theta = \frac{2\sqrt{5}}{5}, \tan \theta = -\frac{1}{2}, \\
 \csc \theta &= -\sqrt{5}, \sec \theta = \frac{\sqrt{5}}{2}, \cot \theta = -2 \quad 41. \sin \theta = \frac{\sqrt{3}}{3}, \\
 \cos \theta &= \frac{\sqrt{6}}{3}, \tan \theta = \frac{\sqrt{2}}{2}, \csc \theta = \sqrt{3}, \sec \theta = \frac{\sqrt{6}}{2}, \cot \theta = \sqrt{2} \\
 43. \sin \theta &= \frac{2\sqrt{5}}{5}, \cos \theta = \frac{\sqrt{5}}{5}, \tan \theta = 2, \csc \theta = \frac{\sqrt{5}}{2}, \\
 \sec \theta &= \sqrt{5}, \cot \theta = \frac{1}{2} \quad 45. \frac{1}{\cos \theta} = \frac{1}{\frac{x}{r}} = \frac{r}{x} = \sec \theta
 \end{aligned}$$

$$\begin{aligned}
 47. \frac{\cos \theta}{\sin \theta} &= \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{y} = \cot \theta \quad 49. \frac{1}{\csc \theta} = \frac{1}{\frac{r}{y}} = \frac{y}{r} = \sin \theta
 \end{aligned}$$

51. (x_1, y_1) , (x_2, y_2) represent two points with the same trigonometric functions. The tangent for the point (x_1, y_1) is $\frac{y_1}{x_1}$.

The tangent for the point (x_2, y_2) is $\frac{y_2}{x_2}$, but $\frac{y_1}{x_1} = \frac{y_2}{x_2}$, since all

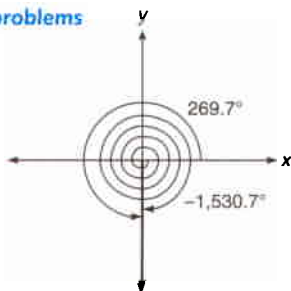
trigonometric functions are the same. Also, $\frac{y_1}{x_1}$ is the slope of the

line through $(0,0)$ and (x_1, y_1) . Likewise for $\frac{y_2}{x_2}$. So $\frac{y_1}{x_1} = \frac{y_2}{x_2} = m$.

Thus, both (x_1, y_1) and (x_2, y_2) lie on the line $y = mx$. They must be on the same terminal side, since all the trigonometric functions have the same sign.

Solutions to trial exercise problems

$$\begin{aligned}
 13. & -1,530.3^\circ \\
 & -1,530 \div 360 = -4.25; \\
 & -1,530.3^\circ + 5(360^\circ) = \\
 & 269.7^\circ \\
 & \text{(See the diagram.)}
 \end{aligned}$$



$$\begin{aligned}
 35. & (-\sqrt{3}, -\sqrt{2}) \\
 r &= \sqrt{(-\sqrt{3})^2 + (-\sqrt{2})^2} = \sqrt{3+2} = \sqrt{5} \\
 \sin \theta &= \frac{y}{r} = \frac{-\sqrt{2}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{\sqrt{10}}{5}; \\
 \csc \theta &= \frac{r}{y} = \frac{\sqrt{5}}{-\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{10}}{2}; \\
 \cos \theta &= \frac{x}{r} = \frac{-\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{\sqrt{15}}{5}; \\
 \sec \theta &= \frac{r}{x} = \frac{\sqrt{5}}{-\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{15}}{3}; \\
 \tan \theta &= \frac{y}{x} = \frac{-\sqrt{2}}{-\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}; \\
 \cot \theta &= \frac{x}{y} = \frac{-\sqrt{3}}{-\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2}
 \end{aligned}$$

$$\begin{aligned}
 38. & (b, -2b) = (x, y) \\
 r &= \sqrt{b^2 + (-2b)^2} = \sqrt{5b^2} = \sqrt{5}\sqrt{b^2} = \sqrt{5}b \\
 \sin \theta &= \frac{y}{r} = \frac{-2b}{\sqrt{5}b} = -\frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{2\sqrt{5}}{5} \\
 \cos \theta &= \frac{x}{r} = \frac{b}{\sqrt{5}b} = \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5} \\
 \tan \theta &= \frac{y}{x} = \frac{-2b}{b} = -2 \\
 \csc \theta &= \frac{1}{\sin \theta} = -\frac{\sqrt{5}}{2} \\
 \sec \theta &= \frac{1}{\cos \theta} = \sqrt{5} \\
 \cot \theta &= \frac{1}{\tan \theta} = -\frac{1}{2}
 \end{aligned}$$

50. As explained in the problem and shown in the figure we have two points (x_1, mx_1) and (x_2, mx_2) on the terminal side of an angle. Note that the value m in each case is the same number, since it is the slope of the line on which both points lie. Using the first point (x_1, mx_1) ,

$$\begin{aligned}
 r_1 &= \sqrt{(x_1)^2 + (mx_1)^2} = \sqrt{x_1^2 + m^2x_1^2} \\
 &= \sqrt{x_1^2(1+m^2)} = |x_1| \sqrt{1+m^2}
 \end{aligned}$$

$$\sin \theta = \frac{y}{r_1} = \frac{mx_1}{|x_1| \sqrt{1+m^2}} = \frac{x_1}{|x_1|} \cdot \frac{m}{\sqrt{1+m^2}}$$

The value of $\frac{x_1}{|x_1|}$ is 1 or -1, depending on the sign of x_1 . Thus,

$$\sin \theta = \begin{cases} \frac{m}{\sqrt{1+m^2}} & \text{if } x_1 > 0 \\ -\frac{m}{\sqrt{1+m^2}} & \text{if } x_1 < 0 \end{cases}$$

Using the second point (x_2, mx_2) ,

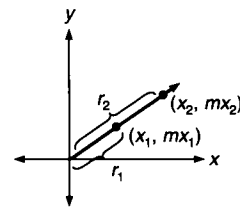
$$\begin{aligned}
 r_2 &= \sqrt{(x_2)^2 + (mx_2)^2} = \sqrt{x_2^2 + m^2x_2^2} \\
 &= \sqrt{x_2^2(1+m^2)} = |x_2| \sqrt{1+m^2}
 \end{aligned}$$

$$\sin \theta = \frac{y}{r_2} = \frac{mx_2}{|x_2| \sqrt{1+m^2}} = \frac{x_2}{|x_2|} \cdot \frac{m}{\sqrt{1+m^2}}$$

As above, the value of $\frac{x_2}{|x_2|}$ is 1 or -1, depending on the sign of x_2 . Thus,

$$\sin \theta = \begin{cases} \frac{m}{\sqrt{1+m^2}} & \text{if } x_2 > 0 \\ -\frac{m}{\sqrt{1+m^2}} & \text{if } x_2 < 0 \end{cases}$$

Since x_1 and x_2 lie in the same quadrant, they have the same sign. Therefore, we obtain the same value for $\sin \theta$ using either point.



Exercise 2-3**Answers to odd-numbered problems**

1. II 3. I 5. IV 7. II 9. IV
 11. III 13. 15.8° 15. 67.1°
 17. 75.3° 19. 49.3° 21. 1°
 23. 80.5° 25. 72° 27. $\frac{\sqrt{2}}{2}$ 29. $-\frac{1}{2}$
 31. $-\sqrt{3}$ 33. $-\frac{\sqrt{3}}{2}$ 35. $\frac{1}{2}$
 37. $-\frac{\sqrt{3}}{3}$ 39. 0 41. 1 43. $\frac{1}{2}$
 45. $-\frac{2\sqrt{3}}{3}$ 47. 0.9178 49. 0.6899
 51. 0.7813 53. 1.0263 55. 0.9967
 57. -1.0367 59. -0.6845
 61. 14.5° 63. 120° 65. -58.0°
 67. -36.2° 69. a. 110.31 volts
 b. 156 volts c. 89.48 volts
 d. -65.93 volts e. 132.73 volts
 f. 0 volts 71. a. 200 lb
 b. 181.3 lb c. 128.6 lb

73. $\sin 60^\circ \stackrel{?}{=} 2 \sin 30^\circ$

$$\frac{\sqrt{3}}{2} \stackrel{?}{=} 2 \cdot \frac{1}{2}$$

$$\frac{\sqrt{3}}{2} \neq 1; \text{ No, it is false.}$$

75. Use the values 30° , 60° , 90° to see if the statement $\sin(\alpha + \beta) = \sin \alpha + \sin \beta$ is true.

$$\sin(30^\circ + 60^\circ) \stackrel{?}{=} \sin 30^\circ + \sin 60^\circ$$

$$\sin 90^\circ \stackrel{?}{=} \sin 30^\circ + \sin 60^\circ$$

$$1 \stackrel{?}{=} \frac{1}{2} + \frac{\sqrt{3}}{2}$$

$$1 \neq \frac{1 + \sqrt{3}}{2}; \text{ No, it is false.}$$

Solutions to trial exercise problems

6. $\sec \theta > 0$, $\csc \theta < 0$

If $\sec \theta > 0$, then $\cos \theta > 0$, so θ is in quadrant I or quadrant IV. If $\csc \theta < 0$, then $\sin \theta < 0$, so θ is in quadrant III or quadrant IV. Therefore, θ is in quadrant IV.

19. 130.7°

Since $90^\circ < 130.7^\circ < 180^\circ$, this angle terminates in quadrant II. In quadrant II, $\theta' = 180^\circ - \theta = 180^\circ - 130.7^\circ = 49.3^\circ$

43. $\sin(-690^\circ)$

-690° is coterminal with 30° (add $-690^\circ + 2[360^\circ]$), so $\sin(-690^\circ) = \sin 30^\circ = \frac{1}{2}$

52. $\tan 527.2^\circ$

$527.2^\circ - 360^\circ = 167.2^\circ$, so $\tan 527.2^\circ = \tan 167.2^\circ$. Since 167.2° terminates in quadrant II, $\theta' = 180^\circ - 167.2^\circ$

$= 12.8^\circ$. The tangent function is negative in quadrant II, so we know that

$$\tan 167.2^\circ = -\tan 12.8^\circ = -0.2272 \text{ (to four decimal places).}$$

65. $\tan \theta = -\frac{8}{5}$

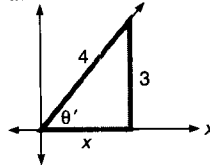
$$\theta = \tan^{-1}\left(-\frac{8}{5}\right) \approx -58.0^\circ$$

$$\text{Display: } \boxed{-57.99461679}$$

Exercise 2-4**Answers to odd-numbered problems**

1. 55.6° 3. 33.3° 5. 200.0°
 7. 358.0° 9. 22.0° 11. 168.7°
 13. 224.4°

15. a. y

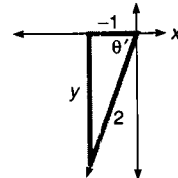


$$\text{b. } \csc \theta = \frac{4}{3}, \cos \theta = \frac{\sqrt{7}}{4},$$

$$\sec \theta = \frac{4\sqrt{7}}{7}, \tan \theta = \frac{3\sqrt{7}}{7},$$

$$\cot \theta = \frac{\sqrt{7}}{3} \quad \text{c. } \theta = \theta' \approx 48.6^\circ$$

17. a. y

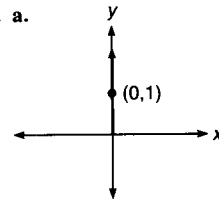


$$\text{b. } \sin \theta = \frac{-\sqrt{3}}{2}, \csc \theta = \frac{-2\sqrt{3}}{3},$$

$$\sec \theta = -2, \tan \theta = \sqrt{3}, \cot \theta = \frac{\sqrt{3}}{3}$$

$$\text{c. } \theta' = 60^\circ, \theta = 240^\circ$$

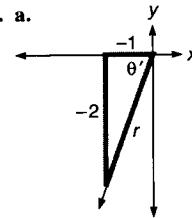
19. a. y



- b. $\cos \theta = 0$, $\tan \theta$ undefined, $\cot \theta = 0$, $\csc \theta = 1$, $\sec \theta$ undefined

- c. $\theta = \theta' = 90^\circ$

21. a. y

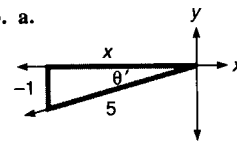


$$\text{b. } \cot \theta = \frac{1}{2}, \sin \theta = \frac{-2\sqrt{5}}{5}, \cos \theta$$

$$= \frac{-\sqrt{5}}{5}, \csc \theta = \frac{-\sqrt{5}}{2}, \sec \theta = -\sqrt{5}$$

$$\text{c. } \theta' = 63.4^\circ, \theta \approx 243.4^\circ$$

23. a. y

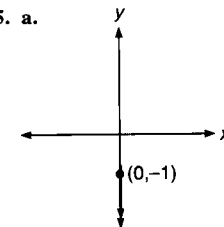


$$\text{b. } \sin \theta = -\frac{1}{5}, \cos \theta = \frac{-2\sqrt{6}}{5},$$

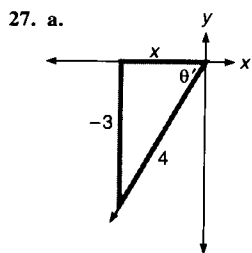
$$\sec \theta = \frac{-5\sqrt{6}}{12}, \tan \theta = \frac{\sqrt{6}}{12}, \cot \theta = 2\sqrt{6}$$

$$\text{c. } \theta' \approx 11.5^\circ, \theta \approx 191.5^\circ$$

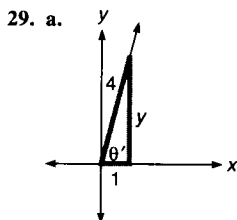
25. a. y



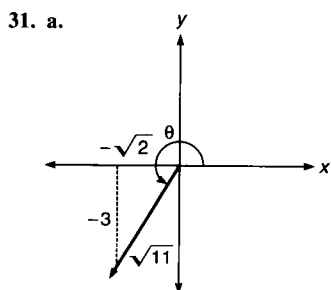
- b. $\sin \theta = -1$, $\cos \theta = 0$, $\sec \theta$ undefined, $\tan \theta$ undefined, $\cot \theta = 0$ c. $\theta' = 90^\circ$, $\theta = 270^\circ$



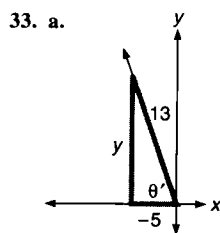
b. $\csc \theta = -\frac{4}{3}$, $\cos \theta = \frac{-\sqrt{7}}{4}$,
 $\sec \theta = \frac{-4\sqrt{7}}{7}$, $\tan \theta = \frac{3\sqrt{7}}{7}$, $\cot \theta = \frac{\sqrt{7}}{3}$
 c. $\theta' \approx 48.6^\circ$, $\theta \approx 228.6^\circ$



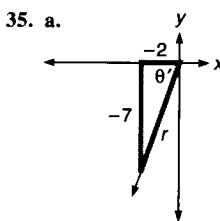
b. $\cos \theta = \frac{1}{4}$, $\sin \theta = \frac{\sqrt{15}}{4}$, $\csc \theta = \frac{4\sqrt{15}}{15}$, $\tan \theta = \sqrt{15}$, $\cot \theta = \frac{\sqrt{15}}{15}$
 c. $\theta = \theta' \approx 75.5^\circ$



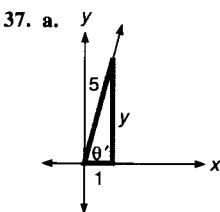
b. $\tan \theta = \frac{3\sqrt{2}}{2}$, $\cos \theta = -\frac{\sqrt{22}}{11}$, $\sin \theta = -\frac{3\sqrt{11}}{11}$, $\csc \theta = -\frac{\sqrt{11}}{3}$, $\sec \theta = -\frac{\sqrt{22}}{2}$
 c. $\theta' \approx 64.8^\circ$, $\theta \approx 244.8^\circ$



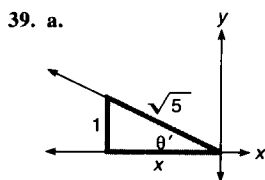
b. $\sec \theta = \frac{-13}{5}$, $\sin \theta = \frac{12}{13}$, $\csc \theta = \frac{13}{12}$, $\tan \theta = \frac{-12}{5}$, $\cot \theta = \frac{-5}{12}$
 c. $\theta' \approx 67.4^\circ$, $\theta \approx 112.6^\circ$



b. $\cot \theta = \frac{2}{7}$, $\sin \theta = \frac{-7\sqrt{53}}{53}$, $\csc \theta = \frac{-\sqrt{53}}{7}$, $\cos \theta = \frac{-2\sqrt{53}}{53}$, $\sec \theta = \frac{-\sqrt{53}}{2}$
 c. $\theta \approx 74.1^\circ$, $\theta \approx 254.1^\circ$



b. $\cos \theta = \frac{1}{5}$, $\sin \theta = \frac{2\sqrt{6}}{5}$, $\csc \theta = \frac{5\sqrt{6}}{12}$, $\tan \theta = 2\sqrt{6}$, $\cot \theta = \frac{\sqrt{6}}{12}$
 c. $\theta = \theta' \approx 78.5^\circ$



b. $\csc \theta = \sqrt{5}$, $\cos \theta = \frac{-2\sqrt{5}}{5}$, $\sec \theta = \frac{-\sqrt{5}}{2}$, $\cot \theta = -2$, $\tan \theta = -\frac{1}{2}$
 c. $\theta' \approx 26.6^\circ$, $\theta \approx 153.4^\circ$

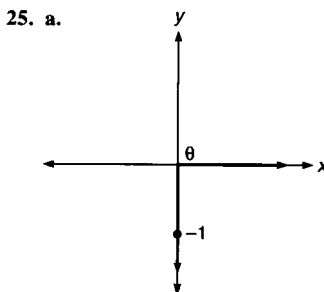
41. a. $\sin \theta = \frac{-5\sqrt{29}}{29}$,
 $\cos \theta = \frac{2\sqrt{29}}{29}$, $\tan \theta = -\frac{5}{2}$,
 $\sec \theta = \frac{\sqrt{29}}{2}$, $\csc \theta = \frac{-\sqrt{29}}{5}$, $\cot \theta = -\frac{2}{5}$ b. 291.8°

43. $(-4.85 \text{ mm}, 4.77 \text{ mm})$
 45. $(-5.77 \text{ cm}, -5.89 \text{ cm})$
 47. 8.8 cm , 60.6° ; -8.8 cm , 119.4° ; -8.8 cm , 240.6° ; 8.8 cm , 299.4°
 49. $y \approx -13.5 \text{ in.}$, $x \approx -22.0 \text{ in.}$

51. $\cot \theta = \frac{1}{u}$, $\sin \theta = \frac{u}{\sqrt{u^2+1}}$,
 $\csc \theta = \frac{\sqrt{u^2+1}}{u}$, $\cos \theta = \frac{1}{\sqrt{u^2+1}}$,
 $\sec \theta = \sqrt{u^2+1}$ 53. $\cot \theta = \frac{1}{u}$,
 $\sin \theta = \frac{-u}{\sqrt{u^2+1}}$, $\csc \theta = \frac{-\sqrt{u^2+1}}{u}$,
 $\cos \theta = \frac{-1}{\sqrt{u^2+1}}$, $\sec \theta = -\sqrt{u^2+1}$
 55. $\sec \theta = \frac{1}{1-u}$, $\sin \theta = \frac{\sqrt{2u-u^2}}{1-u}$,
 $\csc \theta = \frac{1}{\sqrt{2u-u^2}}$, $\tan \theta = \frac{\sqrt{2u-u^2}}{1-u}$,
 $\cot \theta = \frac{1-u}{\sqrt{2u-u^2}}$ 57. 663.4 ft

Solutions to trial exercise problems

13. $\cos \theta = -\frac{5}{7}$, $\tan \theta > 0$
 $\theta' = \cos^{-1} \frac{5}{7} \approx 44.4^\circ$
 $\cos \theta < 0$, $\tan \theta > 0$, so θ terminates in quadrant III. Therefore, $\theta = 180^\circ + \theta' \approx 180^\circ + 44.4^\circ \approx 224.4^\circ$.



b. $\csc \theta = -1$
 $\sin \theta = \frac{1}{\csc \theta} = \frac{1}{-1} = -1$; $\sin \theta = \frac{y}{r} = -1 = \frac{-1}{1}$, so we see that $y = -1$ and $r = 1$ would work. This means

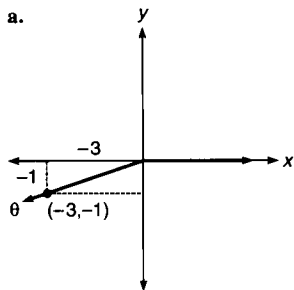
that x must be 0, since $x^2 + y^2 = r^2$.
Thus, the point $(0, -1)$ is on the angle,
which is shown in the diagram. $\cos \theta$
 $= \frac{0}{1} = 0$; $\sec \theta = \frac{1}{\cos \theta}$, which is not

defined in this case. $\tan \theta = \frac{y}{x} = \frac{-1}{0}$

(not defined); $\cot \theta = \frac{x}{y} = \frac{0}{-1} = 0$.

c. We can see that $\theta = 270^\circ$ from the diagram.

34. a.



b. $\cos \theta = -\frac{3}{\sqrt{10}}$, $\sin \theta < 0$; $x = -3$,
 $r = \sqrt{10}$.

Computing, $x^2 + y^2 = r^2$
 $9 + y^2 = 10$
 $y = \pm 1$

Since $\cos \theta < 0$ and $\sin \theta < 0$, θ
terminates in quadrant III, so $y = -1$. (See
the diagram.)

$\sin \theta = \frac{y}{r} = \frac{-1}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{-\sqrt{10}}{10}$; $\tan \theta$
 $= \frac{y}{x} = \frac{-1}{-3} = \frac{1}{3}$; $\csc \theta = \frac{1}{\sin \theta} = -\sqrt{10}$;

$\sec \theta = \frac{1}{\cos \theta} = \frac{-\sqrt{10}}{3}$; $\cot \theta = \frac{1}{\tan \theta} = 3$.

c. $\theta = 180^\circ + \tan^{-1} \frac{1}{3} \approx 180^\circ + 18.4^\circ \approx$
 198.4°

47. $\sin \theta' = \frac{y}{r} = \frac{15.5}{17.8}$ (See diagram.)

$\theta' = 60.6^\circ$. The four angles are

$\theta = \theta' = 60.6^\circ$

$= 180^\circ - 60.6^\circ = 119.4^\circ$

$= 180^\circ + 60.6^\circ = 240.6^\circ$

$= 360^\circ - 60.6^\circ = 299.4^\circ$

We know that $r^2 = x^2 + y^2$, so

$17.8^2 = x^2 + 15.5^2$

$17.8^2 - 15.5^2 = x^2$

$76.59 = x^2$, $8.75 = x$. Thus, $x = \pm 8.8$.

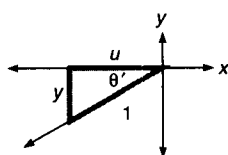
49. 4 ft 3.5 in. $= 51.5$ in.; $r = \frac{51.5}{2}$ in. $=$

25.75 in.

$x = r \cos \theta = 25.75 \cos 211.5^\circ = -22.0$
in.

$y = r \sin \theta = 25.75 \sin 211.5^\circ = -13.5$ in.

52. $\cos \theta = u$, and θ terminates in quadrant
III. (See the diagram.)



Using the Pythagorean theorem, we find

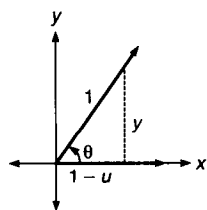
$y = -\sqrt{1 - u^2}$. $\sin \theta = \frac{y}{r} = -\sqrt{1 - u^2}$,

$\tan \theta = \frac{y}{x} = \frac{-\sqrt{1 - u^2}}{u}$, $\csc \theta =$

$\frac{1}{\sin \theta} = \frac{-1}{\sqrt{1 - u^2}}$, $\sec \theta = \frac{1}{\cos \theta} = \frac{1}{u}$,

$\cot \theta = \frac{1}{\tan \theta} = -\frac{u}{\sqrt{1 - u^2}}$

55. $\cos \theta = 1 - u$, θ in quadrant I. (See
the diagram.)

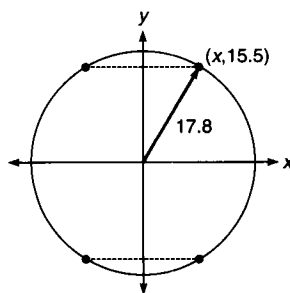


$1^2 = y^2 + (1 - u)^2$

$1 - (1 - 2u + u^2) = y^2$

$2u - u^2 = y^2$

$\sqrt{2u - u^2} = y$ ($y > 0$)



$\sec \theta = \frac{1}{\cos \theta} = \frac{1}{1 - u}$;

$\sin \theta = \frac{y}{r} = \frac{\sqrt{2u - u^2}}{\sqrt{2u - u^2}}$;

$\csc \theta = \frac{1}{\sin \theta} = \frac{1}{\sqrt{2u - u^2}}$;

$\tan \theta = \frac{y}{x} = \frac{\sqrt{2u - u^2}}{1 - u}$;

$\cot \theta = \frac{1}{\tan \theta} = \frac{1 - u}{\sqrt{2u - u^2}}$

57. $AB = 512.4$ feet, $AP = 322.6$ feet,
 $b = 28.3^\circ$.

$\sin p = \frac{AB \sin b}{AP} = \frac{512.4(\sin 28.3^\circ)}{322.6}$

≈ 0.7530

$p \approx 48.85^\circ$

$a = 180^\circ - (b + p) \approx 180^\circ - (28.3^\circ +$
 $48.85^\circ) \approx 102.85^\circ$

$BP = \frac{AP \sin a}{\sin b} \approx \frac{322.6(\sin 102.85^\circ)}{\sin 28.3^\circ} \approx 663.4$

feet

Exercise 2-5

Answers to odd-numbered problems

1. $x^2 + y^2 = 1$ 3. $\frac{\pi}{4}$, 0.79

5. $\frac{5\pi}{9}$, 1.75 7. $\frac{-5\pi}{3}$, -5.24

9. $\frac{3\pi}{2}$, 4.71 11. $\frac{127\pi}{180}$, 2.22

13. $\frac{-61\pi}{36}$, -5.32 15. 330°

17. 108° 19. 40° 21. -510°

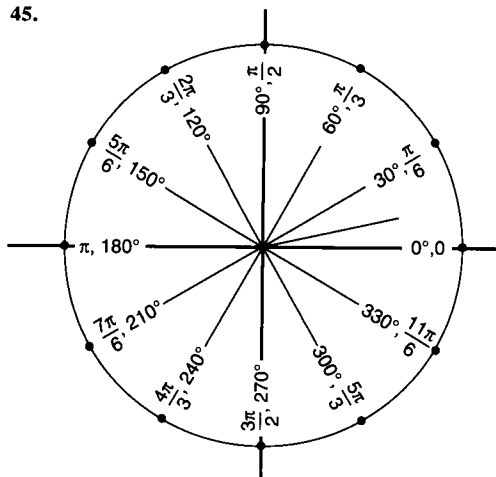
23. $\frac{270^\circ}{\pi}$, 85.94° 25. $\frac{-2,160^\circ}{17\pi}$, -40.44°

27. $\frac{360^\circ}{\pi}$, 114.59° 29. $\frac{-900^\circ}{\pi}$, -286.48°

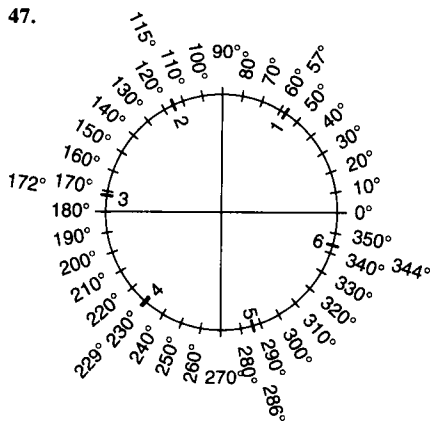
31. $\frac{\pi}{3}$ 33. $\frac{\pi}{6}$ 35. $\frac{\pi}{3}$ 37. $\frac{\pi}{4}$

39. $\frac{\pi}{3}$ 41. $\frac{\pi}{3}$ 43. $\frac{\pi}{6}$

45.



47.



49. 2.7 51. 6.5 inches 53. 24 inches
 55. 98 in.² 57. 33.93 cm² 59. 7.5 in.²
 61. 10.60 mm² 63. 1.24 65. 35.5
 gallons 67. 30 mm

Solutions to trial exercise problems

12. -422°

$$\frac{s}{\pi} = \frac{\theta^\circ}{180^\circ}, \text{ so } \frac{s}{\pi} = \frac{-422^\circ}{180^\circ}, \text{ (Cross multiply.) } s \cdot 180^\circ = \pi(-422^\circ),$$

$$s = \frac{\pi(-422^\circ)}{180^\circ}, s = \frac{-211\pi}{90} \text{ or } -7.37$$

21. $-\frac{17\pi}{6}$

$$\frac{s}{\pi} = \frac{\theta^\circ}{180^\circ}, \text{ so } \frac{-17\pi}{\pi} = \frac{\theta^\circ}{180^\circ}, \text{ (Cross multiply.) } \frac{-17\pi(180^\circ)}{6} = \pi\theta^\circ,$$

$$-17\pi(30^\circ) = \pi\theta^\circ, \frac{-17\pi(30^\circ)}{\pi} = \theta^\circ,$$

$$-17(30^\circ) = \theta^\circ, -510^\circ = \theta^\circ$$

29. -5

$$\frac{s}{\pi} = \frac{\theta^\circ}{180^\circ}, \text{ so } \frac{-5}{\pi} = \frac{\theta^\circ}{180^\circ}, \text{ (Cross multiply.) } -5(180^\circ) = \pi\theta^\circ, \frac{-5(180^\circ)}{\pi}$$

$$= \theta^\circ, \frac{-900^\circ}{\pi} = \theta^\circ, -286.48^\circ = \theta^\circ$$

43. $-\frac{7\pi}{6}$

$-\frac{7\pi}{6} + \pi = -\frac{7\pi}{6} + \frac{1\pi}{6} = \frac{5\pi}{6}$
 Thus, $\frac{5\pi}{6}$ and $-\frac{7\pi}{6}$ are coterminal (and therefore have the same reference angle). Now find in which quadrant $\frac{5\pi}{6}$ terminates.

$$0 < \frac{\pi}{2} < \pi$$

$$0 < \frac{3\pi}{6} < \frac{6\pi}{6}$$

$$\frac{3\pi}{6} < \frac{5\pi}{6} < \frac{6\pi}{6}, \text{ so } \frac{\pi}{2} < \frac{5\pi}{6} < \pi, \text{ so } \frac{5\pi}{6}$$

terminates in quadrant II. Therefore, θ°

$$= \pi - \theta = \frac{6\pi}{6} - \frac{5\pi}{6} = \frac{\pi}{6}.$$

52. $s = \frac{L}{r}$; $L = 14.5$ mm, diameter

$$= 10.3 \text{ mm. } r = \frac{10.3 \text{ mm}}{2} = 5.15 \text{ mm,}$$

$$s = \frac{14.5 \text{ mm}}{5.15 \text{ mm}} = \frac{1,450}{515} = \frac{290}{103} \approx 2.8155$$

$$\frac{s}{\pi} = \frac{\theta^\circ}{180^\circ}, \text{ so } \frac{2.8155}{\pi} \approx \frac{\theta^\circ}{180^\circ}, \text{ (Cross$$

$$\text{multiply.) } 2.8155(180^\circ) \approx \pi\theta^\circ,$$

$$\frac{2.8155(180^\circ)}{\pi} = \theta^\circ, 161.318^\circ \approx \theta^\circ$$

Thus, $\theta^\circ = 161.3^\circ$ or 2.8 (radians).53. Convert 85° to radians.

$$\frac{85^\circ}{180^\circ} = \frac{s}{\pi}$$

$$s = \frac{85\pi}{180} = \frac{17\pi}{36}$$

The radius of the wheel is $32.4'' \div 2 = 16.2''$. The car will move whatever arc length the angle determines on its circumference: $L = rs$

$$L = 16.2'' \cdot \frac{17\pi}{36} = 7.65\pi \approx 24.0$$

inches, or 2 feet

61. 15° , 9 mm

$$15^\circ \text{ is } \frac{\pi}{12} \text{ radians}$$

$$A = \frac{1}{2}sr^2 = \frac{1}{2} \cdot \frac{\pi}{12} \cdot 9^2 = \frac{81\pi}{24} = \frac{27\pi}{8}$$

$$\approx 10.60 \text{ mm}^2$$

Exercise 2-6

Answers to odd-numbered problems

1. $\frac{\sqrt{3}}{2}$ 3. $\frac{\sqrt{3}}{2}$ 5. $-\frac{1}{2}$ 7. $\frac{\sqrt{2}}{2}$

9. $-\sqrt{3}$ 11. $\sqrt{3}$ 13. $\frac{1}{2}$

15. 0.7833 17. 0.5463 19. 1.5523

21. 0.7457 23. 1.4235 25. 1.6709

27. $\frac{4\pi}{3}$ 29. $\frac{5\pi}{3}$ 31. $\frac{7\pi}{4}$ 33. $\frac{11\pi}{6}$

35. 1.9513 37. 3.6259 39. 3.4814

41. 0.17 43. 1.19 45. π

47. $\frac{\pi}{18}$ 49. $\frac{\pi}{2}$ or $\frac{\pi}{3}$ 51. $\frac{\pi}{3}$ or $\frac{\pi}{2}$

53. a. -0.03 b. -0.37 55. 18.18 cm

57. a. 0.099833417 b. 0.479425533

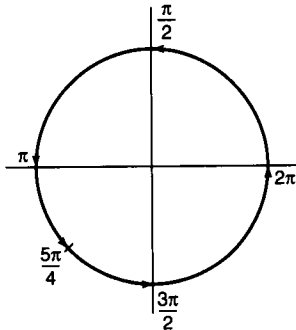
c. 0.841468254 d. 0.499999992

Solutions to trial exercise problems

2. $\tan \frac{5\pi}{4}$

First we must find the quadrant in which $\frac{5\pi}{4}$ terminates.

Remember the following correspondence between values in radians and quadrants, as illustrated in the diagram.



0 to $\frac{1}{2}\pi$: quadrant I; $\frac{1}{2}\pi$ to π : quadrant II; π to $\frac{3}{2}\pi$: quadrant III; $\frac{3}{2}\pi$ to 2π : quadrant IV.

$\frac{5\pi}{4} = 1\frac{1}{4}\pi$, which, therefore, terminates in quadrant III. For a value that corresponds to an angle that terminates in quadrant III, we form the reference angle by subtracting π : $1\frac{1}{4}\pi - \pi = \frac{1}{4}\pi = \frac{\pi}{4}$. $\tan \frac{\pi}{4} = \tan 45^\circ = 1$ so $\tan \frac{5\pi}{4} = \pm 1$.

Since the corresponding angle terminates in quadrant III, where the tangent function is positive, $\tan \frac{5\pi}{4} = 1$.

3. $\cos \frac{11\pi}{6}$

$\frac{11\pi}{6} = 1\frac{5}{6}\pi$, which, therefore, represents an angle that terminates in quadrant IV (see discussion of problem 2). In this quadrant we subtract the value from 2π to obtain the reference angle.

$$2\pi - 1\frac{5}{6}\pi = \frac{1}{6}\pi = \frac{\pi}{6} \cdot \cos \frac{\pi}{6} = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

Since the cosine function is positive in quadrant IV, we know that our result is positive. Thus, $\cos \frac{11\pi}{6} = \frac{\sqrt{3}}{2}$.

6. $\sin \frac{5\pi}{6}$

$\frac{5\pi}{6} = \frac{5}{6}\pi$, which corresponds to an angle that terminates in quadrant II. Therefore, the reference angle is $\pi - \frac{5}{6}\pi = \frac{1}{6}\pi$. $\sin \frac{\pi}{6} = \frac{1}{2}$, and $\frac{5\pi}{6}$ terminates in quadrant II where

the sine function is positive, so $\sin \frac{5\pi}{6} = \frac{1}{2}$.

24. $\sec 5.2 = \frac{1}{\cos 5.2} \approx 2.1344$

5.2 $\cos$ $1/x$

TI-81

($\cos$ 5.2) x^{-1} ENTER

Display:

2.134395767

Make sure calculator is in radian mode.

34. $\sin \theta = -0.5624$

$\theta' = \sin^{-1} 0.5624$

Use the positive value.

$\theta' \approx 0.5973$

$\sin \theta < 0$, $\tan \theta > 0$, so θ terminates in quadrant III. Thus, $\theta = \pi + \theta' \approx 3.74$.

48. $2 \sin \theta - 1 = 0$ or $\sin \theta - 1 = 0$

Zero product property: If $ab = 0$ then $a = 0$ or $b = 0$.

$2 \sin \theta = 1$ or $\sin \theta = \frac{1}{2}$

$\sin \theta = \frac{1}{2}$

$\theta = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$ or $\theta = \sin^{-1} 1 = \frac{\pi}{2}$

$\theta = \frac{\pi}{6}$ or $\frac{\pi}{2}$

55. $V = \frac{\sqrt{a^2 + b^2 - 2ab \cos \theta}}{\sin \theta}$, $a = 6.2$ cm, $b = 3.5$ cm,

$\theta = 2.6$.

$V = \frac{\sqrt{6.2^2 + 3.5^2 - 2(6.2)(3.5) \cos 2.6}}{\sin 2.6} \approx 18.18$ cm

To use a calculator, make sure it is in radian angle mode.

A

6.2 x^2 + 3.5 x^2 - 2 $\times$ 6.2 $\times$

3.5 $\times$ 2.6 $\cos$ = $\sqrt{x}$ $\div$ 2.6

SIN =

P

6.2 x^2 3.5 x^2 + 2 ENTER 6.2 $\times$ 3.5

$\times$

2.6 $\cos$ $\times$ - $\sqrt{x}$ 2.6 SIN $\div$

TI-81

$\sqrt{x}$ (6.2 x^2 + 3.5 x^2 - 2

$\times$ 6.2 $\times$ 3.5 $\times$ $\cos$ 2.6)

$\div$ SIN 2.6 ENTER

Display: 18.18497291

Chapter 2 review

1. a. yes b. $D = \{1, 4, 6, 7\}$,

$R = \{7, 5, 10\}$ c. not one to one

2. a. yes

b. $D = \{-2, -1, 1, 2\}$, $R = \{-3, 1, 2, 5\}$

c. $\{(-3, -2), (1, -1), (2, 1), (5, 2)\}$

3. Not a function 4. a. $(-1, 6)$

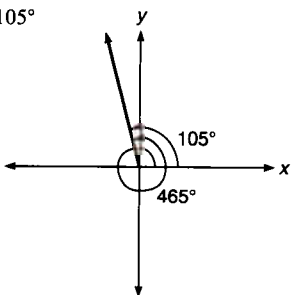
b. $(0, 4)$ c. $(\sqrt{5}, 4 - 2\sqrt{5})$ d. $(\frac{1}{3}, \frac{10}{3})$

5. a. $(-1, 8)$ b. $(0, 3)$

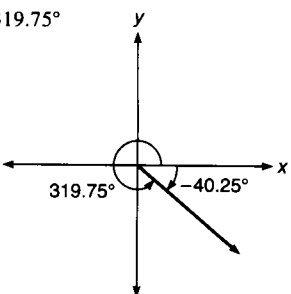
c. $(\sqrt{5}, 18 - 2\sqrt{5})$ d. $(\frac{1}{3}, \frac{8}{3})$

6. a. $(-1, -4)$ b. $(0, -5)$ c. $(\sqrt{5}, 20)$
 d. $(\frac{1}{3}, -4\frac{80}{81})$ 7. a. $(-1, \frac{3}{2})$ b. $(0, 0)$
 c. $(\sqrt{5}, \frac{15 + 3\sqrt{5}}{4})$ d. $(\frac{1}{3}, -\frac{3}{2})$

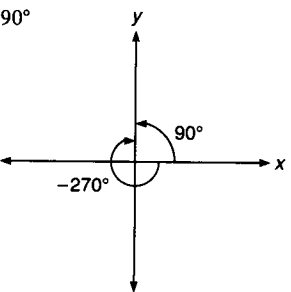
8. 105°



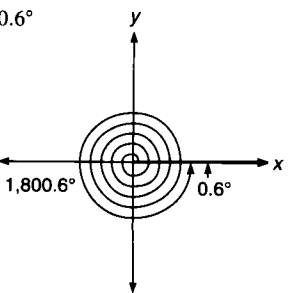
9. 319.75°



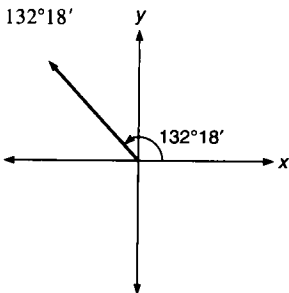
10. 90°



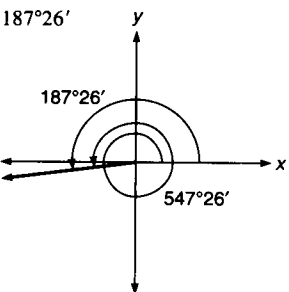
11. 0.6°



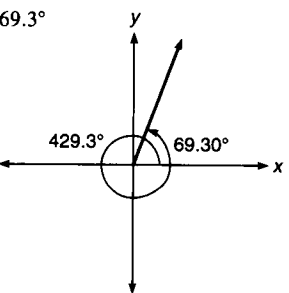
12. $132^\circ 18'$



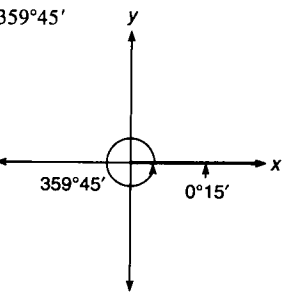
13. $187^\circ 26'$



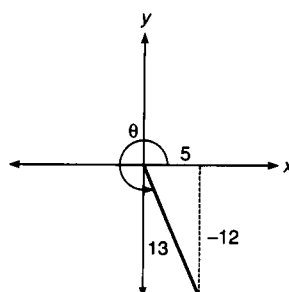
14. 69.3°



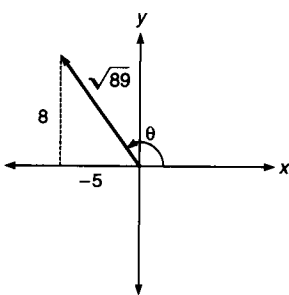
15. $359^\circ 45'$



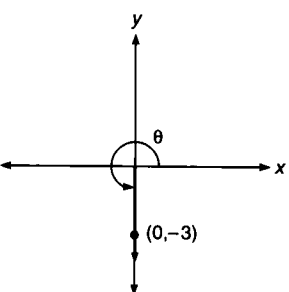
16. $\sin \theta = \frac{-12}{13}$, $\csc \theta = \frac{-13}{12}$, $\cos \theta = \frac{5}{13}$,
 $\sec \theta = \frac{13}{5}$, $\tan \theta = -\frac{12}{5}$, $\cot \theta = -\frac{5}{12}$



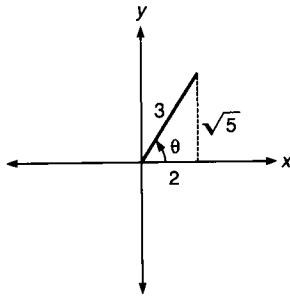
17. $\sin \theta = \frac{8\sqrt{89}}{89}$, $\csc \theta = \frac{\sqrt{89}}{8}$,
 $\cos \theta = \frac{-5\sqrt{89}}{89}$, $\sec \theta = \frac{-\sqrt{89}}{5}$,
 $\tan \theta = -\frac{8}{5}$, $\cot \theta = \frac{-5}{8}$



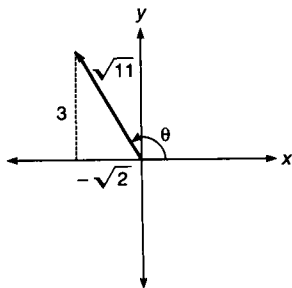
18. $\cos \theta = 0$, $\sec \theta$ undefined, $\sin \theta = -1$, $\csc \theta = -1$, $\tan \theta$ undefined, $\cot \theta = 0$



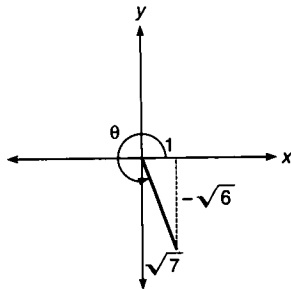
19. $\cos \theta = \frac{2}{3}$, $\sec \theta = \frac{3}{2}$, $\sin \theta = \frac{\sqrt{5}}{3}$,
 $\csc \theta = \frac{3\sqrt{5}}{5}$, $\tan \theta = \frac{\sqrt{5}}{2}$, $\cot \theta = \frac{2\sqrt{5}}{5}$



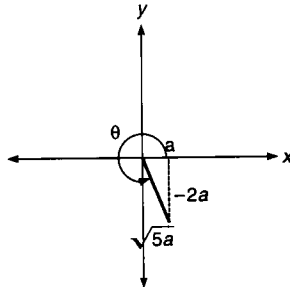
20. $\cos \theta = \frac{-\sqrt{22}}{11}$, $\sec \theta = \frac{-\sqrt{22}}{2}$,
 $\sin \theta = \frac{3\sqrt{11}}{11}$, $\csc \theta = \frac{\sqrt{11}}{3}$, $\tan \theta = \frac{-3\sqrt{2}}{2}$, $\cot \theta = \frac{-\sqrt{2}}{3}$



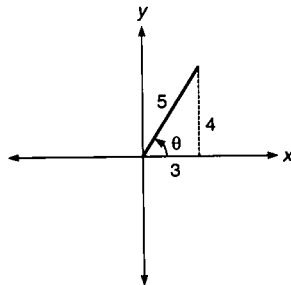
21. $\cos \theta = \frac{\sqrt{7}}{7}$, $\sec \theta = \sqrt{7}$,
 $\sin \theta = \frac{-\sqrt{42}}{7}$, $\csc \theta = \frac{-\sqrt{42}}{6}$,
 $\tan \theta = -\sqrt{6}$, $\cot \theta = \frac{-\sqrt{6}}{6}$



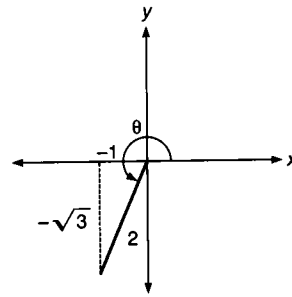
22. $r = \sqrt{a^2 + (-2a)^2} = \sqrt{5a^2} = \sqrt{5}a$
 $\sin \theta = \frac{-2a}{\sqrt{5}a} = \frac{-2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$,
 $\cos \theta = \frac{a}{\sqrt{5}a} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$,
 $\tan \theta = \frac{-2a}{a} = -2$, $\csc \theta = -\frac{\sqrt{5}}{2}$, $\sec \theta = \sqrt{5}$, $\cot \theta = -\frac{1}{2}$



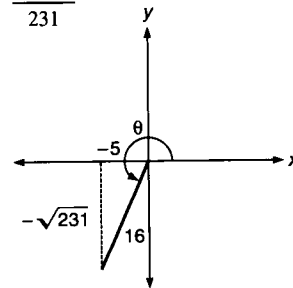
23. III 24. III 25. IV 26. IV
 27. II 28. II 29. III 30. IV
 31. 46.3° 32. $36^\circ 44'$ 33. $61^\circ 48'$
 34. 22.7° 35. 68.7° 36. 74.85°
 37. 62.57° 38. -0.5505 39. $-\frac{1}{2}$
 40. 2.1842 41. 2 42. 1.5818
 43. 0.6720 44. 2 45. -0.6862
 46. 0.9159 47. 161° 48. 141.0°
 49. 212.2° 50. 294.7°
 51. 119.0° 52. 193.7° 53. 120°
 54. 296.6° 55. 112.6°
 56. $\cos \theta = \frac{3}{5}$, $\sec \theta = \frac{5}{3}$, $\csc \theta = \frac{5}{4}$,
 $\tan \theta = \frac{4}{3}$, $\cot \theta = \frac{3}{4}$



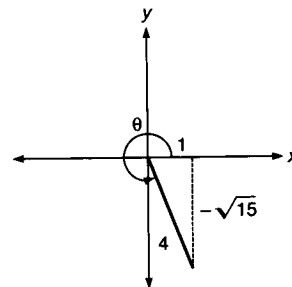
57. $\sec \theta = -2$, $\sin \theta = \frac{-\sqrt{3}}{2}$,
 $\csc \theta = \frac{-2\sqrt{3}}{3}$, $\tan \theta = \sqrt{3}$, $\cot \theta = \frac{\sqrt{3}}{3}$



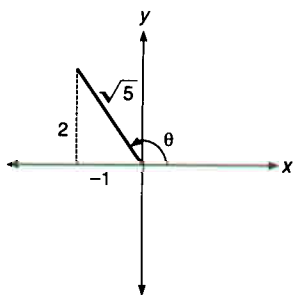
58. $\sec \theta = -\frac{16}{5}$, $\sin \theta = \frac{-\sqrt{231}}{16}$, $\csc \theta = \frac{-16\sqrt{231}}{231}$, $\tan \theta = \frac{\sqrt{231}}{5}$, $\cot \theta = \frac{5\sqrt{231}}{231}$



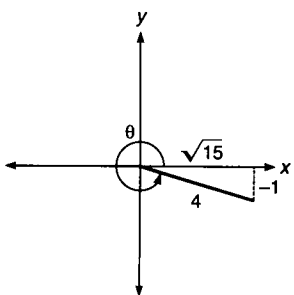
59. $\sec \theta = 4$, $\sin \theta = \frac{-\sqrt{15}}{4}$, $\csc \theta = \frac{-4\sqrt{15}}{15}$, $\tan \theta = -\sqrt{15}$, $\cot \theta = \frac{-\sqrt{15}}{15}$



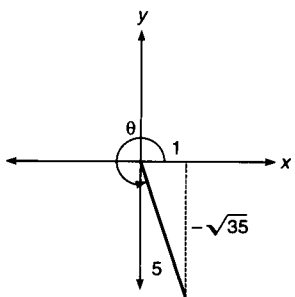
$$60. \cot \theta = -\frac{1}{2}, \sin \theta = \frac{2\sqrt{5}}{5}, \csc \theta = \frac{\sqrt{5}}{2}, \cos \theta = -\frac{\sqrt{5}}{5}, \sec \theta = -\sqrt{5}$$



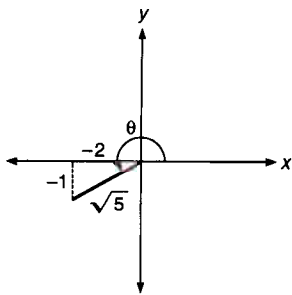
$$61. \cos \theta = \frac{\sqrt{15}}{4}, \sec \theta = \frac{4\sqrt{15}}{15}, \cot \theta = -\sqrt{15}, \tan \theta = \frac{-\sqrt{15}}{15}, \sin \theta = \frac{-1}{4}$$



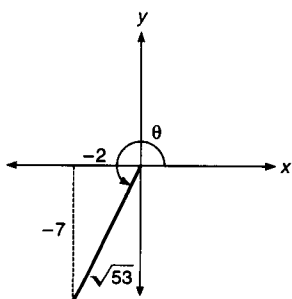
$$62. \cos \theta = \frac{1}{6}, \sin \theta = \frac{-\sqrt{35}}{6}, \csc \theta = \frac{-6\sqrt{35}}{35}, \tan \theta = -\sqrt{35}, \cot \theta = \frac{-\sqrt{35}}{35}$$



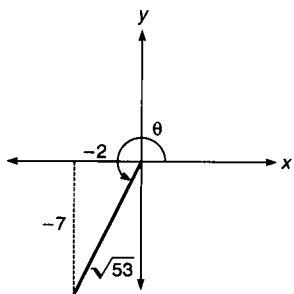
$$63. \tan \theta = \frac{1}{2}, \sin \theta = \frac{-\sqrt{5}}{5}, \cos \theta = \frac{-2\sqrt{5}}{5}, \csc \theta = -\sqrt{5}, \sec \theta = \frac{-\sqrt{5}}{2}$$



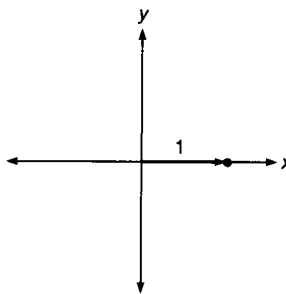
$$64. \cot \theta = \frac{2}{7}, \sin \theta = \frac{-7\sqrt{53}}{53}, \cos \theta = \frac{-2\sqrt{53}}{53}, \csc \theta = \frac{-\sqrt{53}}{7}, \sec \theta = \frac{-\sqrt{53}}{2}$$



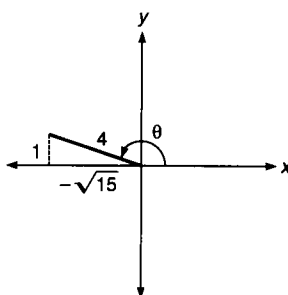
$$65. \tan \theta = \frac{7}{2}, \sin \theta = \frac{-7\sqrt{53}}{53}, \cos \theta = \frac{-2\sqrt{53}}{53}, \csc \theta = \frac{-\sqrt{53}}{7}, \sec \theta = \frac{-\sqrt{53}}{2}$$



$$66. \cot \theta \text{ undefined}, \sin \theta = 0, \cos \theta = 1, \csc \theta \text{ undefined}, \sec \theta = 1$$



$$67. \sin \theta = \frac{1}{4}, \cos \theta = \frac{-\sqrt{15}}{4}, \tan \theta = \frac{-\sqrt{15}}{15}, \sec \theta = \frac{-4\sqrt{15}}{15}, \csc \theta = 4, \cot \theta = -\sqrt{15}$$



$$68. \tan \theta = \frac{-z}{\sqrt{1-z^2}}$$

$$69. \cos \theta = \frac{1}{\sqrt{1+z^2}} \quad 70. \text{ a. 31.06 volts}$$

$$\text{b. 103.92 volts} \quad \text{c. 118.46 volts}$$

$$\text{d. 10.46 volts} \quad \text{e. 0 volts} \quad \text{f. -60 volts}$$

$$71. 337.4 \text{ m} \quad 72. x = -5.2 \text{ mm}, y \approx 7.3 \text{ mm}$$

$$73. x \approx 7.9 \text{ in.}, y \approx 1.4 \text{ in.}$$

$$74. x \approx 5.1 \text{ ft}, y \approx -4.6 \text{ ft}$$

$$75. \frac{2\pi}{3}, 2.09 \quad 76. \frac{-43\pi}{36}, -3.75$$

$$77. \frac{43\pi}{18}, 7.50 \quad 78. 630^\circ \quad 79. 660^\circ$$

$$80. \frac{-900^\circ}{7}, -128.57^\circ \quad 81. \frac{450^\circ}{\pi}, 143.24^\circ$$

$$82. \frac{-756^\circ}{\pi}, -240.64^\circ \quad 83. 12.3 \text{ inches}$$

$$84. 2.3 \quad 85. \frac{25\pi}{4} \text{ mm} \quad 86. 100.8 \text{ inches}$$

$$87. \frac{27\pi}{4} \text{ in.}^2, 21.21 \text{ in.}^2 \quad 88. \frac{128\pi}{3} \text{ mm}^2,$$

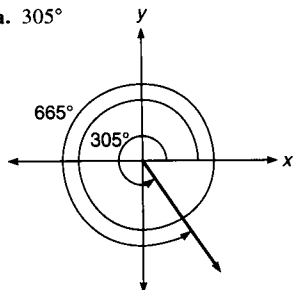
$$134.04 \text{ mm}^2 \quad 89. 16.50 \text{ mm}^2$$

$$90. \frac{49\pi}{5} \text{ in.}^2, 30.79 \text{ in.}^2 \quad 91. 0.9463$$

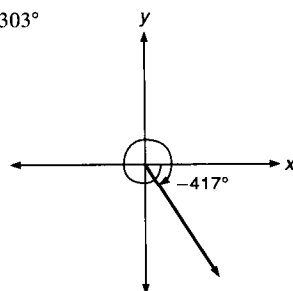
92. -1.3561 93. 4.6373 94. $-\frac{\sqrt{3}}{2}$
 95. -1 96. $-\frac{\sqrt{3}}{2}$ 97. $\frac{\sqrt{3}}{2}$
 98. $-\sqrt{3}$ 99. $-\frac{1}{2}$ 100. $\sqrt{2}$
 101. $\frac{11\pi}{6}$ 102. 2.07 103. 0.43
 104. 0.5 105. $\frac{\pi}{3}$ or 0 106. 0.1074

Chapter 2 test

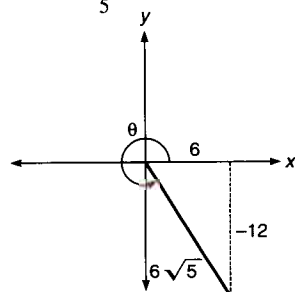
1. a. 305°



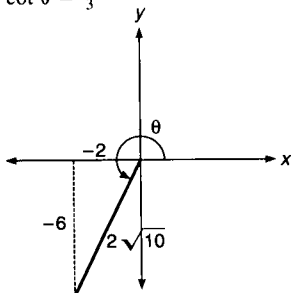
b. 303°



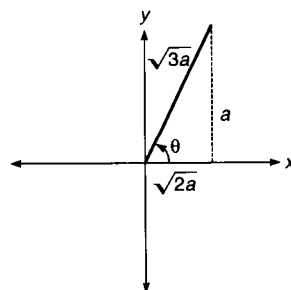
2. $\sin \theta = \frac{-2\sqrt{5}}{5}$, $\sec \theta = \sqrt{5}$,
 $\tan \theta = -2$, $\csc \theta = \frac{-\sqrt{5}}{2}$, $\cot \theta = -\frac{1}{2}$,
 $\cos \theta = \frac{\sqrt{5}}{5}$



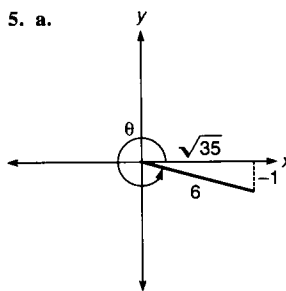
3. $\sin \theta = \frac{-3\sqrt{10}}{10}$,
 $\csc \theta = \frac{-\sqrt{10}}{3}$, $\cos \theta = -\frac{\sqrt{10}}{10}$,
 $\sec \theta = -\sqrt{10}$, $\tan \theta = 3$,
 $\cot \theta = \frac{1}{3}$



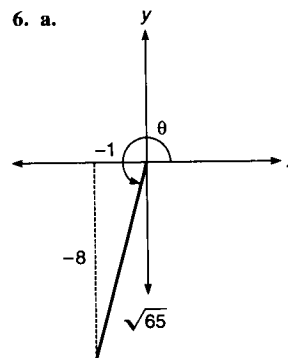
4. $r = \sqrt{(\sqrt{2}a)^2 + a^2} = \sqrt{3a^2} = \sqrt{3}a$
 $\sin \theta = \frac{a}{\sqrt{3}a} = \frac{\sqrt{3}}{3}$, $\cos \theta = \frac{\sqrt{2}a}{\sqrt{3}a} = \frac{\sqrt{6}}{3}$,
 $\tan \theta = \frac{a}{\sqrt{2}a} = \frac{\sqrt{2}}{2}$, $\csc \theta = \sqrt{3}$,
 $\sec \theta = \frac{\sqrt{6}}{2}$, $\cot \theta = \sqrt{2}$



5. a.



b. $\csc \theta = -6$, $\cos \theta = \frac{\sqrt{35}}{6}$,
 $\sec \theta = \frac{6\sqrt{35}}{35}$, $\tan \theta = \frac{-\sqrt{35}}{35}$,
 $\cot \theta = -\sqrt{35}$ c. 350.4°



b. $\cot \theta = \frac{1}{8}$, $\sin \theta = \frac{-8\sqrt{65}}{65}$,
 $\cos \theta = \frac{-\sqrt{65}}{65}$, $\csc \theta = \frac{-\sqrt{65}}{8}$,
 $\sec \theta = -\sqrt{65}$ c. 262.9° 7. 66.2°
 8. 55.6° 9. 0.8957 10. -0.2642
 11. 1.1064 12. 197.2° 13. 119.0°
 14. 5.1 amperes
 15. $x \approx -3.4$ cm, $y \approx -22.3$ cm
 16. a. 108.8° b. 655.1 in.
 17. $\frac{83\pi}{36}$, 7.24 18. 105°
 19. 20.5 inches 20. 14.66 cm
 21. 1.6709 22. $-\frac{\sqrt{3}}{2}$
 23. 134.04 mm² 24. a. yes b. yes
 c. $\{(-3,2), (5,3), (6,5), (12,10)\}$
 25. a. $(-4,33)$ b. $(\sqrt{2}, 7 - 3\sqrt{2})$
 26. 5.94 27. $\frac{4\pi}{3}$ 28. 1.16
 29. 0.38 30. 0 or $\frac{\pi}{6}$ 31. 0.0589

Chapter 3

Exercise 3-1

Answers to odd-numbered problems

1. a. See figure 3-3. b. See figure 3-6.
 c. See figure 3-9. 3. a. $\frac{\pi}{2} + 2k\pi$
 b. $\frac{3\pi}{2} + 2k\pi$ c. $k\pi$ 5. $k\pi$ 7. a. $-\frac{1}{2}$
 b. $\frac{\sqrt{3}}{2}$ c. $-\frac{\sqrt{3}}{3}$ 9. a. $\frac{\sqrt{3}}{2}$ b. $\frac{1}{2}$
 c. $\sqrt{3}$ 11. odd 13. even 15. even
 17. odd 19. even 21. even
 23. odd 25. even

Solutions to trial exercise problems

9. a. $\sin\left(-\frac{5\pi}{3}\right) = -\sin\frac{5\pi}{3}$

Sine is an odd function.

$$\sin\frac{5\pi}{3} = -\sin\frac{\pi}{3}$$

Sine is negative in quadrant IV.

$$= -\frac{\sqrt{3}}{2},$$

$$\text{so } \sin\left(-\frac{5\pi}{3}\right) = -\sin\frac{5\pi}{3}$$

$$= -\left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}.$$

b. $\cos\left(-\frac{5\pi}{3}\right) = \cos\frac{5\pi}{3}$

Cosine is an even function.

$$\cos\frac{5\pi}{3} = \cos\frac{\pi}{3}$$

Cosine is positive in quadrant IV.

$$= \frac{1}{2},$$

$$\text{so } \cos\left(-\frac{5\pi}{3}\right) = \cos\frac{5\pi}{3} = \frac{1}{2}.$$

c. $\tan\left(-\frac{5\pi}{3}\right) = -\tan\frac{5\pi}{3}$

Tangent is an odd function.

$$\tan\frac{5\pi}{3} = -\tan\frac{\pi}{3}$$

Tangent is negative in quadrant IV.

$$= -\sqrt{3},$$

$$\text{so } \tan\left(-\frac{5\pi}{3}\right) = -\tan\frac{5\pi}{3}$$

$$= -(-\sqrt{3}) = \sqrt{3}.$$

17. $f(x) = 3x - 2x^3$

$$f(-x) = 3(-x) - 2(-x)^3$$

$$= -3x + 2x^3$$

$$-f(x) = -(3x - 2x^3)$$

$$= -3x + 2x^3$$

so $f(-x) = -f(x)$; therefore, f is an odd function.

Exercise 3-2

Answers to odd-numbered problems

1. See figures 3-11, 3-12, and 3-14.

3. $f(x) = \csc(x) = \frac{1}{\sin x}$

$$f(-x) = \frac{1}{\sin(-x)}$$

$$= \frac{1}{-\sin x}$$

$$= -\frac{1}{\sin x}$$

$$= -\csc(x)$$

$$= -f(x)$$

5. $f(x) = \cot x = \frac{1}{\tan x}$

$$f(-x) = \cot(-x) = \frac{1}{\tan(-x)}$$

$$= \frac{1}{-\tan(x)}$$

$$= -\frac{1}{\tan x}$$

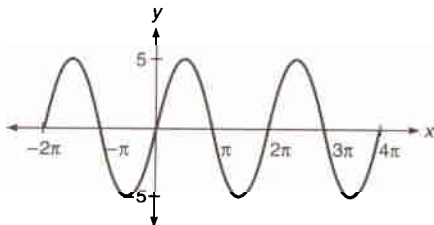
$$= -\cot x$$

$$= -f(x)$$

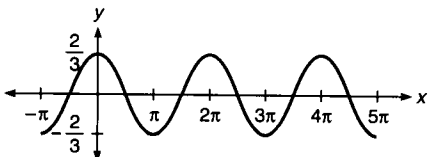
Exercise 3-3

Answers to odd-numbered problems

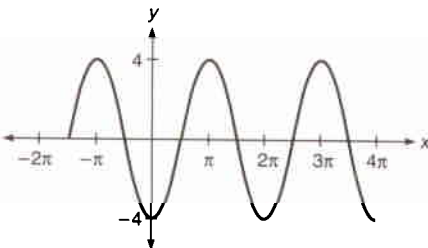
1.



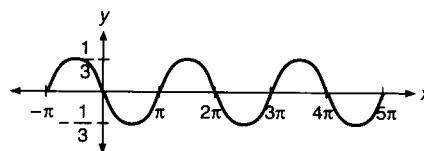
3.



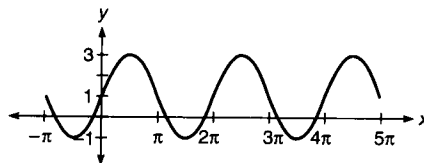
5.



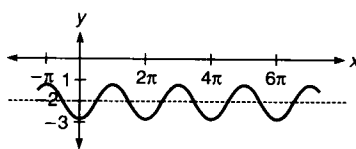
7.



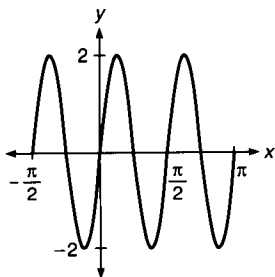
9.



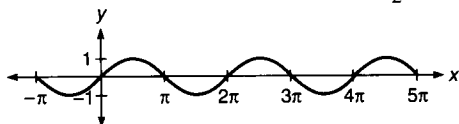
11.



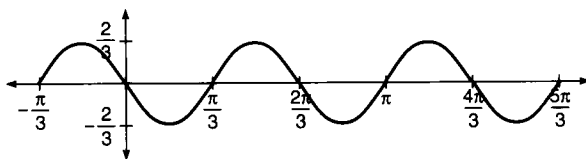
13. amplitude = 2, period = $\frac{\pi}{2}$, phase shift = 0



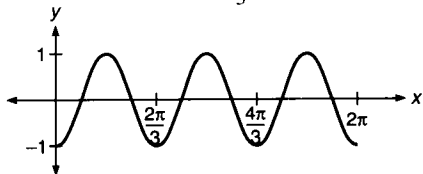
15. amplitude = 1, period = 2π , phase shift = $\frac{\pi}{2}$



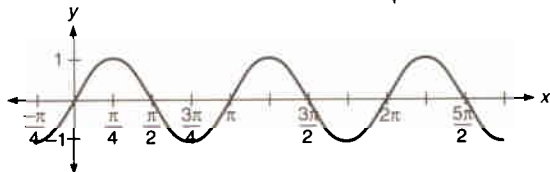
17. amplitude = $\frac{2}{3}$, period = $\frac{2\pi}{3}$, phase shift = $-\frac{\pi}{3}$



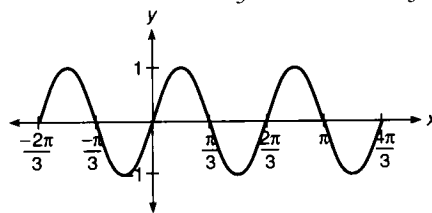
19. amplitude = 1, period = $\frac{2\pi}{3}$, phase shift = 0



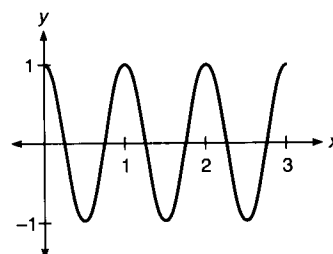
21. amplitude = 1, period = π , phase shift = $-\frac{\pi}{4}$



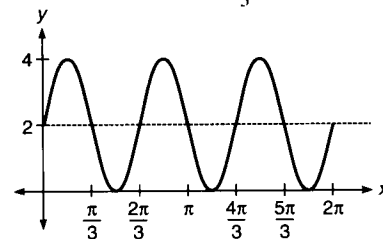
23. amplitude = 1, period = $\frac{2\pi}{3}$, phase shift = $-\frac{2\pi}{3}$



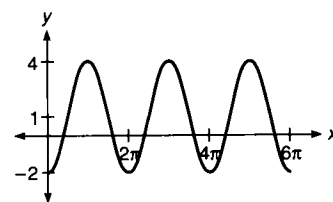
25. amplitude = 1, period = 1, phase shift = 0



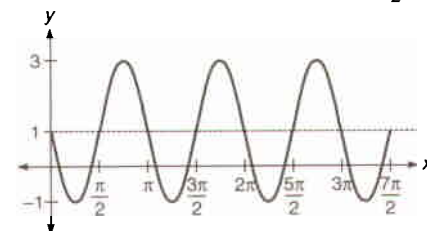
27. amplitude = 2, period = $\frac{2\pi}{3}$, phase shift = 0



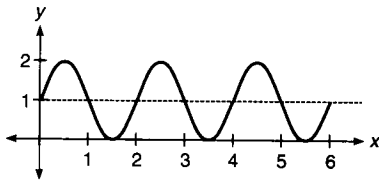
29. amplitude = 3, period = 2π , phase shift = 0



31. amplitude = 2, period = π , phase shift = $\frac{\pi}{2}$



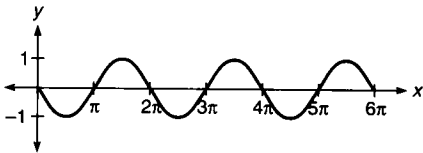
33. amplitude = 1, period = 2, phase shift = 0



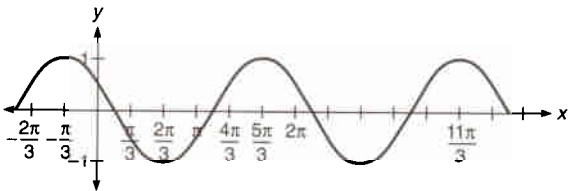
35. $y = -\sin 2x$ 37. $y = -\cos 3x$ 39. $y = -\sin(x + 3)$

41. $y = -\sin x - 3$ 43. $y = -3 \cos\left(2x - \frac{\pi}{2}\right)$

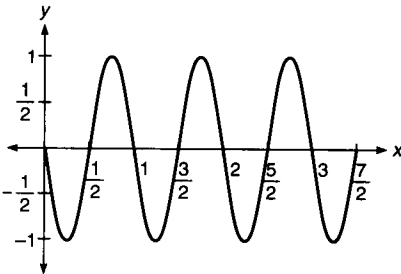
45. $y = -\sin x$



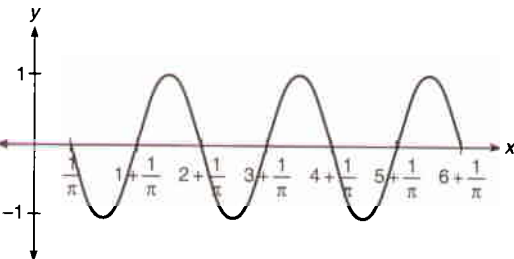
47. $y = \cos\left(x + \frac{\pi}{3}\right)$



49. $y = \sin(2\pi x - \pi)$



51. $y = -\sin(\pi x - 1)$

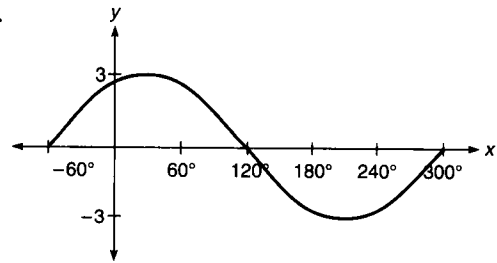


53. $A = 3, B = 4, C = 0, D = 0$ 55. $A = 2, B = \frac{2\pi}{5},$

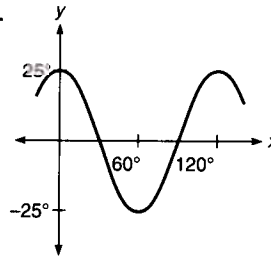
$C = \frac{-2\pi}{5}, D = 0$ 57. $A = 3, B = 4, C = \frac{-\pi}{2}, D = 0$

59. $A = 2, B = \frac{2\pi}{5}, C = \frac{-9\pi}{10}, D = 0$

61.



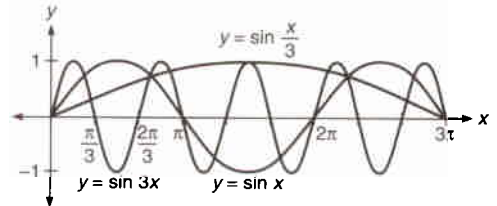
63.



65. $y = 60 \sin\left(\frac{20x}{3} - 600\right)$

67. $y = 0.5 \sin 43x + 23.5$

69.

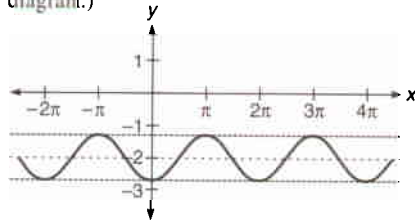


71. Both graphs are the same as figure 3-3, for the sine function.

Solutions to trial exercise problems

11. $y = -\frac{3}{4} \cos x - 2$

Period is 2π because the coefficient of the argument is 1. The amplitude is $\frac{3}{4}$, with a reflection about the horizontal axis. There is a vertical shift of 2 units downward. (See the diagram.)



$$21. y = -\cos\left(2x + \frac{\pi}{2}\right)$$

$$0 \leq 2x + \frac{\pi}{2} \leq 2\pi$$

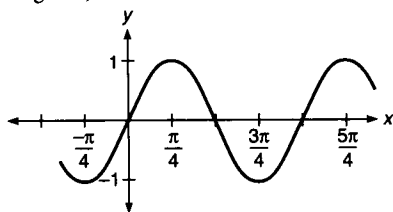
$$\text{Subtract } \frac{\pi}{2}.$$

$$-\frac{\pi}{2} \leq 2x \leq \frac{3\pi}{2}$$

$$\text{Multiply by } \frac{1}{2}.$$

$$-\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$$

Thus we get one complete cycle of the cosine function, reflected about the horizontal axis (because of the negative coefficient), starting at $-\frac{\pi}{4}$ and ending at $\frac{3\pi}{4}$. (See the diagram.)



$$\text{The period is } \frac{3\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{3\pi}{4} + \frac{\pi}{4} = \pi.$$

Phase shift is $-\frac{\pi}{4}$, and amplitude is 1. We mark the x -axis in increments of one half the period, or $\frac{\pi}{2}$, starting at $-\frac{\pi}{4}$.

$$43. y = -3 \cos\left(-2x + \frac{\pi}{2}\right)$$

Since cosine is an even function, we merely change the sign of the argument. $y = -3 \cos\left(2x - \frac{\pi}{2}\right)$.

$$51. y = \sin(-\pi x + 1)$$

Sine is an odd function, so we change both the sign of the coefficient and the sign of the argument. $y = -\sin(\pi x - 1)$.

$$0 \leq \pi x - 1 \leq 2\pi$$

$$\text{Add 1.}$$

$$1 \leq \pi x \leq 2\pi + 1$$

$$\text{Divide by } \pi.$$

$$\frac{1}{\pi} \leq x \leq \frac{2\pi + 1}{\pi}$$

$$\text{Phase shift is } \frac{1}{\pi}, \text{ period is } \frac{2\pi + 1}{\pi} - \frac{1}{\pi} = \frac{2\pi}{\pi} + \frac{1}{\pi} - \frac{1}{\pi} = 2. \text{ Amplitude is 1.}$$

We mark the x -axis in increments of one half the period (1), starting at $\frac{1}{\pi}$. (We also compute decimal approximations as a

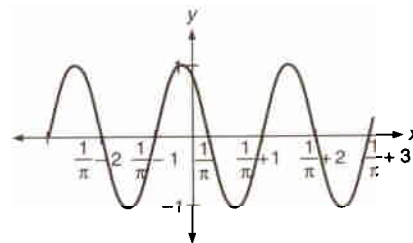
convenience in plotting.) The calculations are as follows:

$$\frac{1}{\pi} + 1 \approx 1.3; \left(\frac{1}{\pi} + 1\right) + 1 = \frac{1}{\pi} + 2 \approx 2.3;$$

$$\left(\frac{1}{\pi} + 2\right) + 1 = \frac{1}{\pi} + 3 \approx 3.3;$$

$$\frac{1}{\pi} + 4 \approx 4.3; \frac{1}{\pi} + 5 \approx 5.3; \left(\frac{1}{\pi} + 1\right) - 1 = \frac{1}{\pi} \approx 0.3;$$

$$\frac{1}{\pi} - 1 \approx -0.7. \quad (\text{See the diagram.})$$



$$56. \text{ Amplitude} = 2. \text{ We have one cycle between } 0 \text{ and } 3\pi.$$

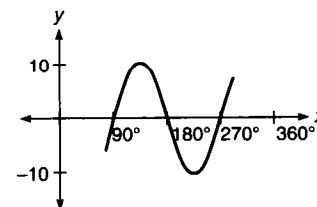
$$0 \leq x \leq 3\pi$$

$$\text{Multiply by } \frac{2}{3}.$$

$$0 \leq \frac{2x}{3} \leq 2\pi$$

$$\text{Vertical shift} = 3, \text{ so the equation is } y = 2 \sin\left(\frac{2x}{3}\right) + 3.$$

$$64. y = 10 \sin(2x - 180^\circ) \quad (\text{See the diagram.})$$



$$0^\circ \leq 2x - 180^\circ \leq 360^\circ$$

$$\text{Add } 180^\circ.$$

$$180^\circ \leq 2x \leq 540^\circ$$

$$\text{Divide by 2.}$$

$$90^\circ \leq x \leq 270^\circ$$

$$67. \text{ Amplitude is } 0.5; 0^\circ \text{ phase shift; period } \left(\frac{360}{43}\right)^\circ; \text{ vertical translation } 23.5.$$

$$0^\circ \leq x \leq \frac{360^\circ}{43}$$

$$0^\circ \leq 43x \leq 360^\circ$$

Thus, the argument is $43x$, and we have for

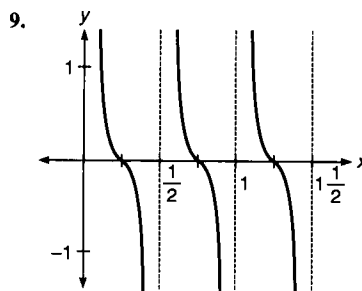
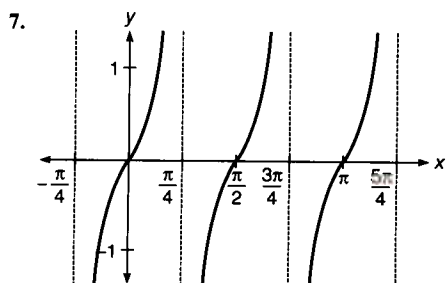
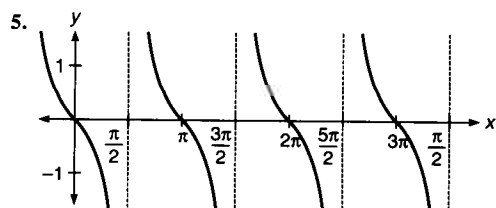
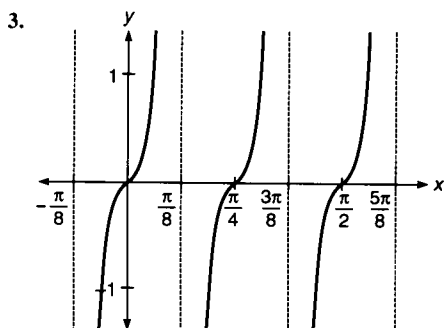
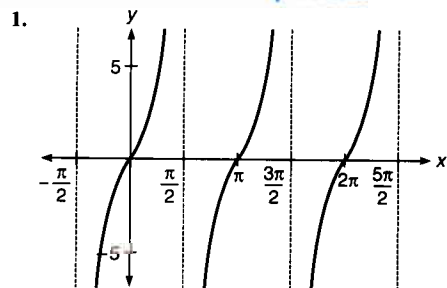
$$y = A \sin(Bx + C) + D,$$

$$A = 0.5, B = 43, C = 0, D = 23.5, \text{ for}$$

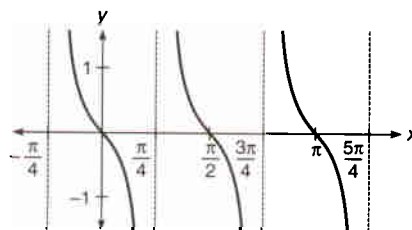
$$y = 0.5 \sin 43x + 23.5$$

Exercise 3–4

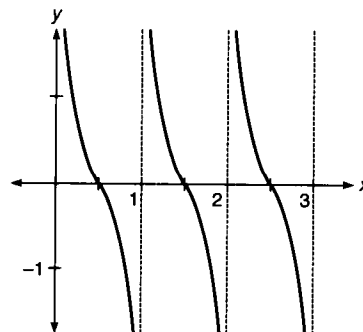
Answers to odd-numbered problems



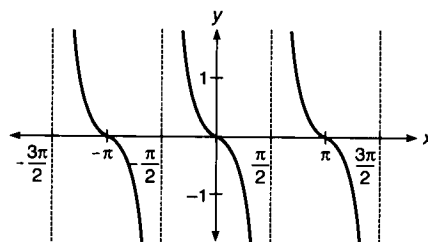
11. $y = -\tan 2x$



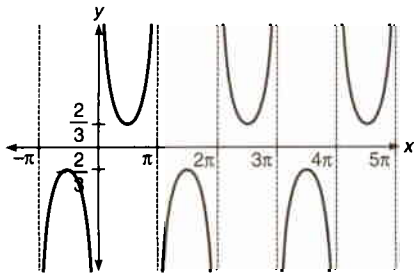
13. $y = \cot \pi x$



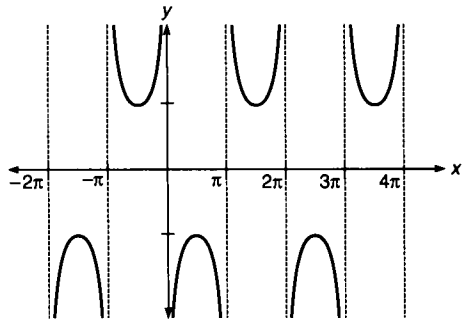
15. $y = -\tan(x + \pi)$



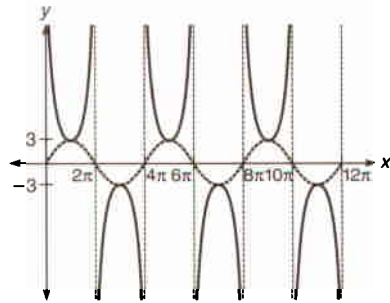
17.



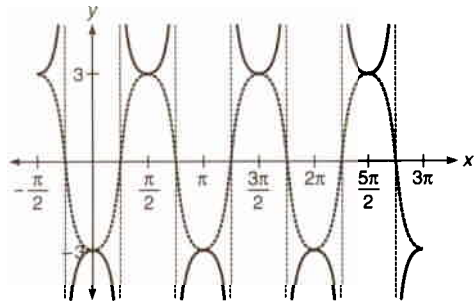
19.



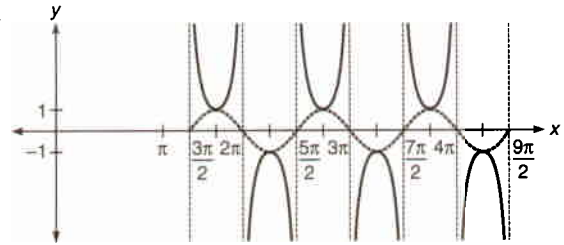
21.



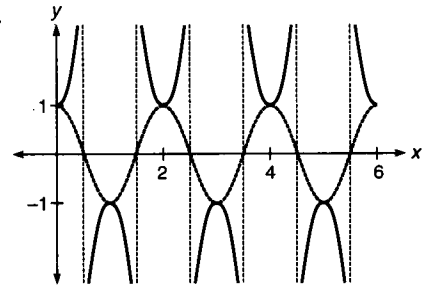
23.



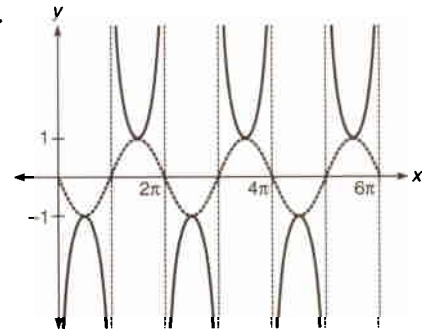
25.



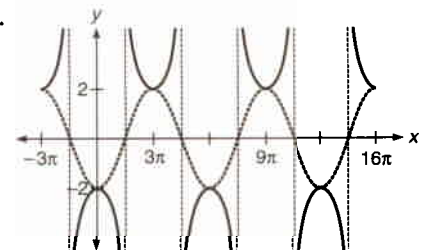
27.



29.



31.



Solutions to trial exercise problems

7. $y = -\cot\left(2x + \frac{\pi}{2}\right)$

There is a reversal about the horizontal axis, because of the negative coefficient. To find the beginning and end points for one cycle,

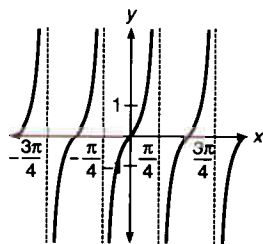
$$0 < 2x + \frac{\pi}{2} < \pi$$

Subtract $\frac{\pi}{2}$.

$$-\frac{\pi}{2} < 2x < \frac{\pi}{2}$$

Multiply by $\frac{1}{2}$.

$$-\frac{\pi}{4} < x < \frac{\pi}{4}$$



Thus, one complete cycle starts at $-\frac{\pi}{4}$ and terminates at $\frac{\pi}{4}$, reversed about the horizontal axis. (See the diagram.)

15. $y = \tan(-x - \pi)$

The tangent function is odd, so we change both the sign of the argument and the coefficient of the function.

$$y = -\tan(x + \pi)$$

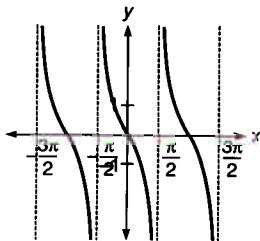
We find the beginning and end points.

$$-\frac{\pi}{2} < x + \pi < \frac{\pi}{2}$$

Subtract π .

$$-1\frac{1}{2}\pi < x < -\frac{1}{2}\pi.$$

See the diagram.



24. $y = \frac{2}{3} \sec(3x + \pi)$

We first graph $y = \frac{2}{3} \cos(3x + \pi)$.

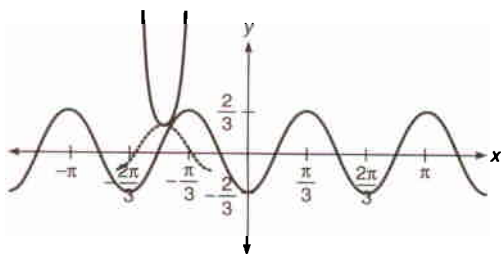
$$0 \leq 3x + \pi \leq 2\pi$$

Subtract π .

$$-\pi \leq 3x \leq \pi$$

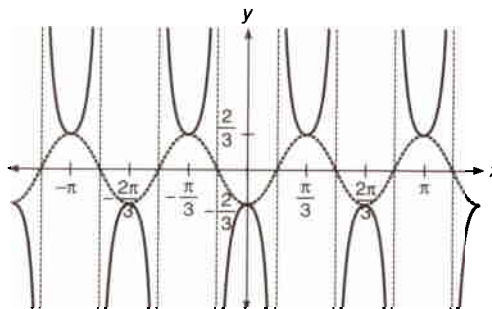
$$-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$$

See the diagram.



Next we construct the graph of the corresponding secant function. This graph touches the cosine graph at its high and

low values and has vertical asymptotes wherever the cosine graph is 0 (crosses the x-axis). (See the diagram.)

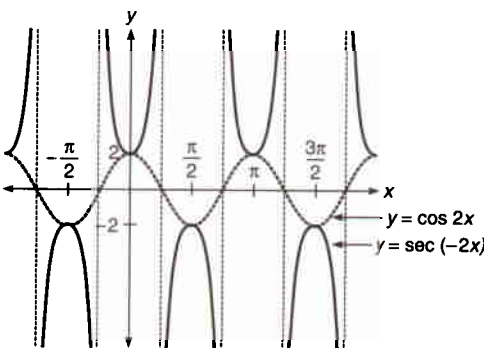


28. $y = \sec(-2x)$

The secant function is even (as is the cosine function), so we simply change the sign of the argument.

$$y = \sec 2x$$

We graph $y = \cos 2x$ first, then use this to construct the secant graph. (See the diagram.)



$$0 \leq 2x \leq 2\pi$$

$$0 \leq x \leq \pi$$

Chapter 3 review

1. See figure 3-3 for the graph; R ; $-1 \leq y \leq 1$; 2π 2. all multiples of 2π : $2k\pi$, k an integer 3. $\frac{\sqrt{3}}{2}$ 4. $-\sqrt{3}$

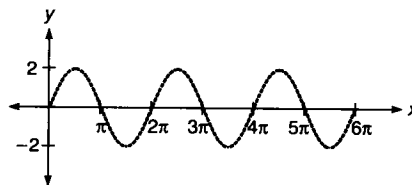
5. $-\frac{1}{2}$ 6. $\sqrt{2}$ 7. odd 8. even 9. odd

10. See figure 3-11. 11. domain: $x \neq k\pi$, k an integer;

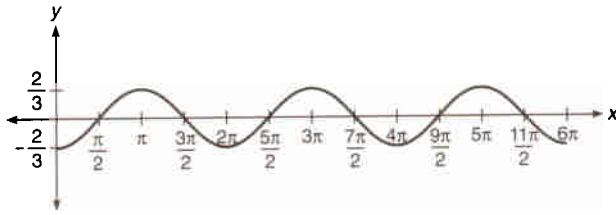
range: R 12. $f(-x) = \frac{-x}{\sec(-x)} = \frac{-x}{\sec x} = -f(x)$

13. $f(-x) = \sec(-x) \cdot [\sin(-x)]^2 + (-x)^4 = \sec x \cdot [-\sin x]^2 + x^4 = \sec x \cdot \sin^2 x + x^4 = f(x)$

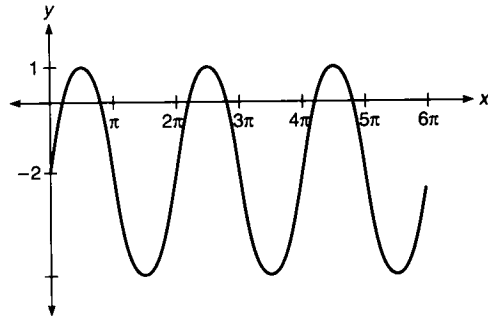
14. amplitude = 2, period = 2π , phase shift = 0



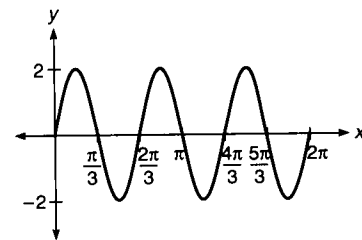
15. amplitude = $\frac{2}{3}$, period = 2π , phase shift = 0



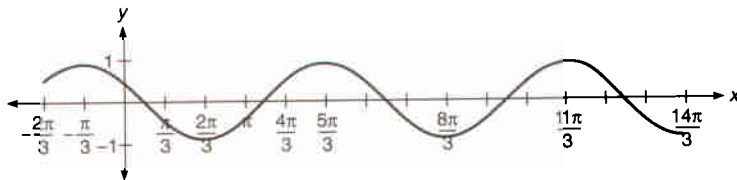
16. amplitude = 3, period = 2π , phase shift = 0



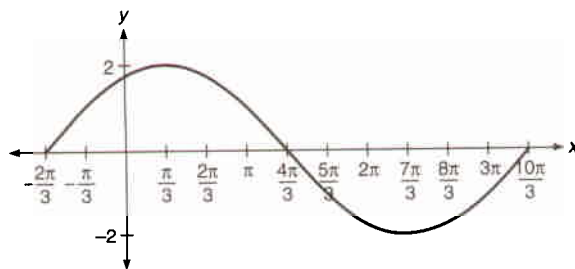
17. amplitude = 2, period = $\frac{2\pi}{3}$, phase shift = 0



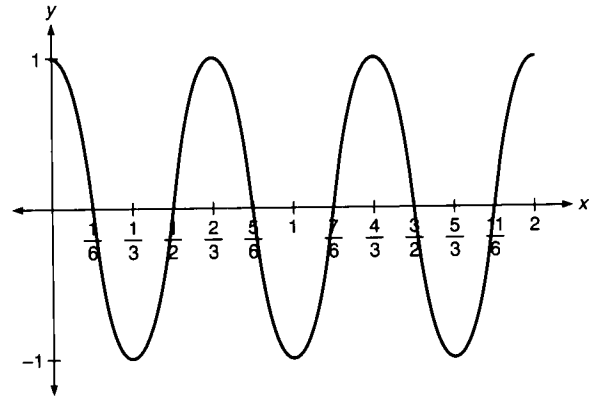
18. amplitude = 1, period = 2π , phase shift = $-\frac{\pi}{3}$



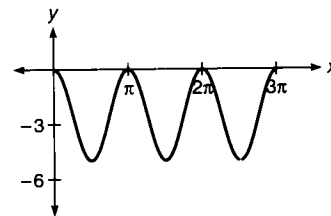
19. amplitude = 2, period = 4π , phase shift = $-\frac{2\pi}{3}$



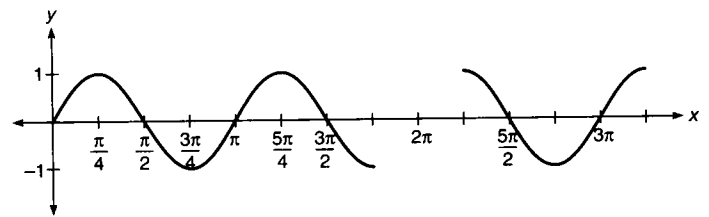
20. amplitude = 1, period = $\frac{2}{3}$, phase shift = 0



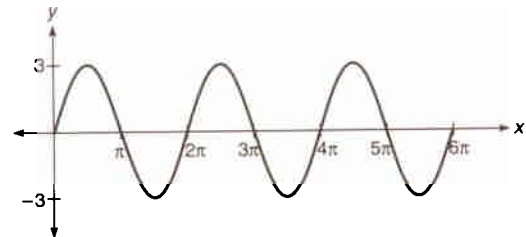
21. amplitude = 3, period = π , phase shift = 0



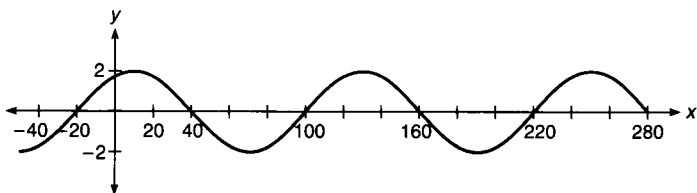
22. $y = \cos\left(2x - \frac{\pi}{2}\right)$



23. $y = -3 \sin(x - \pi)$



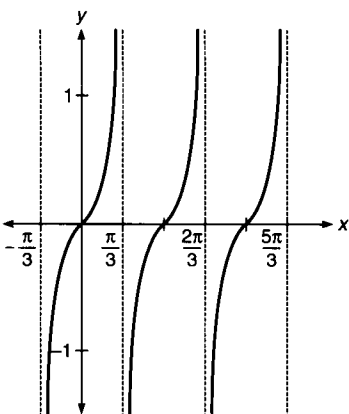
24.


25. amplitude = 2, so $A = 2$; $Bx + C = 2x - \frac{\pi}{3}$, so $B = 2$,

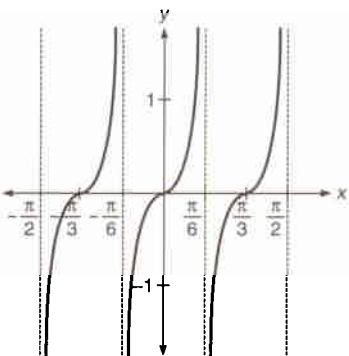
 $C = -\frac{\pi}{3}$, and $D = -1$; $y = 2 \cos\left(2x - \frac{\pi}{3}\right) - 1$

26. $y = \sin\left(\frac{2\pi}{23}x + \frac{20\pi}{23}\right)$

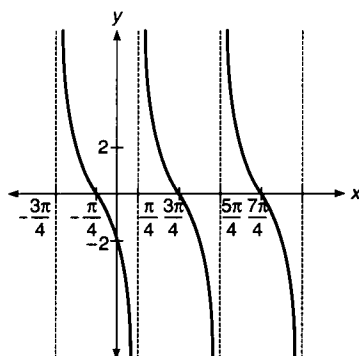
27.



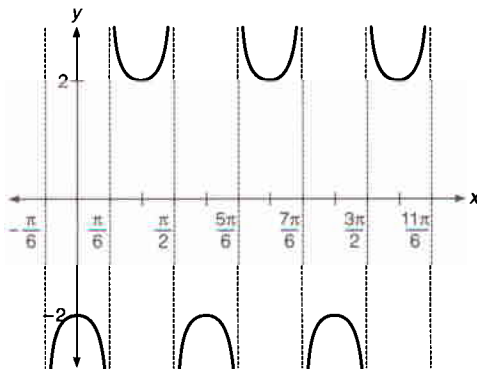
28.



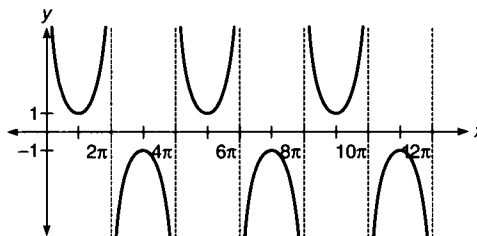
29.



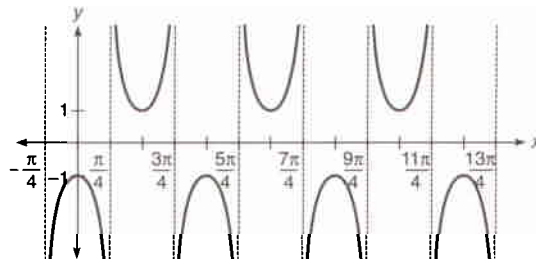
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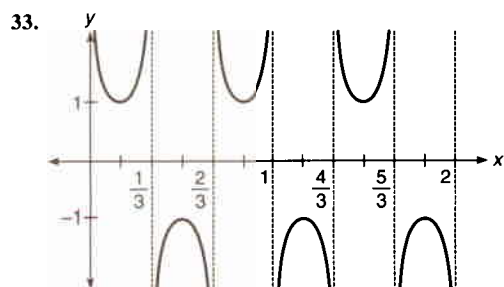


31.



32.

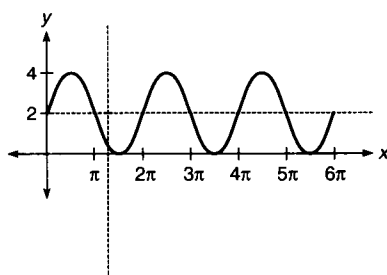




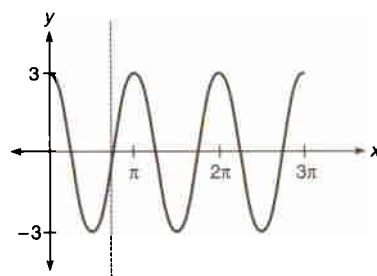
Chapter 3 test

1. $\frac{3\pi}{2} + 2k\pi$, k an integer 2. $\sqrt{3}$ 3. odd

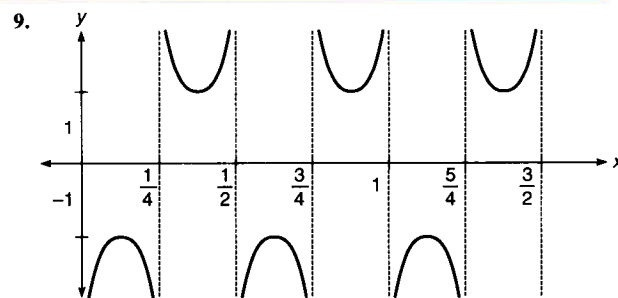
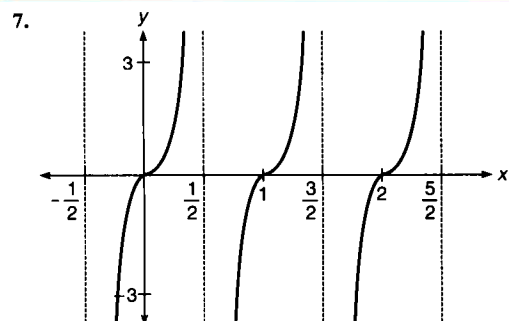
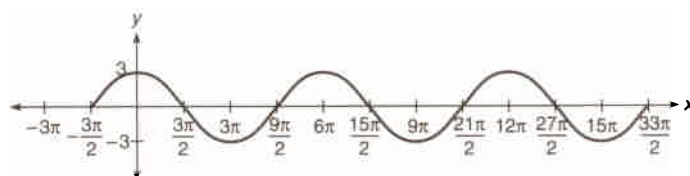
4. amplitude = 2, period = 2π , phase shift = 0



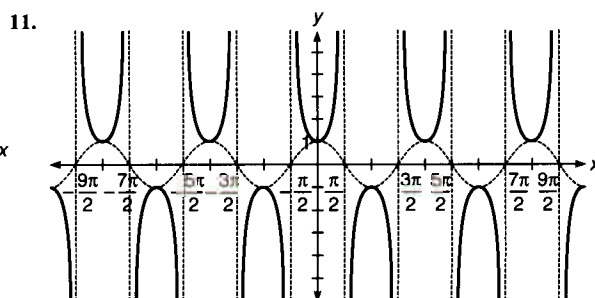
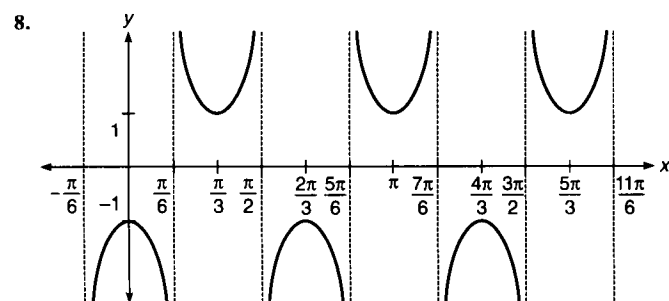
5. amplitude = 3, period = π , phase shift = 0



6. amplitude = 3, period = 6π , phase shift = $-\frac{3\pi}{2}$



10. $A = 2$, $B = 1$, $C = \frac{\pi}{3}$, $D = 0$; $y = 2 \sin\left(x + \frac{\pi}{3}\right)$



12. $f(-x) = \sec(-x) \cdot \sin(-x) + (-x)^3 = \sec x \cdot (-\sin x) - x^3 = -(\sec x \cdot \sin x + x^3) = -f(x)$

$$13. y = \sin\left(\frac{\pi}{14}x - \frac{5\pi}{14}\right)$$

$$14. y = 25 \sin\left(\frac{12}{5}x + 48^\circ\right) + 10$$

Chapter 4

Exercise 4-1

Answers to odd-numbered problems

1. Let $y = f(x) = 2x - 7$. Show $g(y) = x$:

$$\begin{aligned} g(y) &= \frac{1}{2}y + 3\frac{1}{2} \\ &= \frac{1}{2}(2x - 7) + \frac{7}{2} \\ &= x \end{aligned}$$

$$\text{Let } y = g(x) = \frac{1}{2}x + \frac{7}{2}. \text{ Show } f(y) = x:$$

$$f(y) = 2y - 7 = 2\left(\frac{1}{2}x + \frac{7}{2}\right) - 7 = x$$

3. Let $y = f(x) = \frac{1}{3}x + \frac{8}{3}$. Show $g(y) = x$:

$$g(y) = 3y - 8 = 3\left(\frac{1}{3}x + \frac{8}{3}\right) - 8 = x$$

$$\text{Let } y = g(x) = 3x - 8. \text{ Show } f(y) = x:$$

$$f(y) = \frac{1}{3}y + \frac{8}{3} = \frac{1}{3}(3x - 8) + \frac{8}{3} = x$$

5. Let $y = f(x) = 2x - 5$. Show $g(y) = x$:

$$g(y) = \frac{1}{2}(y + 5) = \frac{1}{2}[(2x - 5) + 5] = x$$

$$\text{Let } y = g(x) = \frac{1}{2}(x + 5). \text{ Show } f(y) = x:$$

$$f(y) = 2y - 5 = 2\left[\frac{1}{2}(x + 5)\right] - 5 = x$$

7. Let $y = f(x) = \frac{2}{x - 3}$. Show $g(y) = x$:

$$g(y) = \frac{2}{y} + 3 = \frac{2}{\frac{2}{x - 3}} + 3 = (x - 3) + 3 = x$$

$$\text{Let } y = g(x) = \frac{2}{x} + 3. \text{ Show } f(y) = x:$$

$$f(y) = \frac{2}{y - 3} = \frac{2}{\left(\frac{2}{x} + 3\right) - 3} = \frac{2}{\frac{2}{x}} = x$$

9. Let $y = f(x) = 7 - \frac{3}{x}$. Show $g(y) = x$:

$$g(y) = \frac{3}{7 - y} = \frac{3}{7 - \left(7 - \frac{3}{x}\right)} = \frac{3}{\frac{3}{x}} = x$$

$$\text{Let } y = g(x) = \frac{3}{7 - x}. \text{ Show } f(y) = x:$$

$$f(y) = 7 - \frac{3}{y} = 7 - \frac{3}{\frac{3}{7 - x}} = 7 - (7 - x) = x$$

11. Let $y = f(x) = \frac{x}{x - 1}$. Show $g(y) = x$:

$$g(y) = \frac{y}{y - 1} = \frac{\frac{x}{x - 1}}{\frac{x}{x - 1} - 1}$$

$$= \frac{\frac{x}{x - 1}}{\frac{x}{x - 1} - 1} \cdot \frac{x - 1}{x - 1}$$

$$= \frac{x}{x - (x - 1)} = x$$

$$\text{Let } y = g(x) = \frac{x}{x - 1}. \text{ Show } f(y) = x:$$

$$f(y) = \frac{y}{y - 1} = \frac{\frac{x}{x - 1}}{\frac{x}{x - 1} - 1}$$

$$= \frac{\frac{x}{x - 1}}{\frac{x}{x - 1} - 1} \cdot \frac{x - 1}{x - 1}$$

$$= \frac{x}{x - (x - 1)} = x$$

13. Let $y = f(x) = x^2 - 9$. Show $g(y) = x$:

$$g(y) = \sqrt{y + 9} = \sqrt{(x^2 - 9) + 9}$$

$$= \sqrt{x^2} = x \text{ if } x \geq 0.$$

$$\text{Let } y = g(x) = \sqrt{x + 9}. \text{ Show } f(y) = x:$$

$$f(y) = y^2 - 9 = (\sqrt{x + 9})^2 - 9$$

$$= (x + 9) - 9 = x$$

15. Let $y = f(x) = x^3$. Show $g(y) = x$:

$$g(y) = \sqrt[3]{y} = \sqrt[3]{x^3} = x$$

$$\text{Let } y = g(x) = \sqrt[3]{x}. \text{ Show } f(y) = x:$$

$$f(y) = y^3 = (\sqrt[3]{x})^3 = x$$

17. Let $y = f(x) = x^2 - 2x + 3$. Show $g(y) = x$:

$$g(y) = \sqrt{y - 2} + 1$$

$$= \sqrt{(x^2 - 2x + 3) - 2} + 1$$

$$= \sqrt{x^2 - 2x + 1} + 1$$

$$= \sqrt{(x - 1)^2} + 1 = x - 1 + 1 = x$$

$$\text{Let } y = g(x) = \sqrt{x - 2} + 1. \text{ Show } f(y) = x:$$

$$f(y) = y^2 - 2y + 3$$

$$= (\sqrt{x - 2} + 1)^2 - 2(\sqrt{x - 2} + 1) + 3$$

$$= [(x - 2) + 2\sqrt{x - 2} + 1] - 2\sqrt{x - 2} - 2 + 3$$

$$= x$$

19. Let $y = f(x) = \frac{2x}{x-3}$. Show $g(y) = x$:

$$\begin{aligned} g(y) &= \frac{3y}{y-2} = \frac{3 \cdot \frac{2x}{x-3}}{\frac{2x}{x-3} - 2} \\ &= \frac{3 \cdot \frac{2x}{x-3} \cdot \frac{x-3}{x-3}}{\frac{2x}{x-3} - 2} \\ &= \frac{3(2x)}{2x - 2(x-3)} = \frac{6x}{6} = x \end{aligned}$$

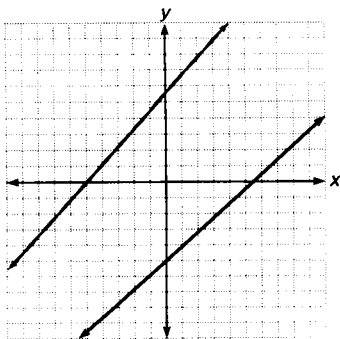
- Let $y = g(x) = \frac{3x}{x-2}$. Show $f(y) = x$:

$$\begin{aligned} f(y) &= \frac{2y}{y-3} = \frac{2 \cdot \frac{3x}{x-2}}{\frac{3x}{x-2} - 3} \\ &= \frac{2 \cdot \frac{3x}{x-2} \cdot \frac{x-2}{x-2}}{\frac{3x}{x-2} - 3} \\ &= \frac{6x}{3x - 3(x-2)} = \frac{6x}{6} = x \end{aligned}$$

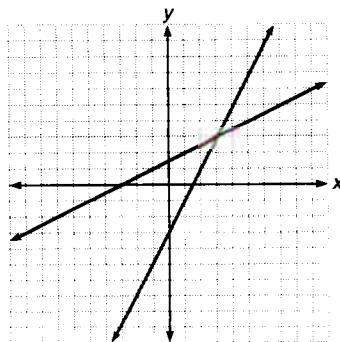
21. function, not one to one

23. not a function

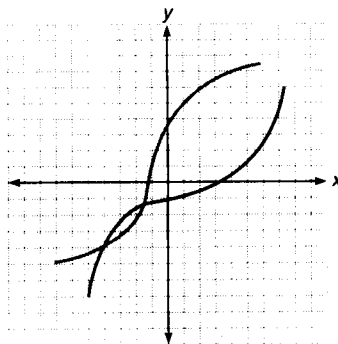
25.



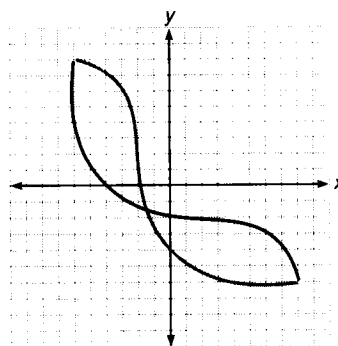
27.



29.



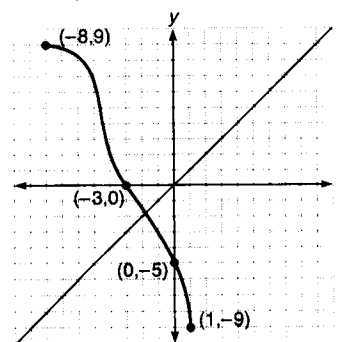
31.



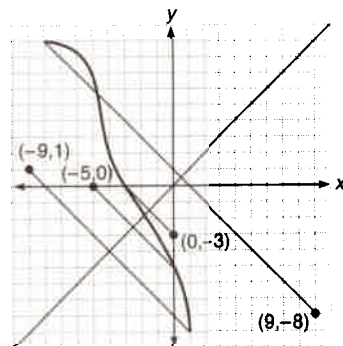
33. $A^{-1}(x) = \frac{x}{4} - 4$ 35. $R^{-1}(x) = \frac{20x}{20-x}$

Solutions to trial exercise problems

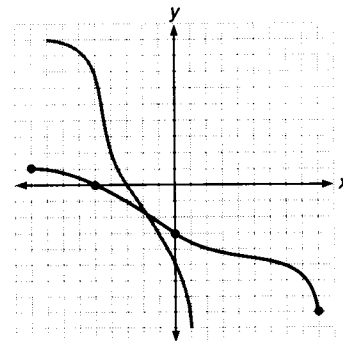
31. We mark four points on the graph. Any points will do, but the end points and intercepts are good choices.



We move these points across the line $y = x$. This is shown in the second figure.



We connect the points and obtain the graph of the inverse function.



34. $d(t) = 16t^2$

$$y = 16t^2$$

$$t = 16y^2$$

$$y^2 = \frac{t}{16}$$

$$y = \pm \sqrt{\frac{t}{16}}$$

$$y = \pm \frac{\sqrt{t}}{4}$$

$y = \frac{\sqrt{t}}{4}$ (choose + among $\pm \frac{\sqrt{t}}{4}$; to see why, consider that the point $(1, 16)$ is in the function d , so the point $(16, 1)$ must be in d^{-1}).

$$d^{-1}(t) = \frac{\sqrt{t}}{4}$$

Exercise 4-2

Answers to odd-numbered problems

1. See figure 4-6. 3. $-\frac{\pi}{6}$, -30°

5. $0, 0^\circ$ 7. $-\frac{\pi}{3}$, -60°

9. 1.08 rad, 61.9° 11. 1.20 rad, 68.8°
13. -1.50 rad, -86.0° 15. -0.26 rad,

-14.9° 17. $\frac{5\sqrt{39}}{39}$ 19. $\frac{3\sqrt{5}}{5}$

21. $\frac{\sqrt{66}}{3}$ 23. $\frac{\sqrt{91}}{10}$ 25. $\sqrt{1-z^2}$

27. $\frac{1+z}{\sqrt{-2z-z^2}}$ 29. $\frac{\sqrt{1-2z}}{1-2z}$ 31. $\frac{\pi}{6}$

33. $-\frac{\pi}{6}$ 35. $\frac{\pi}{2}$ 37. Let f be a

periodic function; then $f(x) = y = f(x + kp) = y$ gives ordered pairs (x, y) , $(x + kp, y)$; second element repeats, so not one to one.

Solutions to trial exercise problems

7. $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$

We want the angle in quadrant I or quadrant IV whose sine value is $-\frac{\sqrt{3}}{2}$.

Since the argument is negative, we know that we want an angle in quadrant IV, where the sine function is negative. Since we have memorized the fact that $\sin 60^\circ = \frac{\sqrt{3}}{2}$, the reference angle is 60° . Therefore, the angle is -60° , or $-\frac{\pi}{3}$ (radians).

(Remember that we use negative values for the inverse sine function in quadrant IV.)

13. $\sin^{-1}(-0.9976)$

Calculator:

The keystrokes are the same for radian and degree mode.

.9976 $\boxed{+/-}$ $\boxed{INV}$ $\boxed{SIN}$

$\boxed{TI-81}$ $\boxed{SIN^{-1}}$ $\boxed{(-)}$.9976

$\boxed{ENTER}$

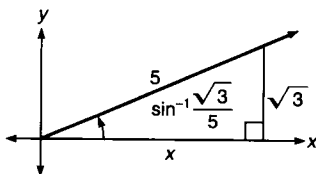
$(= -1.501500431 = -1.50$ to two decimal places)

$(= -86.02963761^\circ = -86.0^\circ$ to one decimal place)

21. $\cot\left(\sin^{-1}\frac{\sqrt{3}}{5}\right)$

We want the cotangent of a first quadrant angle whose sine is $\frac{\sqrt{3}}{5}$.

Such an angle is shown in the diagram. The Pythagorean theorem shows that $x = \sqrt{22}$, so the cotangent of this angle is $\frac{\text{adj}}{\text{opp}} = \frac{\sqrt{22}}{\sqrt{3}} = \frac{\sqrt{66}}{3}$. (Multiply numerator and denominator by $\sqrt{3}$.)



27. $\tan[\sin^{-1}(1+z)]$, $1+z < 0$

The argument of the inverse sine function is negative, so this represents an angle in quadrant IV. The diagram shows a reference triangle in quadrant IV, where the sine of the angle in standard position is $1+z$. Now we use the Pythagorean theorem to find x .

$$1^2 = x^2 + (1+z)^2$$

$$1 = x^2 + 1 + 2z + z^2$$

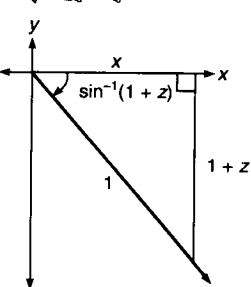
$$-2z - z^2 = x^2$$

$$\sqrt{-2z - z^2} = x$$

Now, the tangent of the angle is

$$\frac{\text{opp}}{\text{adj}} = \frac{1+z}{\sqrt{-2z-z^2}}$$

$$\text{Thus, } \tan[\sin^{-1}(1+z)] = \frac{1+z}{\sqrt{-2z-z^2}}.$$



30. $\cot(\sin^{-1}\sqrt{z-1})$

If $\sqrt{z-1} = 0$, we know $\sin^{-1}0 = 0$, and $\cot 0$ is undefined. Since $\sqrt{z-1} < 0$ is impossible, we assume $\sqrt{z-1} > 0$. In this case, we want a first-quadrant reference triangle. This is shown in the diagram. We find x .

$$x^2 + (\sqrt{z-1})^2 = 1^2$$

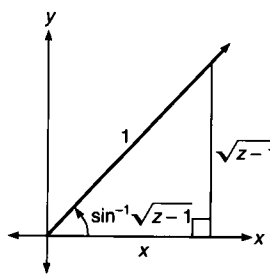
$$x^2 + z - 1 = 1$$

$$x^2 = 2 - z$$

$$x = \sqrt{2-z}$$

The cotangent of this angle is

$$\frac{\text{adj}}{\text{opp}} = \frac{\sqrt{2-z}}{\sqrt{z-1}}.$$



34. $\sin^{-1}\left(\sin\frac{11\pi}{6}\right)$

$$\sin^{-1}\left(\sin\frac{11\pi}{6}\right) = \sin^{-1}\left(-\frac{1}{2}\right)$$

$$= -\frac{\pi}{6} \text{ or } -30^\circ$$

Exercise 4-3

Answers to odd-numbered problems

1. a. See figure 4-6. b. See figure 4-8.

c. See figure 4-10. 3. $\frac{2\pi}{3}$ rad, 120°

5. $\frac{\pi}{2}$ rad, 90° 7. $\frac{\pi}{3}$ rad, 60°

9. $-\frac{\pi}{3}$ rad, -60° 11. $-\frac{\pi}{6}$ rad, -30°

13. 1.08 rad, 61.9° 15. 0.60 rad, 34.4°

17. 0.75 rad, 43.0° 19. 0.92 rad, 52.8°

21. -1.50 rad, -86.0° 23. -0.25 rad, -14.3° 25. 3.00 rad, 172° 27. $\frac{5\sqrt{34}}{34}$

29. $\frac{\sqrt{39}}{8}$ 31. $-\frac{\sqrt{5}}{2}$ 33. $\frac{\sqrt{30}}{6}$

35. $\frac{10\sqrt{109}}{109}$ 37. $\sqrt{1-z^2}$
 39. $\frac{\sqrt{1-z^2}}{z}$ 41. $\frac{1}{\sqrt{1+z^2}}$
 43. $\sqrt{1-9z^2}$ 45. $\sqrt{2+2z+z^2}$
 47. $\frac{1}{\sqrt{1+2z}}$ 49. $\frac{\pi}{4}$
 51. $\frac{\pi}{6}$ 53. 0 55. $\sin^{-1}\frac{m}{r}$
 57. $\tan^{-1}\frac{k}{h}$ 59. $\tan^{-1}\frac{2}{x}$
 61. $\sin^{-1}\frac{3,500}{z}$ 63. $\cos^{-1}(-0.8)$
 65. $\tan^{-1}4.1$ 67. $\tan^{-1}\frac{5}{3}$
 69. $\tan^{-1}50$ 71. $\frac{\tan^{-1}9}{3}$
 73. $\frac{2\sin^{-1}(-0.56)}{3}$ 75. $\frac{\sin^{-1}0.75}{4}$
 77. $\frac{\sin^{-1}\frac{25}{39}}{5}$ 79. $\frac{\tan^{-1}\frac{D}{A} - C}{B}$
 81. $\frac{\sin^{-1}0.6 - 3}{2}$

Solutions to trial exercise problems

11. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

We want the angle in quadrant I or quadrant IV whose tangent is $-\frac{\sqrt{3}}{3}$.

We know that $\tan 30^\circ$ (or $\frac{\pi}{6}$) is $\frac{\sqrt{3}}{3}$.

Thus, the required values are -30° or $-\frac{\pi}{6}$.

23. $\arctan(-0.2553)$

Calculator steps are the same in both degree and radian mode.

.2553 $\boxed{+/-}$ $\boxed{INV}$ $\boxed{TAN}$

$\boxed{TI-81}$ $\boxed{TAN^{-1}}$ $\boxed{(-)}$.2553

$\boxed{ENTER}$

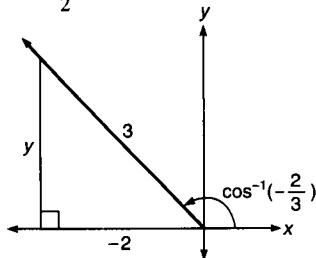
(= -14.32168996°)

(= -0.249960644)

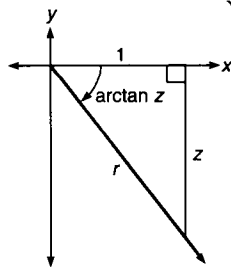
Thus, rounded, $\arctan(-0.2553)$

= -14.3° or -0.25 (radians).

31. $\tan[\cos^{-1}(-\frac{2}{3})]$
 $\cos^{-1}(-\frac{2}{3})$ is an angle in quadrant II whose cosine is $-\frac{2}{3}$. Such an angle is shown in the reference triangle in the diagram. The Pythagorean theorem shows that $y = \sqrt{5}$, and we can then determine that the tangent is $\frac{\text{opp}}{\text{adj}}$
 $= -\frac{\sqrt{5}}{2}$.



41. $\cos(\arctan z)$, $z < 0$
 $\arctan z$ is an angle in quadrant IV if $z < 0$. A reference triangle is shown in the diagram. To find r : $r^2 = 1^2 + z^2$; $r = \sqrt{1+z^2}$ (We know that $r > 0$.) Thus, using the reference triangle,
 $\cos(\arctan z)$, $z < 0$, $= \frac{1}{\sqrt{1+z^2}}$.

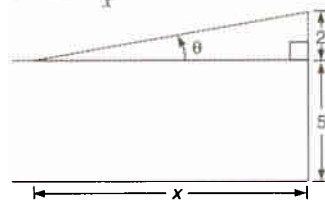


51. $\cos^{-1}\left(\cos\frac{11\pi}{6}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

59. Refer to the diagram.

$\tan \theta = \frac{2}{x}$, so

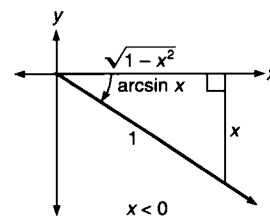
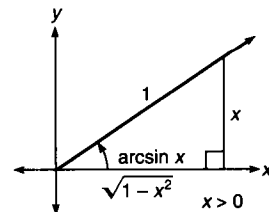
$\theta = \tan^{-1}\frac{2}{x}$.



77. $\frac{6 \sin 5\theta}{5} = \frac{10}{13}$
 $\frac{6 \sin 5\theta}{5} = \frac{10}{13}$
 Multiply by 5.
 $6 \sin 5\theta = \frac{50}{13}$
 Multiply by $\frac{1}{6}$.
 $\sin 5\theta = \frac{50}{78}$
 Reduce.
 $\sin 5\theta = \frac{25}{39}$
 $5\theta = \sin^{-1}\frac{25}{39}$
 Divide by 5.
 $\theta = \frac{\sin^{-1}\frac{25}{39}}{5}$

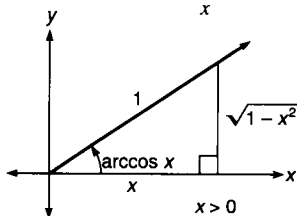
81. $\sin(2x + 3) = 0.6$
 $\sin(2x + 3) = 0.6$
 $2x + 3 = \sin^{-1}0.6$
 Subtract 3.
 $2x = \sin^{-1}0.6 - 3$
 Divide by 2.
 $x = \frac{\sin^{-1}0.6 - 3}{2}$

82. a. The two reference triangles in the first diagram show $\arcsin x$ when $x > 0$ and $x < 0$. In each case we can see that the tangent of this angle is $\frac{x}{\sqrt{1-x^2}}$.



b. In the second diagram we see the reference triangle for $\arccos x$, $x > 0$. It is easy to see that the tangent of the

angle $\arccos x$ is $\frac{\sqrt{1-x^2}}{x}$.

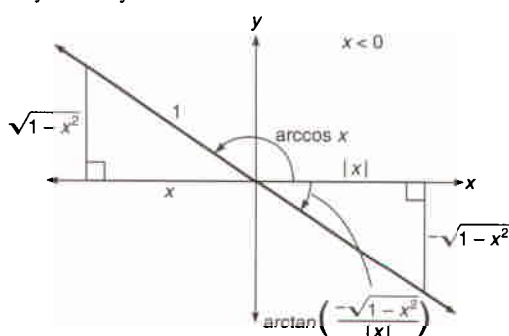


In the third diagram we see both $\arctan \frac{\sqrt{1-x^2}}{x}$, $x < 0$, and $\arccos x$,

$x < 0$. Note that $\frac{\sqrt{1-x^2}}{x}$, $x < 0 =$

$\frac{-\sqrt{1-x^2}}{|x|}$, $x < 0$. We can see that

they differ by π radians.



Exercise 4-4

Answers to odd-numbered problems

1. $\frac{\pi}{6}$ rad, 30° 3. $\frac{\pi}{4}$ rad, 45°
5. $\frac{2\pi}{3}$ rad, 120° 7. $\frac{\pi}{3}$ rad, 60°
9. $\frac{\pi}{2}$ rad, 90° 11. 0.30 rad, 17.3°
13. 1.91, 109.5° 15. 0.19 rad, 10.9°
17. 1.66 rad, 95.2° 19. 0.32 rad, 18.3°
21. $\frac{1}{3}$ 23. $\frac{\sqrt{15}}{15}$ 25. $\sqrt{26}$
27. $\frac{3}{5}$ 29. $-\frac{\sqrt{11}}{5}$ 31. $\frac{1}{z}$
33. $\frac{\sqrt{z^2+1}}{z}$ 35. $\frac{1}{2z}$ 37. $\sqrt{z^2+2z}$
39. $\frac{3}{z}$

Solutions to trial exercise problems

7. $\operatorname{arccsc} \frac{2\sqrt{3}}{3}$

By definition, $\operatorname{arccsc} \frac{2\sqrt{3}}{3}$

$$= \arcsin \frac{1}{2\sqrt{3}}$$

$$= \arcsin \left(\frac{3}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \right)$$

$$= \arcsin \frac{3\sqrt{3}}{2 \cdot 3}$$

$$= \arcsin \frac{\sqrt{3}}{2}$$

$$= 60^\circ \text{ or } \frac{\pi}{3}$$

17. $\sec^{-1}(-11.1261)$

$$\sec^{-1}(-11.1261) = \cos^{-1} \left(\frac{1}{-11.1261} \right)$$

by definition.

11.1261

11.1261

(= 95.15663191 in degree mode)

(= 1.660796532 in radian mode)

so $\sec^{-1}(-11.1261) = 95.2^\circ$ or 1.66 (radians).

27. $\cos[\operatorname{arccsc}(-\frac{5}{4})]$

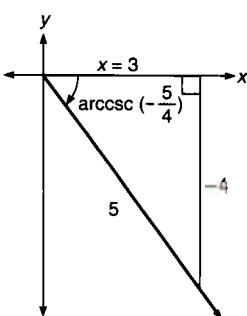
$$\cos[\operatorname{arccsc}(-\frac{5}{4})] = \cos[\arcsin(-\frac{4}{5})]$$

by definition.

$\arcsin(-\frac{4}{5})$ is shown in the diagram.

The cosine of this angle is $\frac{3}{5}$. Thus,

$$\cos[\operatorname{arccsc}(-\frac{5}{4})] = \frac{3}{5}.$$



40. $\sec \left(\cot^{-1} \frac{2}{z+1} \right)$, $z+1 > 0$

$\cot^{-1} \frac{2}{z+1}$, $z+1 > 0$, is an angle in

quadrant I; a reference triangle is shown in the diagram. We can find r .

$$r^2 = 2^2 + (z+1)^2$$

$$r^2 = z^2 + 2z + 5$$

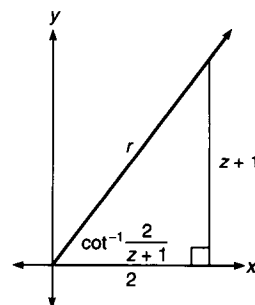
$$r = \sqrt{z^2 + 2z + 5}$$

The cosine of the angle is

$$\frac{2}{\sqrt{z^2 + 2z + 5}}, \text{ and so the secant,}$$

which is the reciprocal of the cosine,

$$\text{is } \frac{\sqrt{z^2 + 2z + 5}}{2}.$$



Chapter 4 review

1. Let $y = f(x) = 3x - 5$; show that $g(y) = x$.

$$g(y) = \frac{y+5}{3} = \frac{(3x-5)+5}{3} = x$$

Let $y = g(x) = \frac{x+5}{3}$; show that $f(y) = x$.

$$f(y) = 3y - 5 = 3\left(\frac{x+5}{3}\right) - 5 = x$$

2. Let $y = f(x) = \frac{3x}{x+1}$; show that $g(y) = x$.

$$g(y) = \frac{-y}{y-3} = \frac{-\frac{3x}{x+1}}{\frac{3x}{x+1} - 3}$$

$$= \frac{-\frac{3x}{x+1} \cdot \frac{x+1}{x+1}}{\frac{3x}{x+1} - 3} = \frac{-3x}{3x - 3(x+1)} = x$$

Let $y = g(x) = \frac{-x}{x-3}$; show that $f(y) = x$.

$$f(y) = \frac{3y}{y+1} = \frac{3 \cdot \frac{-x}{x-3}}{\frac{-x}{x-3} + 1} = \frac{3 \cdot \frac{-x}{x-3} \cdot \frac{x-3}{x-3}}{\frac{-x}{x-3} + 1} = \frac{-3x}{-x+1(x-3)} = x$$

3. no 4. yes 5. See figure 4–6.

6. $\text{domain}_{\sin^{-1}} 1 \leq x \leq 1$; $\text{range}_{\sin^{-1}} -\frac{\pi}{2} \leq$

$y \leq \frac{\pi}{2}$ 7. $-\frac{\pi}{6}$ rad, -30°

8. $\frac{\pi}{3}$ rad, 60° 9. $\frac{\pi}{6}$ rad, 30°

10. $-\frac{\pi}{4}$ rad, -45° 11. 0 rad, 0°

12. $-\frac{\pi}{2}$ rad, -90° 13. 1.34 rad, 76.8°

14. -0.51 rad, -29.2° 15. $\frac{4}{5}$

16. $\frac{\sqrt{15}}{15}$ 17. $\frac{3\sqrt{5}}{5}$ 18. $\frac{z\sqrt{1-z^2}}{1-z^2}$

19. $\sqrt{-z^2-2z}$ 20. See figure 4–8.

21. $\text{domain}_{\tan^{-1}} R$; $\text{range}_{\tan^{-1}} -\frac{\pi}{2} < y < \frac{\pi}{2}$

22. $\frac{2\pi}{3}$ rad, 120°

23. $\frac{\pi}{6}$ rad, 30° 24. $\frac{\pi}{3}$ rad, 60°

25. $-\frac{\pi}{4}$ rad, -45° 26. $\frac{\pi}{3}$ rad, 60°

27. $-\frac{\pi}{4}$ rad, -45° 28. 1.00 rad, 57.3°

29. 1.06 rad, 60.8° 30. 1.88 rad, 107.8°

31. $\frac{5\sqrt{34}}{34}$ 32. 4 33. $-\frac{\sqrt{5}}{2}$

34. $\frac{2\pi}{3}$ rad, 120° 35. $\frac{\sqrt{1-z^2}}{z}$

36. $\frac{1}{\sqrt{z^2+2z+2}}$ 37. $\sin^{-1} \frac{j}{k}$

38. $\sin^{-1} \frac{7}{11}$ 39. $\tan^{-1} \frac{35}{x}$

40. $\cos^{-1} 0.89$ 41. $\sin^{-1}(-0.88)$

42. $\sin^{-1}(-0.2)$ 43. $\tan^{-1} 5$

44. $\frac{\sin^{-1} 0.76}{2}$ 45. $\frac{\cos^{-1} 0.7}{3}$

46. $\frac{\cos^{-1}(0.6) - 3}{2}$ 47. $\frac{\sin^{-1} 0.6}{2}$

48. $\frac{\pi}{3}$ rad, 60° 49. $-\frac{\pi}{3}$ rad, -60°

50. $-\frac{\pi}{6}$ rad, -30° 51. $\frac{5\pi}{6}$ rad, 150°

52. 0.23 rad, 13.2° 53. 0.57 rad, 32.7°

54. 1.82 rad, 104.3° 55. -0.15 rad,

-8.6° 56. $\frac{\sqrt{2}}{4}$ 57. $\frac{\sqrt{15}}{15}$ 58. $\frac{\sqrt{26}}{26}$

59. $\frac{7\sqrt{33}}{33}$ 60. $\frac{\sqrt{z^2+1}}{z^2+1}$ 61. $\sqrt{z^2-1}$

62. $\sqrt{z^2+2z}$ 63. $\frac{1-z}{\sqrt{-2z+z^2}}$

Chapter 4 test

1. See figure 4–10. 2. $\text{domain}_{\csc^{-1}} |x|$

≥ 1 ; $\text{range}_{\csc^{-1}} -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$

3. $\frac{2\pi}{3}$ rad, 120° 4. $\frac{\pi}{4}$ rad, 45°

5. $\frac{\pi}{6}$ rad, 30° 6. $\frac{\pi}{3}$ rad, 60°

7. 0.97 rad, 55.6° 8. 2.00 rad, 114.6°

9. 2.31 rad, 132.4° 10. 0.29 rad, 16.6°

11. $\frac{4}{3}$ 12. $\frac{-4\sqrt{17}}{17}$ 13. $-2\sqrt{2}$

14. $\frac{1}{z}$ 15. $\frac{1}{\sqrt{z^2+2z+2}}$

16. $\sqrt{4z^2-1}$ 17. $\frac{1-z}{\sqrt{z^2-2z}}$

18. $\sin^{-1} \frac{x}{z}$ 19. $\sin^{-1} \frac{52}{x}$

20. $\sec^{-1} 2.25$ 21. $\sin^{-1}(-0.94)$

22. $\frac{\sin^{-1} 0.75}{3}$ 23. $\frac{3 + \cos^{-1} 0.6}{2}$

Chapter 5

Exercise 5–0

Solutions to all problems

- $\sin \theta (\sin \theta - 1)$
- $\cos \theta (\cos^2 \theta + 3)$
- $\cos^2 \theta (\cos^2 \theta - 1)$
 $\cos^2 \theta (\cos \theta - 1)(\cos \theta + 1)$
- $\sin^3 \theta (\sin^2 \theta - 1)$
 $\sin^3 \theta (\sin \theta - 1)(\sin \theta + 1)$
- $(\cos x - 4)(\cos x + 5)$
- $(\tan x + 6)(\tan x - 4)$
- $(2 \sin x - 1)(\sin x - 3)$
- $3 \cos \theta (3 \cos^2 \theta - 5 \cos \theta - 2)$
 $3 \cos \theta (3 \cos \theta + 1)(\cos \theta - 2)$
- $(2 \csc \theta - 1)(3 \csc \theta - 1)$
- $\tan \theta (\tan \theta - 1) = 0$
 $\tan \theta = 0$ or $\tan \theta - 1 = 0$
 $\tan \theta = 0$ or $\tan \theta = 1$
- $\tan^2 \theta - \tan \theta - 2 = 0$
 $(\tan \theta - 2)(\tan \theta + 1) = 0$
 $\tan \theta - 2 = 0$ or $\tan \theta + 1 = 0$
 $\tan \theta = 2$ or $\tan \theta = -1$
- $(2 \sin \theta + 1)(3 \sin \theta + 1) = 0$
 $2 \sin \theta + 1 = 0$ or $3 \sin \theta + 1 = 0$
 $2 \sin \theta = -1$ or $3 \sin \theta = -1$
 $\sin \theta = -\frac{1}{2}$ or $\sin \theta = -\frac{1}{3}$
- $(\sec^2 \theta - 4)(\sec^2 \theta - 1) = 0$
 $(\sec \theta - 2)(\sec \theta + 2)(\sec \theta - 1)$
 $(\sec \theta + 1) = 0$
 $\sec \theta - 2 = 0$ or $\sec \theta + 2 = 0$ or
 $\sec \theta - 1 = 0$ or $\sec \theta + 1 = 0$
 $\sec \theta = \pm 2$ or ± 1
- $(4 \sin^2 \theta - 1)(9 \sin^2 \theta - 1) = 0$
 $(2 \sin \theta - 1)(2 \sin \theta + 1)$
 $(3 \sin \theta - 1)(3 \sin \theta + 1) = 0$
 $2 \sin \theta - 1 = 0$ or $2 \sin \theta + 1 = 0$ or
 $3 \sin \theta - 1 = 0$ or $3 \sin \theta + 1 = 0$
 $2 \sin \theta = 1$ or $2 \sin \theta = -1$ or $3 \sin \theta$
 $= 1$ or $3 \sin \theta = -1$
 $\sin \theta = \pm \frac{1}{2}$ or $\pm \frac{1}{3}$

$$\begin{aligned}
 15. \quad & 3 \sin \theta \left(\frac{2}{3} \sin \theta \right) + 3 \sin \theta \left(\frac{1}{3 \sin \theta} \right) \\
 &= 1(3 \sin \theta) \\
 &2 \sin^2 \theta + 1 = 3 \sin \theta \\
 &2 \sin^2 \theta - 3 \sin \theta + 1 = 0 \\
 &(\sin \theta - 1)(2 \sin \theta - 1) = 0 \\
 &\sin \theta - 1 = 0 \text{ or } 2 \sin \theta - 1 = 0 \\
 &\sin \theta = 1 \text{ or } 2 \sin \theta = 1 \\
 &\sin \theta = 1 \text{ or } \frac{1}{2} \\
 16. \quad & \sin \theta = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-5)}}{2(1)} \\
 &\sin \theta = \frac{3 \pm \sqrt{29}}{2} \\
 17. \quad & \sec \theta = \frac{-3 \pm \sqrt{3^2 - 4(2)(-7)}}{2(2)} \\
 &\sec \theta = \frac{-3 \pm \sqrt{65}}{4} \\
 18. \quad & u = 2x - 6 \\
 &u^3 - u^2 = 0 \\
 &u^2(u - 1) = 0 \\
 &u^2 = 0 \text{ or } u - 1 = 0 \\
 &u = 0 \text{ or } u = 1 \\
 &2x - 6 = 0 \text{ or } 2x - 6 = 1 \\
 &2x = 6 \text{ or } 2x = 7 \\
 &x = 3 \text{ or } x = \frac{7}{2}
 \end{aligned}$$

$$\begin{aligned}
 19. \quad & u = \frac{x}{3} - 1 \\
 &12u^2 - 5u - 2 = 0 \\
 &(3u - 2)(4u + 1) = 0 \\
 &3u - 2 = 0 \text{ or } 4u + 1 = 0 \\
 &u = \frac{2}{3} \text{ or } u = -\frac{1}{4} \\
 &\frac{x}{3} - 1 = \frac{2}{3} \text{ or } \frac{x}{3} - 1 = -\frac{1}{4} \\
 &x - 3 = 2 \text{ or } x - 3 = -\frac{3}{4} \\
 &x = 5 \text{ or } x = 2\frac{1}{4} \\
 20. \quad & 5\left(\frac{\pi}{2} - 3\right) + 3\left[\left(\frac{\pi}{2} - 3\right) - 7\right] \\
 &- \left(\frac{\pi}{2} - 3\right) \\
 &u = \frac{\pi}{2} - 3 \\
 &5u + 3(u - 7) - u \\
 &5u + 3u - 21 - u \\
 &7u - 21 \\
 &7\left(\frac{\pi}{2} - 3\right) - 21 \\
 &\frac{7\pi}{2} - 21 - 21 \\
 &\frac{7\pi}{2} - 42
 \end{aligned}$$

$$\begin{aligned}
 21. \quad & u = 3x - \frac{1}{2} \\
 & \frac{2u^2 - 3u + 1}{u^2 - 1} \\
 & \frac{(u - 1)(2u - 1)}{(u - 1)(u + 1)} \\
 & \frac{2u - 1}{u + 1} \\
 & \frac{2(3x - \frac{1}{2}) - 1}{3x - \frac{1}{2} + 1} \\
 & \frac{6x - 1 - 1}{3x + \frac{1}{2}} \\
 & \frac{6x - 2}{3x + \frac{1}{2}} \cdot \frac{2}{2} \\
 & \frac{12x - 4}{6x + 1}
 \end{aligned}$$

Exercise 5-1

Answers to odd-numbered problems

$$\begin{aligned}
 1. \quad & \frac{\sin \theta}{\sin \theta} \cdot \frac{1}{\sin \theta} \\
 & \frac{\sin \theta \cos \theta}{\sin \theta} \\
 & \cos \theta \\
 5. \quad & \frac{\cos^2 \theta}{\sin^2 \theta} \sin^2 \theta \\
 & \cos^2 \theta \\
 9. \quad & \frac{\sec^2 \theta - 1}{\sin^2 \theta} \\
 & \frac{\tan^2 \theta}{\sin^2 \theta} \\
 & \frac{\sin^2 \theta}{\cos^2 \theta} \\
 & \frac{1}{\sin^2 \theta} \\
 & \frac{1}{\cos^2 \theta} \\
 & \sec^2 \theta \\
 3. \quad & \frac{\sin \theta}{\cos \theta} \cdot \frac{1}{\sin \theta} \\
 & \frac{1}{\cos \theta} \\
 & \sec \theta \\
 7. \quad & \sec^2 \theta \cos^2 \theta \\
 & \frac{1}{\cos^2 \theta} \cos^2 \theta \\
 & 1 \\
 11. \quad & \frac{\frac{1}{\sin \theta} \sin \theta}{\cot \theta} \\
 & \frac{1}{\cot \theta} \\
 & \tan \theta
 \end{aligned}$$

$$\begin{aligned}
 13. \quad & \cos \theta \sec \theta - \cos^2 \theta \\
 & \cos \theta \frac{1}{\cos \theta} - \cos^2 \theta \\
 & 1 - \cos^2 \theta \\
 & \sin^2 \theta \\
 15. \quad & \csc^2 \theta \sin^2 \theta \\
 & \frac{1}{\sin^2 \theta} \sin^2 \theta \\
 & 1 \\
 17. \quad & \frac{\cos^2 \theta}{\sin^2 \theta} - \frac{1}{\sin^2 \theta} \\
 & \frac{\cos^2 \theta - 1}{\sin^2 \theta} \\
 & \frac{-\sin^2 \theta}{\sin^2 \theta} \\
 & -1 \\
 19. \quad & \tan^2 \theta \cot^2 \theta + \tan^2 \theta \\
 & \tan^2 \theta \cdot \frac{1}{\tan^2 \theta} + \tan^2 \theta \\
 & 1 + \tan^2 \theta \\
 & \sec^2 \theta \\
 21. \quad & \sec^2 \theta \cot^2 \theta \\
 & \frac{1}{\cos^2 \theta} \cdot \frac{\cos^2 \theta}{\sin^2 \theta} \\
 & \frac{1}{\sin^2 \theta} \\
 & \csc^2 \theta \\
 23. \quad & \frac{\cos \theta \cdot 1}{\sin \theta \cos \theta} \\
 & \frac{1}{\sin \theta} \\
 & \frac{1}{\sin \theta} \\
 & \frac{1}{\sin \theta} \\
 & 1
 \end{aligned}$$

$$\begin{aligned}
 25. \quad & \sin x + \cos x \frac{\cos x}{\sin x} \\
 & \frac{\sin^2 x + \cos^2 x}{\sin x} \\
 & \frac{1}{\sin x} \\
 & \csc x \\
 27. \quad & \frac{\sin x}{\cos x} \cdot \frac{1}{\sin x} \cos x \\
 & 1 \\
 29. \quad & \frac{1}{\cos x} - \frac{\sin x}{\cos x} \sin x \\
 & \frac{1 - \sin^2 x}{\cos x} \\
 & \frac{\cos^2 x}{\cos x} \\
 & \cos x \\
 31. \quad & \csc x - \csc x \cos x + \cot x - \cot x \cos x \\
 & \frac{1}{\sin x} - \frac{1}{\sin x} \cos x + \frac{\cos x}{\sin x} - \frac{\cos x}{\sin x} \cos x \\
 & \frac{1 - \cos x + \cos x - \cos^2 x}{\sin x} \\
 & \frac{1 - \cos^2 x}{\sin x} \\
 & \frac{\sin^2 x}{\sin x} \\
 & \sin x
 \end{aligned}$$

33. $\frac{\csc \theta + \cot \theta}{\frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta}} = \frac{1 + \cos \theta}{\sin \theta}$
35. $\frac{\csc \theta}{\sec \theta + \tan \theta} = \frac{1}{\frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta}} = \frac{1}{\frac{1 + \cos \theta}{\sin \theta}} = \frac{\sin \theta}{1 + \cos \theta} = \frac{\cos \theta}{\sin \theta + \sin^2 \theta}$
37. $\frac{1 + \csc \theta}{1 + \sec \theta} = \frac{1 + \frac{1}{\sin \theta}}{1 + \frac{1}{\cos \theta}} = \frac{1 + \frac{1}{\sin \theta}}{1 + \frac{1}{\cos \theta}} \cdot \frac{\sin \theta \cos \theta}{\sin \theta \cos \theta} = \frac{\sin \theta \cos \theta + \cos \theta}{\sin \theta \cos \theta + \sin \theta} = \frac{\cos \theta (\sin \theta + 1)}{\sin \theta (\cos \theta + 1)} = \frac{\cos \theta}{\sin \theta} \cdot \frac{\sin \theta + 1}{\cos \theta + 1} = \cot \theta \left(\frac{1 + \sin \theta}{1 + \cos \theta} \right)$
39. $\frac{\tan^2 \theta + \sec^2 \theta}{\sec^2 \theta} = \frac{\tan^2 \theta}{\sec^2 \theta} + \frac{\sec^2 \theta}{\sec^2 \theta} = \tan^2 \theta \cos^2 \theta + 1 = \frac{\sin^2 \theta}{\cos^2 \theta} \cdot \cos^2 \theta + 1 = \sin^2 \theta + 1$
41. $\frac{\sec y - \cos y}{\frac{1}{\cos y} - \cos y} = \frac{1 - \cos^2 y}{\cos y} = \frac{\sin^2 y}{\cos y} = \frac{\sin y}{\cos y} \sin y = \tan y \sin y$
43. $\frac{1 + \cot^2 \theta}{\tan^2 \theta} = 1 + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta} = \frac{1}{\cos^2 \theta} = \sec^2 \theta$
45. $\frac{1}{\sec \theta - \cos \theta} = \frac{1}{\frac{1}{\cos \theta} - \cos \theta} = \frac{1}{\frac{1 - \cos^2 \theta}{\cos \theta}} = \frac{\cos \theta}{1 - \cos^2 \theta} = \frac{\cos \theta}{\sin^2 \theta} = \frac{\cos \theta}{\sin \theta} \cdot \frac{1}{\sin \theta} = \cot \theta \csc \theta$
47. $\frac{\tan \theta}{\csc \theta} = \frac{\frac{\sin \theta}{\cos \theta}}{\frac{1}{\sin \theta}} = \frac{\sin^2 \theta}{\cos \theta} = \sin \theta \frac{\sin \theta}{\cos \theta} = \sin \theta \tan \theta$
49. $\frac{\tan^2 \theta}{\sec \theta - 1} = \frac{\sec^2 \theta \sin^2 \theta}{\sec \theta - 1} = \frac{\sec^2 \theta - 1}{\sec \theta - 1} = \frac{(\sec \theta + 1)(\sec \theta - 1)}{\sec \theta - 1} = \sec \theta + 1$
51. $\frac{\cot x + 1}{\cot x - 1} = \frac{\frac{\cos x}{\sin x} + 1}{\frac{\cos x}{\sin x} - 1} = \frac{\cos x + \sin x}{\cos x - \sin x} = \frac{\cos x + \sin x}{\cos x - \sin x} \cdot \frac{\cos x + \sin x}{\cos x + \sin x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x - \sin^2 x} = \frac{1}{\cos^2 x - \sin^2 x}$
53. $\frac{1 - \cos x}{\sin x} = \frac{(1 - \cos x)(1 + \cos x)}{\sin x(1 + \cos x)} = \frac{1 - \cos^2 x}{\sin x(1 + \cos x)} = \frac{\sin^2 x}{\sin x(1 + \cos x)} = \frac{\sin x}{1 + \cos x}$
55. $\frac{\cos x}{\sec x - \tan x} = \frac{\cos x}{\frac{1}{\cos x} - \frac{\sin x}{\cos x}} = \frac{\cos x}{\frac{1 - \sin x}{\cos x}} = \frac{\cos^2 x}{1 - \sin x}$
57. $\frac{1}{1 - \sin x} + \frac{1}{1 + \sin x} = \frac{1 + \sin x + 1 - \sin x}{(1 - \sin x)(1 + \sin x)} = \frac{2}{1 - \sin^2 x} = \frac{2}{\cos^2 x} = 2 \sec^2 x$
59. $\sin^4 x - \cos^4 x = (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) = (1)(1 - \cos^2 x - \cos^2 x) = 1 - 2 \cos^2 x$
61. $\cot^2 x \cos^2 x = \frac{\cos^2 x}{\sin^2 x} (1 - \sin^2 x) = \frac{\cos^2 x}{\sin^2 x} - \cos^2 x = \cot^2 x - \cos^2 x$
63. $\frac{\tan y - \cot y}{\tan y + \cot y} = \frac{\frac{\tan^2 y - 1}{\tan y}}{\frac{\tan^2 y + 1}{\tan y}} = \frac{\tan^2 y - 1}{\tan^2 y + 1} = \frac{\tan^2 y - 1}{\tan^2 y + 1} \cdot \frac{1}{\sec^2 y} = \frac{\tan^2 y - 1}{\sec^2 y}$
65. $\frac{1 - \sin x}{1 + \sin x} = \frac{(1 - \sin x)(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} = \frac{1 - 2 \sin x + \sin^2 x}{1 - \sin^2 x} = \frac{1 - 2 \sin x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 2 \left(\frac{\sin x}{\cos x} \right) \left(\frac{1}{\cos x} \right) + \frac{\sin^2 x}{\cos^2 x} = \sec^2 x - 2 \tan x \sec x + \tan^2 x = (\tan x - \sec x)^2$
67. $\sec^4 x - \sec^2 x = \sec^2 x (\sec^2 x - 1) = (1 + \tan^2 x)(\tan^2 x) = \tan^2 x + \tan^4 x$
69. $2 \cos^2 y - 1 = \cos^2 y + \cos^2 y - 1 = \cos^2 y - (1 - \cos^2 y) = \cos^2 y - \sin^2 y$

71. Let $\theta = \frac{\pi}{6}$:

$$\begin{aligned} 1 - \cos \theta &= 1 - \cos \frac{\pi}{6} \\ &= 1 - \frac{\sqrt{3}}{2} \\ &= \frac{2 - \sqrt{3}}{2} \end{aligned}$$

$$\sin \frac{\pi}{6} = \frac{1}{2} \neq \frac{2 - \sqrt{3}}{2}$$

73. Let $\theta = \frac{\pi}{4}$:

$$\sec \frac{\pi}{4} \stackrel{?}{=} \frac{1}{\csc \frac{\pi}{4}}$$

$$\sqrt{2} \neq \frac{1}{\sqrt{2}}$$

75. Let $\theta = \frac{\pi}{2}$:

$$\sin^2 \theta - 2 \cos \theta \sin \theta + \cos^2 \theta$$

$$\sin^2 \frac{\pi}{2} - 2 \cos \frac{\pi}{2} \sin \frac{\pi}{2} + \cos^2 \frac{\pi}{2}$$

$$1 - 2(0)(1) + 0 = 1$$

$$1 \neq 2$$

77. $\csc \theta + \sec \theta \cot \theta$

$$\csc \frac{\pi}{6} + \sec \frac{\pi}{6} \cot \frac{\pi}{6}$$

$$2 + \left(\frac{2\sqrt{3}}{3}\right)\left(\frac{3\sqrt{3}}{3}\right)$$

$$2 + 2 = 4$$

$$4 \neq 2$$

79. $\frac{1 - \cos \theta}{1 + \cos \theta}$

$$\frac{1 - \cos \frac{\pi}{4}}{1 + \cos \frac{\pi}{4}} = \frac{1 - \frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}} = \frac{2 - \sqrt{2}}{2 + \sqrt{2}}$$

$$\sin^2 \frac{\pi}{4} = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} = \frac{1}{2}$$

$$\frac{2 - \sqrt{2}}{2 + \sqrt{2}} \neq \frac{1}{2}$$

81. a. $\left(1 - \csc^2 \frac{\pi}{6}\right)\left(1 - \sec^2 \frac{\pi}{6}\right)$

$$(1 - 2^2)\left[1 - \left(\frac{2}{\sqrt{3}}\right)^2\right]$$

$$(-3)\left(-\frac{1}{3}\right) = 1$$

b. $\left(1 - \csc^2 \frac{\pi}{4}\right)\left(1 - \sec^2 \frac{\pi}{4}\right)$

$$[1 - (\sqrt{2})^2][1 - (\sqrt{2})^2]$$

$$(1 - 2)(1 - 2) = 1$$

c. yes

83. a. $2 \sin^2 \frac{\pi}{6} + \sin \frac{\pi}{6}$

$$2\left(\frac{1}{2}\right)^2 + \frac{1}{2} = 1$$

b. $2 \sin^2 \frac{3\pi}{2} + \sin \frac{3\pi}{2}$

$$2(-1)^2 + (-1) = 1$$

c. no; $\theta = \frac{\pi}{4}$ is a counterexample

Solutions to trial exercise problems

18. $\frac{\sin^2 \theta}{\cos^2 \theta} - \frac{1}{\cos^2 \theta}$

$$\frac{\sin^2 \theta - 1}{\cos^2 \theta}$$

$$\frac{-(1 - \sin^2 \theta)}{\cos^2 \theta}$$

$$\frac{-\cos^2 \theta}{\cos^2 \theta}$$

$$-1$$

32. $\frac{\sec^4 y - \tan^4 y}{\sec^2 y + \tan^2 y}$

$$\frac{(\sec^2 y + \tan^2 y)(\sec^2 y - \tan^2 y)}{\sec^2 y + \tan^2 y}$$

$$\sec^2 y - \tan^2 y$$

$$(\tan^2 y + 1) - \tan^2 y$$

$$1$$

36. $\frac{\sec \theta}{\csc \theta + \cot \theta}$

$$\frac{1}{\cos \theta}$$

$$\frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta}$$

$$\frac{1}{\cos \theta}$$

$$\frac{1 + \cos \theta}{\sin \theta}$$

$$\frac{1}{\cos \theta} \cdot \frac{\sin \theta}{1 + \cos \theta}$$

$$\frac{\tan \theta}{1 + \cos \theta}$$

$$\frac{\tan \theta}{1 + \cos \theta}$$

$$\frac{\tan \theta}{1 + \cos \theta}$$

52. $\frac{\csc y + 1}{\csc y - 1}$

$$\frac{1}{\sin y} + 1$$

$$\frac{1}{\sin y} - 1$$

$$\frac{1 + \sin y}{\sin y}$$

$$\frac{1 - \sin y}{\sin y}$$

$$\frac{1 + \sin y}{\sin y}$$

$$\frac{1 - \sin y}{\sin y} \cdot \frac{\sin y}{\sin y}$$

$$\frac{1 + \sin y}{1 - \sin y}$$

84. a. $\tan^4 \theta - \tan^2 \theta = 6$

$$\theta = \frac{\pi}{3}: (\sqrt{3})^4 - (\sqrt{3})^2 = 6$$

$$9 - 3 = 6$$

b. $\theta = \frac{4\pi}{3}: (\sqrt{3})^4 - (\sqrt{3})^2 = 6$

$$9 - 3 = 6$$

c. no;

$$\text{let } \theta = 0: 0^4 - 0^2 \neq 6$$

Exercise 5-2

Answers to odd-numbered problems

1. $\cos 72^\circ$ 3. $\cot 82^\circ$ 5. $\csc \frac{\pi}{6}$

7. $\sin\left(-\frac{\pi}{3}\right)$ 9. $\csc \frac{5\pi}{4}$ 11. 1

13. 1 15. 1 17. -1 19. 1

21. 1 23. 1 25. 1

27. $\frac{\sqrt{6} + \sqrt{2}}{4}$ 29. $\frac{\sqrt{6} + \sqrt{2}}{4}$

31. $\frac{\sqrt{6} + \sqrt{2}}{4}$ 33. $\frac{\sqrt{6} - \sqrt{2}}{4}$

35. $\frac{\sqrt{2} - \sqrt{6}}{4}$ 37. $2 + \sqrt{3}$

39. $\frac{2\sqrt{14} + 3}{12}$ 41. $\frac{-15 - 12\sqrt{7}}{36 - 5\sqrt{7}}$

43. $\frac{13}{85}$ 45. $\frac{-\sqrt{5} - 4\sqrt{2}}{9}$

47. $\frac{(8 + \sqrt{5})}{51}\sqrt{17}$ 49. $-\frac{119}{120}$

51. $-\frac{\sqrt{5}}{5}$ 53. $\frac{-4\sqrt{6} + \sqrt{5}}{15}$

55. $2 + \sqrt{3}$

57. $\sin(\pi - \theta) = \sin \pi \cos \theta - \cos \pi \sin \theta$
 $= 0 \cos \theta - (-1) \sin \theta$
 $= \sin \theta$

59. $\cos(\pi - \theta) = \cos \pi \cos \theta + \sin \pi \sin \theta$
 $= (-1) \cos \theta + 0 \sin \theta$
 $= -\cos \theta$

61. $\tan(\pi - \theta) = \frac{\tan \pi - \tan \theta}{1 + \tan \pi \tan \theta}$
 $= \frac{0 - \tan \theta}{1 + 0 \tan \theta}$
 $= -\tan \theta$

63. $\sin(\theta + 2\pi) = \sin \theta \cos 2\pi + \cos \theta \sin 2\pi$
 $= \sin \theta(1) + \cos \theta(0)$
 $= \sin \theta$

65. $\tan(\theta + \pi) = \frac{\tan \theta + \tan \pi}{1 - \tan \theta \tan \pi}$
 $= \frac{\tan \theta + 0}{1 - \tan \theta(0)}$
 $= \tan \theta$

$$67. \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

$$\frac{1}{2}[\sin \alpha \cos \beta + \cos \alpha \sin \beta - (\sin \alpha \cos \beta - \cos \alpha \sin \beta)]$$

$$\frac{1}{2}[2 \cos \alpha \sin \beta]$$

$$\cos \alpha \sin \beta$$

$$69. \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\frac{1}{2}[\cos \alpha \cos \beta + \sin \alpha \sin \beta - (\cos \alpha \cos \beta - \sin \alpha \sin \beta)]$$

$$\frac{1}{2}[2 \sin \alpha \sin \beta]$$

$$\sin \alpha \sin \beta$$

$$71. \text{a. } \frac{11}{29} \quad \text{b. } 3\frac{12}{29}$$

$$73. \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

This was shown true in the text.

$$\cos(\alpha + (-\beta)) = \cos \alpha \cos(-\beta) - \sin \alpha \sin(-\beta)$$

Replace β by $-\beta$. This is valid since the identity is true for all angles and α .

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta - \sin \alpha [-\sin \beta]$$

$$\alpha + (-\beta) = \alpha - \beta; \cos(-\theta) = \cos \theta; \sin(-\theta) = -\sin \theta.$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

This statement is true since the preceding statements are true.

$$75. \cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$$

Proved true above.

$$\text{Let } \alpha = \frac{\pi}{2} - \theta. \text{ Then } \theta = \frac{\pi}{2} - \alpha.$$

$$\cos \alpha = \sin\left(\frac{\pi}{2} - \alpha\right)$$

Substitution of expression (section 5-0).

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

The variable name α or θ is unimportant.

$$77. \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

Identity [3].

$$\sin(\alpha + (-\beta)) = \sin \alpha \cos(-\beta) + \cos \alpha \sin(-\beta)$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta + \cos \alpha [-\sin \beta]$$

Cosine is an even function, sine is odd.

$$= \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

This is identity [4].

$$79. \cot\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan\left(\frac{\pi}{2} - \theta\right)} = \frac{1}{\cot \theta} = \tan \theta.$$

$$81. \csc\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\sin\left(\frac{\pi}{2} - \theta\right)} = \frac{1}{\cos \theta} = \sec \theta.$$

Solutions to trial exercise problems

$$8. \sin\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{2} - \left(-\frac{\pi}{3}\right)\right) = \cos \frac{5\pi}{6}$$

$$17. \tan^2 8^\circ - \csc^2 82^\circ$$

$$\tan^2 8^\circ - \sec^2(90^\circ - 82^\circ)$$

$$\tan^2 8^\circ - \sec^2 8^\circ$$

$$-(\sec^2 8^\circ - \tan^2 8^\circ) = -1$$

$$28. \tan \frac{\pi}{12}$$

$$\tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$\frac{\tan \frac{\pi}{4} - \tan \frac{\pi}{6}}{1 + \tan \frac{\pi}{4} \tan \frac{\pi}{6}}$$

$$\frac{1 - \frac{\sqrt{3}}{3}}{1 + 1\left(\frac{\sqrt{3}}{3}\right)} \cdot \frac{3}{3} = \frac{3 - \sqrt{3}}{3 + \sqrt{3}} \cdot \frac{3 - \sqrt{3}}{3 - \sqrt{3}}$$

$$\frac{12 - 6\sqrt{3}}{6} = 2 - \sqrt{3}$$

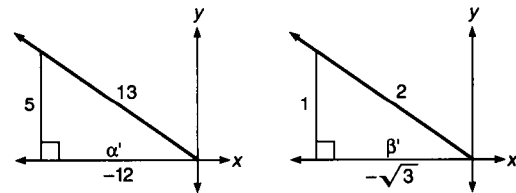
$$40. \cos \alpha = -\frac{12}{13}, \text{ quadrant II; } \sin \beta = \frac{1}{2}, \text{ quadrant II.}$$

Find $\cos(\alpha - \beta)$.

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

Find $\cos \alpha$, $\sin \alpha$, $\cos \beta$, $\sin \beta$ from the reference triangles in the figure.

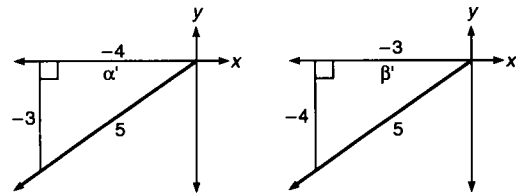
$$= -\frac{12}{13} \cdot \left(-\frac{\sqrt{3}}{2}\right) + \frac{5}{13} \cdot \frac{1}{2} = \frac{12\sqrt{3} + 5}{26}.$$



$$48. \tan \alpha = \frac{3}{4}, \text{ quadrant III; } \sin \beta = -\frac{4}{5}, \text{ quadrant III. Find } \cos(\alpha - \beta).$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$= -\frac{4}{5}\left(-\frac{3}{5}\right) + \left(-\frac{3}{5}\right)\left(-\frac{4}{5}\right) = \frac{24}{25}.$$



70. For angle α , the hypotenuse is $\sqrt{164} = 2\sqrt{41}$, and for angle β , the hypotenuse is $\sqrt{136} = 2\sqrt{34}$. Thus, $\sin \alpha = \frac{8}{2\sqrt{41}}$
 $= \frac{4}{\sqrt{41}}$; $\cos \alpha = \frac{10}{2\sqrt{41}} = \frac{5}{\sqrt{41}}$; $\sin \beta = \frac{6}{2\sqrt{34}} = \frac{3}{\sqrt{34}}$;
 $\cos \beta = \frac{10}{2\sqrt{34}} = \frac{5}{\sqrt{34}}$

$$\begin{aligned}\text{Therefore, } \sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\ &= \left(\frac{4}{\sqrt{41}}\right)\left(\frac{5}{\sqrt{34}}\right) - \left(\frac{5}{\sqrt{41}}\right)\left(\frac{3}{\sqrt{34}}\right) \\ &= \frac{20}{\sqrt{1,394}} - \frac{15}{\sqrt{1,394}} = \frac{5}{\sqrt{1,394}} \\ &= \frac{5\sqrt{1,394}}{1,394}\end{aligned}$$

$$76. \sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

We know this is true.

Replace θ by $\alpha + \beta$.

$$\sin(\alpha + \beta) = \cos\left[\frac{\pi}{2} - (\alpha + \beta)\right]$$

$$\sin(\alpha + \beta) = \cos\left[\left(\frac{\pi}{2} - \alpha\right) - \beta\right]$$

$$\text{Regroup } \frac{\pi}{2} - \alpha - \beta.$$

$$= \cos\left(\frac{\pi}{2} - \alpha\right)\cos \beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin \beta$$

Use the identity for $\cos(\alpha - \beta)$.

$$= \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

Use cofunction identities.

$$\text{Thus, } \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

Exercise 5-3

Answers to odd-numbered problems

1. $\sin \frac{\pi}{2}$ 3. $\cos 6\pi$ 5. $\cos \frac{\pi}{5}$

7. $3 \tan 20^\circ$ 9. $\sin 120$ 11. $3 \cos 100$

13. $5 \tan 60$ 15. $2 \cos 140$

17. $3 \cos 60$ 19. 70° 21. $\frac{5\pi}{12}$

23. 35° 25. 20° 27. $\frac{\pi}{8}$ 29. $\frac{4\pi}{5}$

31. $\frac{24}{25}, \frac{7}{25}, \frac{24}{7}$ 33. $-\frac{24}{25}, \frac{7}{25}, -\frac{24}{7}$

35. $\frac{5\sqrt{39}}{32}, \frac{7}{32}, \frac{5\sqrt{39}}{7}$

37. $\frac{\sqrt{70}}{10}, -\frac{\sqrt{30}}{10}, -\frac{\sqrt{21}}{3}$

39. $\frac{\sqrt{50 - 20\sqrt{5}}}{10}, -\frac{\sqrt{50 + 20\sqrt{5}}}{10}, 2 - \sqrt{5}$

41. $\frac{\sqrt{2 - \sqrt{3}}}{2}, \frac{\sqrt{2 + \sqrt{3}}}{2}, 2 - \sqrt{3}$

43. $\frac{1}{2}\sqrt{2 + \sqrt{3}}, \frac{1}{2}\sqrt{2 - \sqrt{3}}, 2 + \sqrt{3}$

45. $\frac{\sqrt{4 + \sqrt{6} + 2\sqrt{3} + 2\sqrt{2}} - \sqrt{4 + \sqrt{6} - 2\sqrt{3} - 2\sqrt{2}}}{4}$

47. $\frac{\sqrt{2 - \sqrt{2 + \sqrt{3}}}}{2}$

49. $(\sin \theta + \cos \theta)^2$
 $\sin^2 + 2 \sin \theta \cos \theta + \cos^2 \theta$
 $2 \sin \theta \cos \theta + 1$
 $\sin 2\theta + 1$

51. $\cos^4 \theta - \sin^4 \theta$
 $(\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta)$
 $(1)(\cos 2\theta)$
 $\cos 2\theta$

53. $\frac{(\cos^2 \theta + \sin^2 \theta) + (\cos^2 \theta - \sin^2 \theta)}{(\cos^2 \theta + \sin^2 \theta) - (\cos^2 \theta - \sin^2 \theta)}$

$$\frac{2 \cos^2 \theta}{2 \sin^2 \theta}$$

$$\frac{\cot^2 \theta}{1}$$

55. $\frac{2 \cos 2\theta}{\sin 2\theta}$

$$\frac{2(\cos^2 \theta - \sin^2 \theta)}{2 \sin \theta \cos \theta}$$

$$\frac{\cos^2 \theta}{\sin \theta \cos \theta} - \frac{\sin^2 \theta}{\sin \theta \cos \theta}$$

$$\frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta - \tan \theta$$

57. $\sin 2\theta \cos 2\theta$

$$2 \sin \theta \cos \theta (\cos^2 \theta - \sin^2 \theta)$$

$$2 \sin \theta \cos \theta (\cos^2 \theta) - 2 \sin^3 \theta \cos \theta$$

$$2 \sin \theta \cos \theta (1 - \sin^2 \theta) - 2 \sin^3 \theta \cos \theta$$

$$2 \sin \theta \cos \theta - 2 \sin^3 \theta \cos \theta - 2 \sin^3 \theta \cos \theta$$

$$2 \sin \theta \cos \theta - 4 \sin^3 \theta \cos \theta$$

59. $\frac{2}{1 - \cos 2\theta}$

$$\frac{2}{1 - (\cos^2 \theta - \sin^2 \theta)}$$

$$\frac{2}{1 - \cos^2 \theta + \sin^2 \theta}$$

$$\frac{2}{\sin^2 \theta + \sin^2 \theta}$$

$$\frac{2}{2 \sin^2 \theta}$$

$$\frac{1}{\sin^2 \theta}$$

$$\csc^2 \theta$$

61. $\frac{2(\tan \theta + \tan^3 \theta)}{1 - \tan^4 \theta}$

$$\frac{2 \tan \theta (1 + \tan^2 \theta)}{(1 - \tan^2 \theta)(1 + \tan^2 \theta)}$$

$$\frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\tan 2\theta$$

63. $2 \csc 2\theta \sin \theta \cos \theta$

$$2\left(\frac{1}{\sin 2\theta}\right)\sin \theta \cos \theta$$

$$\frac{2 \sin \theta \cos \theta}{\sin 2\theta}$$

$$\frac{2 \sin \theta \cos \theta}{2 \sin \theta \cos \theta}$$

$$1$$

65. $\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$

$$1 - \frac{\sin^2 \theta}{\cos^2 \theta}$$

$$1 + \frac{\sin^2 \theta}{\cos^2 \theta}$$

$$\frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta}$$

$$\frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta}$$

$$\frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin^2 \theta}$$

$$\frac{\cos^2 \theta - \sin^2 \theta}{1}$$

$$\cos^2 \theta - \sin^2 \theta$$

$$\cos 2\theta$$

$$67. \cos^2 \frac{\theta}{2}$$

$$\begin{aligned} & \left(\pm \sqrt{\frac{1 + \cos \theta}{2}} \right)^2 \\ & \frac{1 + \cos \theta}{2} \\ & \frac{1 + \cos \theta}{2} \cdot \frac{1 - \cos \theta}{1 - \cos \theta} \\ & \frac{1 - \cos^2 \theta}{2 - 2 \cos \theta} \end{aligned}$$

$$69. \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2}$$

$$\begin{aligned} & \left(\pm \sqrt{\frac{1 + \cos \theta}{2}} \right)^2 \left(\pm \sqrt{\frac{1 - \cos \theta}{2}} \right)^2 \\ & \left(\pm \sqrt{\frac{1 - \cos^2 \theta}{4}} \right)^2 \\ & \frac{\sin^2 \theta}{4} \end{aligned}$$

$$71. \tan^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2}$$

$$\begin{aligned} & \left(\pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} \right)^2 + \left(\pm \sqrt{\frac{1 + \cos \theta}{2}} \right)^2 \\ & \frac{1 - \cos \theta}{1 + \cos \theta} + \frac{1 + \cos \theta}{2} \\ & \frac{2 - 2 \cos \theta + 1 + 2 \cos \theta + \cos^2 \theta}{2(1 + \cos \theta)} \end{aligned}$$

$$\frac{\cos^2 \theta + 3}{2 + 2 \cos \theta}$$

$$73. \frac{\csc \theta - \cot \theta}{2 \csc \theta}$$

$$\begin{aligned} & \frac{\frac{1}{\sin \theta} - \frac{\cos \theta}{\sin \theta}}{\frac{2}{\sin \theta}} \\ & \frac{1 - \cos \theta}{\sin \theta} \\ & \frac{2}{\sin \theta} \\ & \frac{1 - \cos \theta}{2} \\ & \left(\pm \sqrt{\frac{1 - \cos \theta}{2}} \right)^2 \\ & \sin^2 \frac{\theta}{2} \end{aligned}$$

$$75. \frac{1 + \sec \theta}{\sec \theta}$$

$$\begin{aligned} & \frac{1 + \frac{1}{\cos \theta}}{\frac{1}{\cos \theta}} \\ & \frac{\cos \theta + 1}{\cos \theta} \\ & \frac{1}{\cos \theta} \\ & \frac{\cos \theta + 1}{2 \left(\frac{1 + \cos \theta}{2} \right)} \\ & \frac{2 \left(\pm \sqrt{\frac{1 + \cos \theta}{2}} \right)^2}{2 \cos^2 \frac{\theta}{2}} \end{aligned}$$

$$77. 2 \cos^2 \frac{\theta}{2} - \cos \theta$$

$$\begin{aligned} & 2 \left(\pm \sqrt{\frac{1 + \cos \theta}{2}} \right)^2 - \cos \theta \\ & 1 + \cos \theta - \cos \theta \\ & 1 \end{aligned}$$

$$79. \cos 3\theta = \cos(2\theta + \theta)$$

$$\begin{aligned} & = \cos(2\theta)\cos \theta - \sin(2\theta)\sin \theta \\ & (\cos^2 \theta - \sin^2 \theta)\cos \theta - 2 \sin \theta \cos \theta \sin \theta \\ & \cos^3 \theta - \sin^2 \theta \cos \theta - 2 \sin^2 \theta \cos \theta \\ & \cos^3 \theta - 3 \sin^2 \theta \cos \theta \\ & \cos^2 \theta - 3(1 - \cos^2 \theta)\cos \theta \\ & \cos^3 \theta - 3 \cos \theta + 3 \cos^3 \theta \\ & 4 \cos^3 \theta - 3 \cos \theta \end{aligned}$$

$$81. \text{ We know that } \sin 4\theta = 4 \cos^3 \theta \sin \theta - 4 \sin^3 \theta \cos \theta \text{ and } \cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1 \text{ from problem 80.}$$

$$\text{a. } \sin 5\theta$$

$$\begin{aligned} & = \sin(\theta + 4\theta) \\ & = \sin \theta \cos 4\theta + \cos \theta \sin 4\theta \\ & = \sin \theta (8 \cos^4 \theta - 8 \cos^2 \theta + 1) + \cos \theta (4 \sin \theta \cos \theta \\ & \quad - 8 \sin^3 \theta \cos \theta) \\ & = 4 \sin \theta \cos^2 \theta - 8 \sin^3 \theta \cos^2 \theta + 8 \sin \theta \cos^4 \theta \\ & \quad - 8 \sin \theta \cos^2 \theta + \sin \theta \end{aligned}$$

$$\text{We know } \cos^2 \theta = 1 - \sin^2 \theta, \text{ so that } \cos^4 \theta = (1 - \sin^2 \theta)^2 = 1 - 2 \sin^2 \theta + \sin^4 \theta. \text{ Replace } \cos^2 \theta \text{ and } \cos^4 \theta \text{ in the equation above:}$$

$$\begin{aligned} & = 4 \sin \theta (1 - \sin^2 \theta) - 8 \sin^3 \theta (1 - \sin^2 \theta) \\ & \quad + 8 \sin \theta (1 - 2 \sin^2 \theta + \sin^4 \theta) - 8 \sin \theta (1 - \sin^2 \theta) \\ & \quad + \sin \theta \\ & = 4 \sin \theta - 4 \sin^3 \theta - 8 \sin^3 \theta + 8 \sin^5 \theta + 8 \sin \theta \\ & \quad - 16 \sin^3 \theta + 8 \sin^5 \theta - 8 \sin \theta + 8 \sin^3 \theta + \sin \theta \\ & = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta \end{aligned}$$

$$\text{b. } \cos 5\theta$$

$$\begin{aligned} & = \cos(\theta + 4\theta) \\ & = \cos \theta \cos 4\theta - \sin \theta \sin 4\theta \\ & = \cos \theta (8 \cos^4 \theta - 8 \cos^2 \theta + 1) - \sin \theta (4 \cos^3 \theta \sin \theta \\ & \quad - 4 \sin^3 \theta \cos \theta) \\ & = 8 \cos^5 \theta - 8 \cos^3 \theta + \cos \theta - 4 \cos^3 \theta \sin^2 \theta \\ & \quad + 4 \sin^4 \theta \cos \theta \\ & = 8 \cos^5 \theta - 8 \cos^3 \theta + \cos \theta - 4 \cos^3 \theta (1 - \cos^2 \theta) + \\ & \quad 4(1 - \cos^2 \theta)^2 \cos \theta \\ & = 8 \cos^5 \theta - 8 \cos^3 \theta + \cos \theta - 4 \cos^3 \theta + 4 \cos^5 \theta + \\ & \quad 4 \cos \theta - 8 \cos^3 \theta + 4 \cos^5 \theta \\ & = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta \end{aligned}$$

$$83. \tan \frac{\alpha}{2} = \frac{1 - \cos \alpha}{\sin \alpha}; \text{ let } \frac{\alpha}{2} = \theta, \text{ so } \alpha = 2\theta.$$

$$\begin{aligned} \tan \theta &= \frac{1 - \cos 2\theta}{\sin 2\theta} \\ &= \frac{1 - (1 - 2\sin^2\theta)}{2\sin\theta\cos\theta} \\ &= \frac{2\sin^2\theta}{2\sin\theta\cos\theta} \\ &= \frac{\sin\theta}{\cos\theta} \\ &= \tan\theta \end{aligned}$$

If every instance of θ above were replaced by $\frac{\alpha}{2}$, then the statements would still be true, thus proving the identity.

$$85. \text{ a. Problem 41 shows that } \sin 15^\circ = \frac{\sqrt{2} - \sqrt{3}}{2}.$$

$$\begin{aligned} \text{b. } \sin 15^\circ &= \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$

$$\text{c. } \frac{\sqrt{6} - \sqrt{2}}{4} \stackrel{?}{=} \frac{\sqrt{2} - \sqrt{3}}{2}$$

$$\left(\frac{\sqrt{6} - \sqrt{2}}{4}\right)^2 \quad \left(\frac{\sqrt{2} - \sqrt{3}}{2}\right)^2$$

Square both values.

$$\begin{aligned} &\left(\frac{(\sqrt{6} - \sqrt{2})^2}{16}\right) \quad \frac{2 - \sqrt{3}}{4} \\ &\frac{6 - 2\sqrt{12} + 2}{16} \\ &\frac{8 - 4\sqrt{3}}{16} \\ &\frac{4(2 - \sqrt{3})}{16} \\ &\frac{2 - \sqrt{3}}{4} \end{aligned}$$

87. Identity [5]

$$\begin{aligned} \cos 2\theta &= 2\cos^2\theta - 1 \\ \cos 2\theta + 1 &= 2\cos^2\theta \\ \cos^2\theta &= \frac{1 + \cos 2\theta}{2} \end{aligned}$$

$$\cos \theta = \pm \sqrt{\frac{1 + \cos 2\theta}{2}}$$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

Identity [6-a]

$$\begin{aligned} \tan \frac{\alpha}{2} &= \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} = \frac{\pm \sqrt{\frac{1 - \cos \alpha}{2}}}{\pm \sqrt{\frac{1 + \cos \alpha}{2}}} \\ &= \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} \\ &= \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} \end{aligned}$$

$$89. x = 13\frac{37}{47}$$

$$\begin{aligned} 91. \sin 2\alpha + \sin 2\beta &= 2\sin(\alpha + \beta) \cdot \cos(\alpha - \beta) \\ &= 2\sin\alpha\cos\beta + 2\sin\beta\cos\alpha \\ &= 2(\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\ &= 2(\sin\alpha\cos^2\beta\cos\alpha + \sin^2\alpha\cos\beta\sin\beta \\ &\quad + \cos^2\alpha\sin\beta\cos\beta + \cos\alpha\sin^2\beta\sin\alpha) \\ &= 2[\sin\alpha\cos^2\beta\cos\alpha + \cos\alpha\sin^2\beta\sin\alpha \\ &\quad + \cos^2\alpha\sin\beta\cos\beta + \sin^2\alpha\cos\beta\sin\beta] \\ &= 2[\sin\alpha\cos\alpha(\cos^2\beta + \sin^2\beta) \\ &\quad + \sin\beta\cos\beta(\cos^2\alpha + \sin^2\alpha)] \\ &= 2[\sin\alpha\cos\alpha(1) + \sin\beta\cos\beta(1)] \\ &= 2\sin\alpha\cos\alpha + 2\sin\beta\cos\beta \end{aligned}$$

$$\begin{aligned} 93. \cos 2\alpha + \cos 2\beta &= 2\cos(\alpha + \beta) \cdot \cos(\alpha - \beta) \\ &= (2\cos^2\alpha - 1) + (2\cos^2\beta - 1) \\ &= 2(\cos\alpha\cos\beta - \sin\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\ &= 2\cos^2\alpha + 2\cos^2\beta - 2 \\ &= 2(\cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta) \\ &= 2[\cos^2\alpha\cos^2\beta - (1 - \cos^2\alpha)(1 - \cos^2\beta)] \\ &= 2[\cos^2\alpha\cos^2\beta - (1 - \cos^2\beta - \cos^2\alpha + \cos^2\alpha\cos^2\beta)] \\ &= 2[-1 + \cos^2\beta + \cos^2\alpha] \\ &= 2\cos^2\alpha + 2\cos^2\beta - 2 \end{aligned}$$

$$\begin{aligned} 95. \cot 2\theta &= \frac{\cos 2\theta}{\sin 2\theta} \\ &= \frac{\cos^2\theta - \sin^2\theta}{2\sin\theta\cos\theta} \\ &= \frac{1}{2} \cdot \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta} \\ &= \frac{1}{2} \left(\frac{\cos^2\theta}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta} \right) \\ &= \frac{1}{2} \left(\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} \right) \\ &= \frac{1}{2} (\cot\theta - \tan\theta) \\ &= \frac{1}{2} \left(\cot\theta - \frac{1}{\cot\theta} \right) \end{aligned}$$

Solutions to trial exercise problems

$$2 \tan \frac{\pi}{6}$$

$$6. \frac{1 - \tan^2 \frac{\pi}{6}}{1 - \tan^2 \frac{\pi}{6}}$$

$$\frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \tan 2\alpha; \text{ Let } \alpha = \frac{\pi}{6}, \text{ so } 2\alpha = \frac{\pi}{3},$$

$$\text{and } \tan 2\alpha \text{ becomes } \tan \frac{\pi}{3}.$$

$$8. 8 \cos^2 \frac{\pi}{2} - 4$$

$$2\cos^2\alpha - 1 = \cos 2\alpha$$

$$8\cos^2\alpha - 1 = 4\cos 2\alpha$$

Multiply each member by 4.

$$\text{Let } \alpha = \frac{\pi}{2}, \text{ so } 2\alpha = \pi, \text{ and } 4\cos 2\alpha = 4\cos \pi.$$

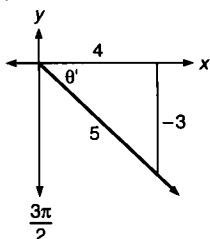
$$27. \cos \theta = \sqrt{\frac{1}{2} \left(1 + \cos \frac{\pi}{4} \right)}$$

$$\cos \frac{\alpha}{2} = \sqrt{\frac{1 + \cos \alpha}{2}}$$

$$\alpha = \frac{\pi}{4}, \text{ so } \frac{\alpha}{2} = \frac{\pi}{8}. \text{ Thus, } \theta \text{ is } \frac{\pi}{8}, \text{ and}$$

$$\cos \frac{\pi}{8} = \sqrt{\frac{1}{2} \left(1 + \cos \frac{\pi}{4} \right)}.$$

$$34. \tan \theta = -\frac{3}{4}, \frac{3\pi}{2} < \theta < 2\pi$$

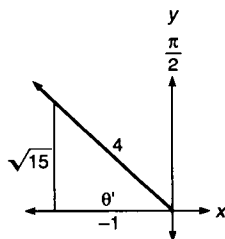


$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \left(-\frac{3}{5} \right) \cdot \frac{4}{5} = -\frac{24}{25}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \left(\frac{4}{5} \right)^2 - \left(-\frac{3}{5} \right)^2 = \frac{7}{25}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = -\frac{24}{7}$$

$$38. \tan \theta = -\sqrt{15}, \frac{\pi}{2} < \theta < \pi; \cos \theta = -\frac{1}{4}$$



$$\frac{\pi}{4} \leq \frac{\theta}{2} \leq \frac{\pi}{2}, \left(\frac{\theta}{2} \text{ in quadrant I} \right) \text{ so}$$

$$\sin \frac{\theta}{2} > 0, \cos \frac{\theta}{2} > 0, \tan \frac{\theta}{2} > 0.$$

$$\sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{2}} = \sqrt{\frac{1 - (-\frac{1}{4})}{2}} = \sqrt{\frac{5}{8}}$$

$$= \frac{\sqrt{5}}{2\sqrt{2}} = \frac{\sqrt{10}}{4}$$

$$\cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}} = \sqrt{\frac{1 + (-\frac{1}{4})}{2}} = \sqrt{\frac{3}{8}}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{6}}{4}$$

$$\tan \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} = \sqrt{\frac{1 - (-\frac{1}{4})}{1 + (-\frac{1}{4})}}$$

$$= \sqrt{\frac{5}{8} \div \frac{3}{4}} = \sqrt{\frac{5}{3}} = \frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{15}}{3}$$

$$42. 22.5^\circ, \text{ or } \frac{\pi}{8}$$

$$\text{a. } \sin 22.5^\circ = \sqrt{\frac{1 - \cos 45^\circ}{2}} = \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}}$$

$$= \sqrt{\frac{2 - \sqrt{2}}{4}} = \frac{\sqrt{2 - \sqrt{2}}}{2}$$

$$\text{b. } \cos 22.5^\circ = \sqrt{\frac{1 + \cos 45^\circ}{2}} = \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}}$$

$$= \sqrt{\frac{2 + \sqrt{2}}{4}} = \frac{\sqrt{2 + \sqrt{2}}}{2}$$

$$\text{c. } \tan 22.5^\circ = \sqrt{\frac{1 - \cos 45^\circ}{1 + \cos 45^\circ}} = \frac{1 - \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{2 - \sqrt{2}}{\sqrt{2}}$$

$$= \frac{2\sqrt{2} - 2}{2} = \sqrt{2} - 1$$

$$44. \sin 37.5^\circ = \sin(15^\circ + 22.5^\circ)$$

$$= \sin 15^\circ \cos 22.5^\circ + \cos 15^\circ \sin 22.5^\circ$$

$$= \frac{\sqrt{2 - \sqrt{3}}}{2} \cdot \frac{\sqrt{2 + \sqrt{2}}}{2} + \frac{\sqrt{2 + \sqrt{3}}}{2} \cdot \frac{\sqrt{2 - \sqrt{2}}}{2}$$

$$= \frac{\sqrt{4 - \sqrt{6} - 2\sqrt{3} + 2\sqrt{2}} + \sqrt{4 - \sqrt{6} + 2\sqrt{3} - 2\sqrt{2}}}{4}$$

$$\text{Note: } \sqrt{2 - \sqrt{3}} \cdot \sqrt{2 + \sqrt{2}} = \sqrt{(2 - \sqrt{3})(2 + \sqrt{2})}$$

$$= \sqrt{4 + 2\sqrt{2} - 2\sqrt{3} - \sqrt{6}}$$

The calculation for $\sqrt{2 + \sqrt{3}} \cdot \sqrt{2 - \sqrt{2}}$ is similar.

80. Find identities for $\sin 4\theta$ in terms of $\sin x$ and $\cos x$, and for $\cos 4\theta$ in terms of $\cos \theta$.

$$\text{a. } \sin 4\theta = \sin 2(2\theta)$$

$$= 2 \sin 2\theta \cos 2\theta$$

$$= 2(2 \sin \theta \cos \theta)(1 - 2 \sin^2 \theta)$$

$$= 4 \sin \theta \cos \theta - 8 \sin^3 \theta \cos \theta$$

Depending on how $\cos 2\theta$ is expanded, other possible answers are

$$\sin 4\theta = 8 \cos^3 \theta \sin \theta - 4 \cos \theta \sin \theta$$

$$\sin 4\theta = 4 \cos^3 \theta \sin \theta - 4 \sin^3 \theta \cos \theta$$

$$\text{b. } \cos 4\theta = \cos 2(2\theta)$$

$$= 2 \cos^2(2\theta) - 1$$

$$= 2[\cos 2\theta]^2 - 1$$

$$= 2[2 \cos^2 \theta - 1]^2 - 1$$

$$= 2[4 \cos^4 \theta - 4 \cos^2 \theta + 1] - 1$$

$$= 8 \cos^4 \theta - 8 \cos^2 \theta + 1$$

$$89. \cos \theta_2 = \cos \frac{\theta}{2} = \frac{8}{9}; \cos \theta = \frac{8}{x}$$

$$\cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}}$$

The angles are acute, so we need the positive value.

$$\frac{8}{9} = \sqrt{\frac{1 + \frac{8}{x}}{2}}$$

Substitute values.

$$\frac{64}{81} = \frac{1 + \frac{8}{x}}{2}$$

Square both members.

$$128 = 81 \left(1 + \frac{8}{x} \right)$$

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc.$$

$$128 = 81 + \frac{648}{x}$$

$$47 = \frac{648}{x}$$

$$x = \frac{648}{47} = 13\frac{37}{47}$$

Exercise 5-4

Answers to odd-numbered problems

$$1. \frac{3\pi}{4}(135^\circ), \frac{7\pi}{4}(315^\circ) \quad 3. \frac{\pi}{3}(60^\circ), \frac{5\pi}{3}(300^\circ)$$

$$5. \frac{\pi}{6}(30^\circ), \frac{7\pi}{6}(210^\circ) \quad 7. \frac{\pi}{3}(60^\circ), \frac{5\pi}{3}(300^\circ)$$

$$9. \frac{\pi}{2}(90^\circ), \frac{3\pi}{2}(270^\circ) \quad 11. 0(0^\circ), \pi(180^\circ) \quad 13. 0(0^\circ), \frac{3\pi}{2}(270^\circ)$$

$$15. \frac{\pi}{4}(45^\circ), \frac{3\pi}{4}(135^\circ), \frac{5\pi}{4}(225^\circ), \frac{7\pi}{4}(315^\circ)$$

$$17. 0(0^\circ), \pi(180^\circ), \frac{\pi}{2}(90^\circ) \quad 19. 0(0^\circ), \pi(180^\circ), \frac{\pi}{3}(60^\circ),$$

$$\frac{4\pi}{3}(240^\circ) \quad 21. \frac{3\pi}{2}(270^\circ), \frac{\pi}{6}(30^\circ), \frac{5\pi}{6}(150^\circ)$$

$$23. 0(0^\circ), \pi(180^\circ), \frac{\pi}{4}(45^\circ), \frac{3\pi}{4}(135^\circ), \frac{5\pi}{4}(225^\circ), \frac{7\pi}{4}(315^\circ)$$

$$25. 0(0^\circ), \pi(180^\circ), \frac{\pi}{3}(60^\circ), \frac{5\pi}{3}(300^\circ)$$

$$27. \frac{5\pi}{6}(150^\circ), \frac{11\pi}{6}(330^\circ), \frac{\pi}{2}(90^\circ), \frac{3\pi}{2}(270^\circ)$$

$$29. 0(0^\circ), \pi(180^\circ), \frac{\pi}{2}(90^\circ), \frac{3\pi}{2}(270^\circ)$$

$$31. \frac{\pi}{6}(30^\circ), \frac{5\pi}{6}(150^\circ), \frac{3\pi}{2}(270^\circ) \quad 33. \frac{3\pi}{4}(135^\circ), \frac{7\pi}{4}(315^\circ)$$

$$35. \frac{\pi}{3}(60^\circ), \frac{5\pi}{3}(300^\circ), \pi(180^\circ)$$

$$37. \frac{\pi}{3}(60^\circ), \frac{2\pi}{3}(120^\circ), \frac{4\pi}{3}(240^\circ), \frac{5\pi}{3}(300^\circ)$$

$$39. \frac{\pi}{4}(45^\circ), \frac{3\pi}{4}(135^\circ), \frac{5\pi}{4}(225^\circ), \frac{7\pi}{4}(315^\circ)$$

$$41. 0(0^\circ), \pi(180^\circ), \frac{\pi}{6}(30^\circ), \frac{5\pi}{6}(150^\circ)$$

$$43. 0.65, 2.49, 3.42, 6.01 (37.4^\circ, 142.6^\circ, 195.9^\circ, 344.1^\circ)$$

$$45. 0.27, 2.08, 3.42, 5.22 (15.7^\circ, 119.3^\circ, 195.7^\circ, 299.3^\circ)$$

$$47. 1.26, 2.51, 3.77, 5.03 (72^\circ, 144^\circ, 216^\circ, 288^\circ)$$

$$49. 5.08(290.8^\circ), 0.56(32.3^\circ)$$

$$51. \frac{\pi}{3} + 2k\pi(60^\circ + k \cdot 360^\circ), \frac{5\pi}{3} + 2k\pi(300^\circ + k \cdot 360^\circ)$$

$$53. \frac{5\pi}{6} + k\pi, (150^\circ + k \cdot 180^\circ)$$

$$55. \frac{5\pi}{4} + 2k\pi(225^\circ + k \cdot 360^\circ), \frac{7\pi}{4} + 2k\pi(315^\circ + k \cdot 360^\circ)$$

$$57. \frac{\pi}{4} + k\pi(45^\circ + k \cdot 180^\circ)$$

$$59. \frac{\pi}{6} + 2k\pi(30^\circ + k \cdot 360^\circ), \frac{5\pi}{6} + 2k\pi(150^\circ + k \cdot 360^\circ)$$

$$61. \frac{2\pi}{3} + 4k\pi(120^\circ + k \cdot 720^\circ), \frac{4\pi}{3} + 4k\pi(240^\circ + k \cdot 720^\circ)$$

$$63. \frac{\pi}{3} + \frac{2k\pi}{3}(60^\circ + k \cdot 120^\circ)$$

$$65. \frac{\pi}{6} + k\frac{\pi}{2}(30^\circ + k \cdot 90^\circ)$$

$$67. \frac{\pi}{6} + k\frac{\pi}{2}(30^\circ + k \cdot 90^\circ), \frac{\pi}{3} + k\frac{\pi}{2}(60^\circ + k \cdot 90^\circ)$$

$$69. \frac{\pi}{3} + k\pi(60^\circ + k \cdot 180^\circ), \frac{2\pi}{3} + k\pi(120^\circ + k \cdot 180^\circ)$$

$$71. \frac{\pi}{12} + k\frac{\pi}{2}(15^\circ + k \cdot 90^\circ)$$

$$73. \frac{\pi}{12} + k\pi(15^\circ + k \cdot 180^\circ), \frac{5\pi}{12} + k\pi(75^\circ + k \cdot 180^\circ)$$

$$75. \frac{\pi}{9} + \frac{2k\pi}{3}(20^\circ + k \cdot 120^\circ), \frac{5\pi}{9} + \frac{2k\pi}{3}(100^\circ + k \cdot 120^\circ)$$

$$77. \frac{2\pi}{3} + 4k\pi(120^\circ + k \cdot 720^\circ)$$

$$79. \frac{7\pi}{6}(210^\circ), \frac{11\pi}{6}(330^\circ), \frac{\pi}{2}(90^\circ)$$

$$81. 0(0^\circ), \pi(180^\circ), \frac{2\pi}{3}(120^\circ), \frac{4\pi}{3}(240^\circ)$$

$$83. 0(0^\circ), \pi(180^\circ), \frac{\pi}{6}(30^\circ), \frac{5\pi}{6}(150^\circ)$$

$$85. 0(0^\circ) \quad 87. \frac{\pi}{3}(60^\circ) \text{ or } \frac{5\pi}{3}(300^\circ)$$

$$89. 0(0^\circ), \frac{3\pi}{2}(270^\circ) \quad 91. \frac{\pi}{2}(90^\circ), \frac{3\pi}{2}(270^\circ), \pi(180^\circ), 0.93(53.1^\circ)$$

$$93. \frac{\pi}{2}, 1.31 \quad 95. 0, 0.55, 0.69, 0.72, 0.66$$

Solutions to trial exercise problems

4. $2 \cos \theta + 1 = 0$

$$2 \cos \theta = -1$$

$$\cos \theta = -\frac{1}{2}$$

$$\theta' = \cos^{-1} \frac{1}{2} = \frac{\pi}{3} (60^\circ)$$

$\cos \theta < 0$ in quadrants II and III, so $\theta = \pi - \theta' =$

$$\pi - \frac{\pi}{3} = \frac{2\pi}{3} (180^\circ - 60^\circ = 120^\circ) \text{ or } \theta = \pi + \theta' =$$

$$\pi + \frac{\pi}{3} = \frac{4\pi}{3} (180^\circ + 60^\circ = 240^\circ).$$

20. $\cos^2 \theta - \frac{1}{2} \cos \theta = 0$

$$\cos \theta (\cos \theta - \frac{1}{2}) = 0$$

Case 1: $\cos \theta = 0$

$$\frac{\pi}{2} (90^\circ), \frac{3\pi}{2} (270^\circ)$$

Case 2: $\cos \theta - \frac{1}{2} = 0$

$$\cos \theta = \frac{1}{2}$$

$\cos \theta > 0$ in quadrants I and IV, so $\theta = \theta' = \frac{\pi}{3} (60^\circ)$

$$\text{or } 2\pi - \theta' = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} (360^\circ - 60^\circ = 300^\circ).$$

27. $\sqrt{3} \tan \theta \cot \theta + \cot \theta = 0$

$$\cot \theta (\sqrt{3} \tan \theta + 1) = 0$$

Case 1: $\cot \theta = 0$

$$\frac{\cos \theta}{\sin \theta} = 0$$

$$\cos \theta = 0$$

$$\frac{\pi}{2} (90^\circ) \text{ or } \frac{3\pi}{2} (270^\circ)$$

Case 2: $\sqrt{3} \tan \theta + 1 = 0$

$$\tan \theta = -\frac{\sqrt{3}}{3}$$

$$\theta' = \tan^{-1} \frac{\sqrt{3}}{3} = \frac{\pi}{6} (30^\circ).$$

$\tan \theta < 0$ in quadrants II and IV, so $\theta = \pi - \theta' =$

$$\pi - \frac{\pi}{6} = \frac{5\pi}{6} (180^\circ - 30^\circ = 150^\circ) \text{ or } 2\pi - \theta' =$$

$$2\pi - \frac{\pi}{6} = \frac{11\pi}{6} (360^\circ - 30^\circ = 330^\circ).$$

33. $\tan x + \cot x = -2$

$$\tan x + \frac{1}{\tan x} = -2$$

$$\tan^2 x + 1 = -2 \tan x$$

Multiply each member by $\tan x$; assume $\tan x \neq 0$.

$$\tan^2 x + 2 \tan x + 1 = 0$$

$$(\tan x + 1)^2 = 0$$

$$\tan x + 1 = 0$$

$$\tan x = -1$$

$$\theta' = \tan^{-1} 1 = \frac{\pi}{4} (45^\circ)$$

Tangent is negative in quadrants II and IV, so $\theta = \pi - \theta' =$

$$\pi - \frac{\pi}{4} = \frac{3\pi}{4} (180^\circ - 45^\circ = 135^\circ) \text{ or } 2\pi - \theta' =$$

$$2\pi - \frac{\pi}{4} = \frac{7\pi}{4} (360^\circ - 45^\circ = 315^\circ).$$

34. $2 - \sin x - \csc x = 0$

$$2 - \sin x - \frac{1}{\sin x} = 0$$

$$2 \sin x - \sin^2 x - 1 = 0$$

$$-\sin^2 x + 2 \sin x - 1 = 0$$

$$\sin^2 x - 2 \sin x + 1 = 0$$

$$(\sin x - 1)^2 = 0$$

$$\sin x - 1 = 0$$

$$\sin x = 1$$

$$x = \frac{\pi}{2} (90^\circ)$$

41. $2 \tan^2 x \sin x = \tan^2 x$

$$2 \tan^2 x \sin x - \tan^2 x = 0$$

$$\tan^2 x (2 \sin x - 1) = 0$$

Case 1: $\tan^2 x = 0$

$$\tan x = 0$$

$$\frac{\sin \theta}{\cos \theta} = 0$$

$$\sin \theta = 0$$

$$0(0^\circ), \pi(180^\circ)$$

Case 2: $2 \sin x - 1 = 0$

$$\sin x = \frac{1}{2}$$

$$\theta' = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$$

$\sin \theta > 0$ in quadrants I and II.

$$\theta = \theta' = \frac{\pi}{6} (30^\circ) \text{ or}$$

$$\pi - \frac{\pi}{6} = \frac{5\pi}{6} (150^\circ)$$

Recall: If $ax^2 + bx + c = 0$, $c \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

46. $\tan^2 x + 5 \tan x + 2 = 0$

$$a = 1, b = 5, c = 2:$$

$$\tan x = \frac{-5 \pm \sqrt{5^2 - 4(2)}}{2} = \frac{-5 \pm \sqrt{17}}{2}$$

Case 1:

$$\tan x = \frac{-5 + \sqrt{17}}{2} \approx -0.4384$$

$$x = \tan^{-1} \left| \frac{-5 + \sqrt{17}}{2} \right| \approx 0.413 (23.7^\circ)$$

$\tan x < 0$ in quadrants II and IV.

$$x = \pi - x' \approx \pi - 0.413 \approx 2.74$$

$$= 2\pi - x' \approx 2\pi - 0.413 \approx 5.87$$

$$x = 180^\circ - x' \approx 180^\circ - 23.7^\circ \approx 156.3^\circ$$

$$= 360^\circ - x' \approx 360^\circ - 23.7^\circ \approx 336.3^\circ$$

Case 2:

$$\tan x = \frac{-5 - \sqrt{17}}{2} \approx -4.5616$$

$$x = \tan^{-1} \left| \frac{-5 - \sqrt{17}}{2} \right| \approx 1.355 (77.6^\circ)$$

$\tan x < 0$ in quadrants II and IV.

$$x = \pi - x' \approx \pi - 1.355 \approx 1.79$$

$$= 2\pi - x' \approx 2\pi - 1.355 \approx 4.93$$

$$x = 180^\circ - x' \approx 180^\circ - 77.6^\circ \approx 102.4^\circ$$

$$= 360^\circ - x' \approx 360^\circ - 77.6^\circ \approx 282.4^\circ$$

57. $\tan x = 1$

$$x = \tan^{-1} 1 = \frac{\pi}{4} (45^\circ)$$

Primary solutions are in quadrants I and III:

$$\frac{\pi}{4} (45^\circ) \text{ and } \frac{5\pi}{4} (225^\circ). \text{ These differ by } \pi (180^\circ), \text{ so we can}$$

write all solutions with one of them: $\frac{\pi}{4} + k\pi (45^\circ + k \cdot 180^\circ)$.

64. $\sec \frac{x}{2} = 1; \cos \frac{x}{2} = 1$

Primary solutions: $\frac{x}{2} = \cos^{-1} 1 = 0 (0^\circ)$

All solutions: $\frac{x}{2} = 0 + 2k\pi (0^\circ + k \cdot 360^\circ)$
 $x = 4k\pi (k \cdot 720^\circ)$

74. $\sin \frac{\theta}{3} = \frac{\sqrt{3}}{2}$

$$\left(\frac{\theta}{3} \right) = \sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3} (60^\circ)$$

Primary solutions: $\frac{\theta}{3} = \frac{\pi}{3} (60^\circ) \text{ or } \frac{2\pi}{3} (120^\circ)$

All solutions: $\frac{\theta}{3} = \frac{\pi}{3} + 2k\pi (60^\circ + k \cdot 360^\circ) \text{ or } \frac{2\pi}{3} + 2k\pi (120^\circ + k \cdot 360^\circ)$

$$\theta = \pi + 6k\pi (180^\circ + k \cdot 1,080^\circ) \text{ or}$$

$$2\pi + 6k\pi (360^\circ + k \cdot 1,080^\circ).$$

81. $\sin 2\theta + \sin \theta = 0$

$$2 \sin \theta \cos \theta + \sin \theta = 0$$

$$\sin \theta (2 \cos \theta + 1) = 0$$

Case 1: $\sin \theta = 0$

$$0 (0^\circ), \pi (180^\circ)$$

Case 2: $2 \cos \theta + 1 = 0$

$$\cos \theta = -\frac{1}{2}$$

$$\frac{2\pi}{3} (120^\circ) \text{ or } \frac{4\pi}{3} (240^\circ)$$

88. $\sin^2 \frac{\theta}{2} = \cos \theta$

$$\left(\pm \sqrt{\frac{1 - \cos \theta}{2}} \right)^2 = \cos \theta$$

$$\frac{1 - \cos \theta}{2} = \cos \theta$$

$$1 - \cos \theta = 2 \cos \theta$$

$$1 = 3 \cos \theta$$

$$\frac{1}{3} = \cos \theta$$

$$\theta' = \cos^{-1} \frac{1}{3}$$

$$\theta = \cos^{-1} \frac{1}{3} \text{ or } 2\pi - \cos^{-1} \frac{1}{3}$$

$$\theta = 1.23 (70.5^\circ) \text{ or } 5.05 (289.5^\circ)$$

92. If $A = 0.855$, $B = 1.052$, and $y = 0$, solve for x to the nearest 0.01.

$$0 = x \cos 0.855 \cos 1.052 - x^2 \cos 0.855 \sin 1.052$$

$$- x^3 \sin 0.855$$

$$0 = 0.32538x - 0.56987x^2 - 0.75457x^3$$

$$x(0.75457x^2 + 0.56987x - 0.32538) = 0$$

$$x = 0 \text{ or } 0.75457x^2 + 0.56987x - 0.32538 = 0$$

Solve the quadratic equation with the quadratic formula.

$$x = 0, -1.14, 0.38$$

93. If $B = 0.7$, $x = 2$, and $y = -8$, find A to the nearest 0.01.

$$-8 = 2 \cos A \cos 0.7 - 4 \cos A \sin 0.7 - 8 \sin A$$

$$-8 = (2 \cos 0.7) \cos A - (4 \sin 0.7) \cos A - 8 \sin A$$

$$-8 = 1.5297 \cos A - 2.5769 \cos A - 8 \sin A$$

$$-8 = -1.0472 \cos A - 8 \sin A$$

$$8 \sin A - 8 = -1.0472 \cos A$$

$$\sin A - 1 = -0.1309 \cos A$$

Divide each member by 8.

$$\sin A = 1 - 0.1309 \cos A$$

$$(\sin A)^2 = (1 - 0.1309 \cos A)^2$$

$$\sin^2 A = 1 - 0.2618 \cos A + 0.017134 \cos^2 A$$

$$1 - \cos^2 A = 1 - 0.2618 \cos A + 0.017134 \cos^2 A$$

$$0 = 1.0171 \cos^2 A - 0.2618 \cos A$$

$$0 = \cos A (1.0171 \cos A - 0.2618)$$

$$\cos A = 0 \text{ or } 1.0171 \cos A - 0.2618 = 0$$

$$A = \cos^{-1} 0 \text{ or } 1.0171 \cos A = 0.2618$$

$$A = \frac{\pi}{2} \text{ or } \cos A = 0.25739$$

$$A \approx 1.310476103$$

Thus, A is $\frac{\pi}{2}$ or 1.31

96. Find θ when $\mathcal{A} = 0.5 \text{ m}^2$. Round the answer to the nearest 0.1° .

$$\sin \theta \sqrt{1.44 - \sin^2 \theta} = 0.5$$

$$(\sin \theta \sqrt{1.44 - \sin^2 \theta})^2 = 0.5^2$$

$$\sin^2 \theta (1.44 - \sin^2 \theta) = 0.25$$

$$1.44 \sin^2 \theta - \sin^4 \theta = 0.25$$

$$\sin^4 \theta - 1.44 \sin^2 \theta + 0.25 = 0$$

$$\text{Let } u = \sin^2 \theta.$$

$$u^2 - 1.44u + 0.25 = 0$$

$$u \approx 1.2381 \quad \text{or} \quad u \approx 0.20193$$

$$\sin^2 \theta \approx 1.2381 \quad \text{or} \quad \sin^2 \theta \approx 0.20193$$

$$\sin \theta \approx \pm \sqrt{1.2381} \quad \text{or} \quad \sin \theta \approx \pm \sqrt{0.20193}$$

$$\sin \theta \approx \pm 1.1127 \quad \text{or} \quad \sin \theta \approx \pm 0.44936$$

$$\text{no solution} \quad \theta \approx \pm 26.7^\circ$$

Because $\theta \geq 0$, we choose the positive value for θ .

$$\theta \approx 26.7^\circ$$

Chapter 5 review

$$1. \frac{\frac{\cos \theta}{\sin \theta}}{\frac{\cos \theta}{\sin \theta}}$$

$$\frac{1}{\sin \theta}$$

$$\csc \theta$$

$$3. \frac{\tan^2 \theta}{\tan^2 \theta}$$

$$1$$

$$\sin^4 \theta \frac{1}{\sin^4 \theta}$$

$$\sin^4 \theta \csc^4 \theta$$

$$5. \frac{\csc \theta \frac{\sin \theta}{\cos \theta}}{\sin \theta}$$

$$\csc \theta \frac{1}{\cos \theta}$$

$$\csc \theta \sec \theta$$

$$7. \frac{\frac{1}{\sin \theta} - \frac{1}{\cos \theta}}{\frac{\cos \theta - \sin \theta}{\sin \theta \cos \theta}}$$

$$9. \frac{\frac{1}{\cos \theta}}{\frac{\sin \theta}{\cos \theta} - \frac{\cos \theta}{\sin \theta}}$$

$$\frac{1}{\cos \theta}$$

$$\frac{1}{\sin^2 \theta - \cos^2 \theta}$$

$$\frac{1}{\cos \theta \sin \theta}$$

$$\frac{1}{\sin^2 \theta - \cos^2 \theta}$$

$$\frac{\sin \theta}{\sin \theta}$$

$$\sin^2 \theta - \cos^2 \theta$$

$$2. \sec \theta \frac{\sin \theta}{\cos \theta}$$

$$\sec \theta \left(\frac{1}{\cos \theta} \right) \sin \theta$$

$$\sec \theta \sec \theta \sin \theta$$

$$\sec^2 \theta \sin \theta$$

$$4. \frac{\cot^2 \theta}{\tan^2 \theta}$$

$$\frac{\cos^2 \theta}{\sin^2 \theta}$$

$$\frac{\sin^2 \theta}{\sin^2 \theta}$$

$$\frac{\cos^2 \theta}{\cos^2 \theta}$$

$$\frac{\cos^4 \theta}{\sin^4 \theta}$$

$$\cot^4 \theta$$

$$6. \frac{\sin^2 \theta - (1 - \sin^2 \theta)}{\sin^2 \theta - 1 + \sin^2 \theta}$$

$$2 \sin^2 \theta - 1$$

$$8. \frac{\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}}{\frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta}}$$

$$\frac{1}{\cos \theta \sin \theta}$$

$$10. \frac{1 - \frac{\cos \theta}{\sin \theta}}{\frac{1}{\sin \theta} + 1}$$

$$\frac{\sin \theta - \cos \theta}{\sin \theta}$$

$$\frac{1 + \sin \theta}{\sin \theta}$$

$$\frac{\sin \theta - \cos \theta}{1 + \sin \theta}$$

$$\frac{\sin \theta - \cos \theta}{1 + \sin \theta}$$

$$\frac{\sin \theta - \cos \theta}{1 + \sin \theta}$$

$$\frac{\sin \theta - \cos \theta}{1 + \sin \theta}$$

$$\frac{\sin \theta - \cos \theta}{1 + \sin \theta}$$

$$11. \frac{\frac{1}{\cos^2 \theta} - 1}{\frac{1}{\cos^2 \theta}}$$

$$\frac{1 - \cos^2 \theta}{\cos^2 \theta}$$

$$\frac{1}{\cos^2 \theta}$$

$$1 - \cos^2 \theta$$

$$13. \csc x - \tan x \cot x$$

$$\csc x - \tan x \left(\frac{1}{\tan x} \right)$$

$$\csc x - 1$$

$$14. \sin^2 x + \sin^2 x \cot^2 x$$

$$\sin^2 x + \sin^2 x \frac{\cos^2 x}{\sin^2 x}$$

$$\sin^2 x + \cos^2 x$$

$$15. \csc^2 x \sec^2 x (\cos^2 x - \sin^2 x)$$

$$\left(\frac{1}{\sin^2 x} \right) \left(\frac{1}{\cos^2 x} \right) (\cos^2 x - \sin^2 x)$$

$$\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x}$$

$$16. \frac{1}{\csc x - \cot x}$$

$$\frac{1}{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}$$

$$\frac{1}{1 - \cos x}$$

$$17. \sec x (\sin x - \cos x)$$

$$\frac{1}{\cos x} (\sin x - \cos x)$$

$$\frac{\sin x}{\cos x} - \frac{\cos x}{\cos x}$$

$$\tan x - 1$$

$$18. \frac{\csc^2 x - 1}{\sin^2 x}$$

$$\frac{\cot^2 x}{\sin^2 x}$$

$$\frac{\cos^2 x}{\sin^2 x}$$

$$\frac{\sin^2 x}{\sin^2 x}$$

$$\frac{\cos^2 x}{\sin^4 x}$$

$$\cos^2 x \csc^4 x$$

$$19. \frac{1}{1 + \csc x} + \frac{1}{1 - \csc x}$$

$$\frac{1 - \csc x + 1 + \csc x}{1 - \csc^2 x}$$

$$\frac{2}{1 - \csc^2 x}$$

$$\frac{2}{-1(\csc^2 x - 1)}$$

$$\frac{-2}{\cot^2 x}$$

$$20. \sin^2 x + \sin^2 x \cos^2 x$$

$$\sin^2 x (1 + \cos^2 x)$$

$$(1 - \cos^2 x)(1 + \cos^2 x)$$

$$1 - \cos^4 x$$

$$21. \frac{\sec^2 x}{\cot^2 x}$$

$$\frac{\tan^2 x + 1}{\cot^2 x}$$

$$\frac{\tan^2 x + 1}{\frac{1}{\tan^2 x}}$$

$$\tan^4 x + \tan^2 x$$

$$22. \frac{1 - \cot x}{1 + \csc x}$$

$$1 - \frac{\cos x}{\sin x}$$

$$1 + \frac{1}{\sin x}$$

$$\frac{\sin x - \cos x}{\sin x}$$

$$\frac{\sin x + 1}{\sin x}$$

$$\frac{\sin x - \cos x}{\sin x + 1}$$

$$23. \sin \frac{\pi}{6} + \cos \frac{\pi}{6}$$

$$\frac{1}{2} + \frac{\sqrt{3}}{2}$$

$$\frac{1 + \sqrt{3}}{2} \neq 1$$

$$24. \tan \frac{\pi}{4} - \sin \frac{\pi}{4} \cos \frac{\pi}{4}$$

$$1 - \left(\frac{\sqrt{2}}{2} \right) \left(\frac{\sqrt{2}}{2} \right)$$

$$1 - \frac{1}{2}$$

$$\frac{1}{2} \neq 0$$

$$25. \sec \frac{\pi}{4} = \sqrt{2}$$

$$\frac{1}{\cot \frac{\pi}{4} - \csc \frac{\pi}{4}} = \frac{1}{1 - \sqrt{2}}$$

$$= \frac{1 + \sqrt{2}}{-1} = -1 - \sqrt{2} \neq \sqrt{2}$$

26. $\frac{\sqrt{2} + \sqrt{6}}{4}$ 27. $\sqrt{3} - 2$
28. $\frac{\sqrt{6} + \sqrt{2}}{4}$ 29. $-2 + \sqrt{3}$
30. a. $\frac{-15 + \sqrt{77}}{24}$ b. $\frac{-15 - \sqrt{77}}{5\sqrt{7} - 3\sqrt{11}}$
31. a. $\frac{3\sqrt{5} + 1}{8}$ b. $\frac{\sqrt{15} + \sqrt{3}}{1 - 3\sqrt{5}}$
32. a. $\frac{204}{8\sqrt{119} + 75}$
b. $\frac{952 - 75\sqrt{119}}{-40\sqrt{119} - 1,785}$
33. $\frac{\sin(\alpha + \beta)}{\cos(\alpha - \beta)}$
 $\frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta + \sin \alpha \sin \beta}$
 $\frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\sin \alpha \sin \beta + \sin \alpha \sin \beta}$
 $\frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta}{\sin \alpha \sin \beta + \sin \alpha \sin \beta}$
 $\frac{\cot \beta + \cot \alpha}{\cot \alpha \cot \beta + 1}$
34. $\cos\left(\frac{3\pi}{2} + \theta\right)$
 $\cos \frac{3\pi}{2} \cos \theta - \sin \frac{3\pi}{2} \sin \theta$
 $(0) \cos \theta - (-1) \sin \theta$
 $\sin \theta$
35. $\sin\left(\frac{\pi}{4} - \theta\right)$
 $\sin \frac{\pi}{4} \cos \theta - \cos \frac{\pi}{4} \sin \theta$
 $\frac{\sqrt{2}}{2} \cos \theta - \frac{\sqrt{2}}{2} \sin \theta$
 $\frac{\sqrt{2}}{2} (\cos \theta - \sin \theta)$
36. $\frac{\cos(\alpha + \beta)}{\sin \alpha \cos \beta}$
 $\frac{\cos \alpha \cos \beta - \sin \alpha \sin \beta}{\sin \alpha \cos \beta}$
 $\frac{\cos \alpha \cos \beta}{\sin \alpha \cos \beta} - \frac{\sin \alpha \sin \beta}{\sin \alpha \cos \beta}$
 $\frac{\cos \alpha}{\sin \alpha} - \frac{\sin \beta}{\cos \beta}$
 $\cot \alpha - \tan \beta$
37. $\cot(\theta + \pi)$
 $\frac{1}{\tan(\theta + \pi)}$
 $\frac{1}{\frac{\tan \theta + \tan \pi}{1 - \tan \theta \tan \pi}}$
 $\frac{1}{\frac{\tan \theta + (0)}{1 - \tan \theta (0)}}$
 $\frac{1}{\tan \theta}$
 $\cot \theta$
38. 124° 39. 10π 40. $\frac{7\pi}{6}$ 41. 12°
42. $3 \sin \theta$ 43. $2 \tan 8\theta$
44. a. $-\frac{5\sqrt{119}}{47}$ b. $\frac{5\sqrt{119}}{72}$
45. a. $-\frac{7}{25}$
46. a. $-\frac{41}{40}$ b. $\frac{9}{40}$
47. $\sin 2x - \cos x$
 $2 \sin x \cos x - \cos x$
 $\cos x(2 \sin x - 1)$
48. $1 + \cos 2x$
 $1 + \cos^2 x - \sin^2 x$
 $1 + \cos^2 x - (1 - \cos^2 x)$
 $1 + \cos^2 x - 1 + \cos^2 x$
 $2 \cos^2 x$
49. $\frac{\cos 2x}{2 - 4 \sin^2 x}$
 $\frac{\cos^2 x - \sin^2 x}{2(1 - 2 \sin^2 x)}$
 $\frac{\cos^2 x - \sin^2 x}{2(\sin^2 x + \cos^2 x - 2 \sin^2 x)}$
 $\frac{\cos^2 x - \sin^2 x}{2(\cos^2 x - \sin^2 x)}$
 $\frac{1}{2}$
50. $\frac{2 \cot x}{\cot^2 x - 1}$
 $\frac{2}{\tan x}$
 $\frac{1}{\tan^2 x} - 1$
 $\frac{2}{\tan x}$
 $\frac{1 - \tan^2 x}{\tan^2 x}$
 $\frac{2 \tan x}{1 - \tan^2 x}$
 $\tan 2x$
51. $\sin 2x - \cos 2x$
 $2 \sin x \cos x - (\cos^2 x - \sin^2 x)$
 $2 \sin x \cos x - \cos^2 x + \sin^2 x$
 $2 \sin x \cos x - \cos^2 x + 1 - \cos^2 x$
 $2 \sin x \cos x - 2 \cos^2 x + 1$
 $2 \cos x(\sin x - \cos x) + 1$
52. $\sqrt{2} - 1$ 53. $\frac{\sqrt{2 + \sqrt{3}}}{2}$
54. $\frac{\sqrt{2 + \sqrt{3}}}{2}$ 55. $\frac{\sqrt{2 - \sqrt{2}}}{2}$
56. a. $\frac{\sqrt{26}}{26}$ b. $\frac{1}{5}$ 57. a. $-\sqrt{\frac{6}{3 - \sqrt{5}}}$
58. $\frac{5}{6}$
59. $\cot \frac{\theta}{2}$
 $\frac{1}{\tan \frac{\theta}{2}}$
 $\frac{1}{\frac{\sin \theta}{1 + \cos \theta}}$
 $\frac{1 + \cos \theta}{\sin \theta}$
60. $\sec^2 \frac{\theta}{2} - \tan^2 \frac{\theta}{2}$
 $\left(1 + \tan^2 \frac{\theta}{2}\right) - \tan^2 \frac{\theta}{2}$
 $1 + \tan^2 \frac{\theta}{2} - \tan^2 \frac{\theta}{2}$
 1
61. $\tan \frac{\theta}{2} \csc^2 \frac{\theta}{2}$
 $\frac{\sin \theta}{1 + \cos \theta} \cdot \frac{2}{1 - \cos \theta}$
 $\frac{2 \sin \theta}{1 - \cos^2 \theta}$
 $\frac{2 \sin \theta}{\sin^2 \theta}$
 $\frac{2}{\sin \theta}$
62. $\frac{\pi}{6}$ 63. $\frac{\pi}{3}$
64. $\frac{\pi}{3}, \frac{\pi}{2}$ 65. $\frac{\pi}{6}, \frac{\pi}{3}$
66. $45^\circ, 90^\circ, 225^\circ, 270^\circ$ 67. $60^\circ, 120^\circ, 240^\circ, 300^\circ$ 68. $0^\circ, 120^\circ, 240^\circ$ 69. $30^\circ, 150^\circ, 270^\circ$ 70. $30^\circ, 90^\circ, 150^\circ, 270^\circ$
71. $120^\circ, 180^\circ, 240^\circ$ 72. $k\pi$

73. $1.31 + k\pi, -0.67 + k\pi$
 74. $3.52 + 2k\pi, 5.90 + 2k\pi$
 75. $\frac{\pi}{3} + 2k\pi, \pi + 2k\pi, \frac{5\pi}{3} + 2k\pi$
 76. $\frac{\pi}{24}, \frac{5\pi}{24}, \frac{13\pi}{24}, \frac{17\pi}{24}, \frac{25\pi}{24}, \frac{29\pi}{24},$
 $\frac{37\pi}{24}, \frac{41\pi}{24}$ 77. $\frac{\pi}{3}$ 78. $\frac{\pi}{2}$ 79. π
 80. $12^\circ, 60^\circ, 84^\circ, 132^\circ, 156^\circ, 204^\circ, 228^\circ,$
 $276^\circ, 300^\circ, 348^\circ$ 81. 240° 82. $7.5^\circ,$
 $37.5^\circ, 67.5^\circ, 97.5^\circ, 127.5^\circ, 157.5^\circ, 187.5^\circ,$
 $217.5^\circ, 247.5^\circ, 277.5^\circ, 307.5^\circ, 337.5^\circ,$
 $22.5^\circ, 52.5^\circ, 82.5^\circ, 112.5^\circ, 142.5^\circ, 172.5^\circ,$
 $202.5^\circ, 232.5^\circ, 262.5^\circ, 292.5^\circ, 322.5^\circ,$
 352.5° 83. $90^\circ, 210^\circ, 270^\circ, 330^\circ$
 84. $\frac{\pi}{3} + 2k\pi, \pi + 2k\pi, \frac{5\pi}{3} + 2k\pi$
 85. $0.13 + k \cdot \frac{\pi}{4}, 1.22 + k \cdot \frac{2\pi}{3}, 1.92 + k$
 $\cdot \frac{2\pi}{3}$ 86. $\frac{\pi}{4} + k\pi, 1.94 + k\pi, 2.78 + k\pi$

Chapter 5 test

1. $\csc^2 x \sin x \cos x$
 $\frac{1}{\sin^2 x} \sin x \cos x$
 $\frac{\cos x}{\sin x}$
 $\cot x$
 $\frac{1}{1} - \frac{1}{1}$
 2. $\frac{\sin x}{\sin x} - \frac{\cos x}{\cos x}$
 $\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$
 $\frac{\cos x - \sin x}{\sin x \cos x}$
 $\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}$
 $\frac{\cos x - \sin x}{\sin^2 x + \cos^2 x}$
 $\frac{\cos x - \sin x}{\cos x - \sin x}$
 3. $\cot x - 2 \tan x$
 $\cot 45^\circ - 2 \tan 45^\circ$
 $1 - 2(1)$
 -1
 $-1 \neq 1$
 4. $\frac{8\sqrt{3} - 15}{34}$ 5. $-\frac{240}{289}$
 6. 3 7. $\frac{2}{\sqrt{2} + \sqrt{2}}$

8. a. $\frac{1 + \cot \theta}{\csc \theta}$
 $\frac{1}{\csc \theta} + \frac{\cot \theta}{\csc \theta}$
 $\frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta}$
 $\frac{1}{\sin \theta} + \frac{\sin \theta}{\sin \theta}$
 $\frac{1 + \sin^2 \theta}{\sin \theta}$
 b. $\frac{\cos^2 x - 1}{\sin^2 x}$
 $\frac{-(1 - \cos^2 x)}{\sin^2 x}$
 $\frac{-\sin^2 x}{\sin^2 x}$
 -1
 c. $\cos\left(\theta - \frac{3\pi}{2}\right)$
 $\cos \theta \cos \frac{3\pi}{2} + \sin \theta \sin \frac{3\pi}{2}$
 $\cos \theta(0) + \sin \theta(-1)$
 $-\sin \theta$
 d. $-1 + 2 \cos x(\cos x - \sin x)$
 $-(\sin^2 x + \cos^2 x) + 2 \cos^2 x$
 $-2 \sin x \cos x$
 $\cos^2 x - \sin^2 x - 2 \sin x \cos x$
 $\cos 2x - \sin 2x$
 9. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
 10. $30^\circ, 120^\circ, 210^\circ, 240^\circ$
 11. $0.17 + 2\pi k, 2.97 + 2\pi k, \frac{3\pi}{2} + 2\pi k$
 12. $18^\circ, 54^\circ, 90^\circ, 126^\circ, 162^\circ, 198^\circ, 234^\circ,$
 $270^\circ, 306^\circ, 342^\circ$ 13. $\pi + 8k\pi,$
 $7\pi + 8k\pi, 3\pi + 8k\pi, 5\pi + 8k\pi$ 14. $\pm m$

Chapter 6

Exercise 6–1

Answers to odd-numbered problems

1. $C = 96^\circ, b \approx 16.4, c \approx 21.7$
 3. $A = 34.4^\circ, b \approx 0.52, c \approx 1.64$
 5. $C = 33^\circ, c \approx 51.2, a \approx 68.7$
 7. $B = 20.6^\circ, b \approx 0.172, c \approx 0.489$
 9. $C = 35^\circ, a \approx 8.58, b \approx 6.16$
 11. $A \approx 45.6^\circ, C \approx 85.4^\circ, c \approx 17.4$
 13. $B \approx 18.0^\circ, C \approx 30.0^\circ, b \approx 1.77$
 15. $C \approx 47.4^\circ, A \approx 88.9^\circ, a \approx 133.9$
 $C \approx 132.6^\circ, A \approx 3.7^\circ, a \approx 8.7$
 17. no solution

19. $B \approx 85.0^\circ, A \approx 52.7^\circ, a \approx 5.07$
 $B \approx 95.01^\circ, A \approx 42.7^\circ, a \approx 4.32$
 21. $C \approx 26.23^\circ, A \approx 108.77^\circ, a \approx 10.71$
 23. 32.3 miles 25. 843,400 miles
 27. 15.8 knots 29. 25 miles
 31. By the definitions of section 2–3,

$\tan A = \frac{y}{x}$ in each figure. By the

trigonometric ratios (for a right triangle) it can be seen in each case

that $\tan C = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{b-x}$. (Note

that x is negative in the right figure, so that $b-x$ is larger than b itself.) Also, as noted in the problem, $y = h$. Putting these values in the expression

$\frac{b \cdot \tan A \cdot \tan C}{\tan A + \tan C}$ we obtain:

$$\frac{b \cdot \frac{y}{x} \cdot \frac{y}{b-x}}{\frac{y}{x} + \frac{y}{b-x}} = \frac{\frac{by^2}{x(b-x)}}{\frac{yb}{x(b-x)}} = y = h$$

33. Let (x, y) be the point at B . It is on the terminal side of angle A . Then $\cos A = \frac{x}{r}$, where r is the length of AB . But

then $r = c$, so $\cos A = \frac{x}{c}$, and so c

$\cos A = x$. Using right triangles we see

that in each figure $\cos C = \frac{b-x}{a}$, so

that $a \cos C = b - x$. Note that when A is obtuse (the right-hand figure) x is negative, so $b - x$ is the length of $|b| + |x|$. Proceeding with the information above:

$$\begin{aligned} c \cos A &= x \\ a \cos C &= b - x \\ a \cos C + c \cos A &= x + (b - x) \\ a \cos C + c \cos A &= b \end{aligned}$$

Thus, (2) is true.

(1) can be shown to be true by putting angle B in standard position with angle C on the x -axis and proceeding in the same manner. (2) is done with angle B in standard position and angle A on the x -axis. Actually, this is not really necessary since the labeling in a triangle is arbitrary, and thus, for example, we could obtain (1) by changing the label B to A , C to B , and A to C , and labeling the sides appropriately.

35. Consider any triangle ABC ; place it as shown in the figure for problem 33, so angle A is in standard position. The figure covers the cases where A is acute, right, or obtuse. Then it can be seen that if h is the height of the triangle, then $h = x$. We know that

$$\sin A = \frac{x}{c} = \frac{h}{c}, \text{ so } h = c \sin A.$$

Then the area is $\frac{1}{2}bh = \frac{1}{2}b(c \sin A) = \frac{1}{2}bc \sin A$.

37. a. It can be seen that the sum of the area of the four triangles shown in the figure is $\frac{1}{2}ab \sin A + \frac{1}{2}cd \sin C + \frac{1}{2}ad \sin D + \frac{1}{2}bc \sin B$. This total is twice as large as the total area of the four-sided figure, so the area of the four-sided figure is $\frac{1}{2}(\frac{1}{2}ab \sin A + \frac{1}{2}cd \sin C + \frac{1}{2}ad \sin D + \frac{1}{2}bc \sin B)$ or $\frac{1}{4}(ab \sin A + ad \sin D + bc \sin B + cd \sin C)$.

b. The difference between the Egyptian formula $\frac{1}{4}(ab + ad + bc + cd)$ and the correct formula $\frac{1}{4}(ab \sin A + ad \sin D + bc \sin B + cd \sin C)$ is the factors $\sin A$, $\sin B$, $\sin C$, and $\sin D$. The value of the sine of each angle is between 0 and 1. Thus,

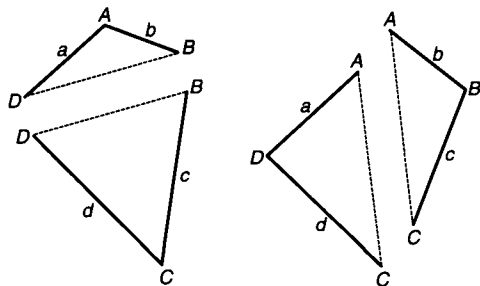
$$ab \geq ab \sin A$$

$$ad \geq ad \sin D$$

$$bc \geq bc \sin B$$

$$cd \geq cd \sin C$$

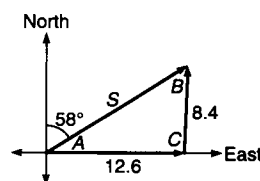
$ab + ad + bc + cd \geq ab \sin A + ad \sin D + bc \sin B + cd \sin C$,
 $\frac{1}{4}(ab + ad + bc + cd) \geq \frac{1}{4}(ab \sin A + ad \sin D + bc \sin B + cd \sin C)$
 If the figure is a rectangle $A = B = C = D = 90^\circ$, and $\sin A = \sin B = \sin C = \sin D = 1$, so both expressions give the same value.



Solutions to trial exercise problems

7. $a = 0.452$, $A = 67.6^\circ$, $C = 91.8^\circ$
 $B = 180^\circ - 67.6^\circ - 91.8^\circ = 20.6^\circ$
 $\frac{\sin 67.6^\circ}{0.452} = \frac{\sin 20.6^\circ}{b} = \frac{\sin 91.8^\circ}{c}$
 $\frac{\sin 67.6^\circ}{0.452} = \frac{\sin 20.6^\circ}{b}$
 $b = \frac{0.452 \sin 20.6^\circ}{\sin 67.6^\circ}$
 $b \approx 0.172$
 $\frac{\sin 67.6^\circ}{0.452} = \frac{\sin 91.8^\circ}{c}$
 $c = \frac{0.452 \sin 91.8^\circ}{\sin 67.6^\circ}$
 $c \approx 0.489$
 13. $a = 4.25$, $c = 2.86$, $A = 132^\circ$
 $\frac{\sin 132^\circ}{4.25} = \frac{\sin B}{b} = \frac{\sin C}{2.86}$
 $\frac{\sin 132^\circ}{4.25} = \frac{\sin C}{2.86}$ so $\sin C = \frac{2.86 \sin 132^\circ}{4.25}$,
 $C' = \sin^{-1} \frac{2.86 \sin 132^\circ}{4.25} \approx 30.01^\circ$
 $C \approx 30.01^\circ$ or $180^\circ - 30.01^\circ \approx 149.99^\circ$
 Case 1: $C \approx 30.01^\circ$
 $B \approx 180^\circ - 132^\circ - 30.01^\circ \approx 17.99^\circ$
 $\frac{\sin 132^\circ}{4.25} = \frac{\sin 17.99^\circ}{b}$
 $b \approx 1.77$
 Thus, the solution is
 $B \approx 18.0^\circ$, $C \approx 30.0^\circ$, $b \approx 1.77$.
 Case 2: $C \approx 149.99^\circ$
 $B = 180^\circ - 132^\circ - 149.99^\circ \approx -101.99^\circ$
 (No solution.)

15. $b = 92.5$, $c = 98.6$, $B = 43.7^\circ$
 $\frac{\sin A}{a} = \frac{\sin 43.7^\circ}{92.5} = \frac{\sin C}{98.6}$
 $\frac{\sin 43.7^\circ}{92.5} = \frac{\sin C}{98.6}$, $\sin C = \frac{98.6 \sin 43.7^\circ}{92.5}$
 $C' = \sin^{-1} \frac{98.6 \sin 43.7^\circ}{92.5} \approx 47.429^\circ$
 $C \approx 47.429^\circ$ or $180^\circ - 47.429^\circ \approx 132.571^\circ$
 Case 1: $C \approx 47.429^\circ$
 $A = 180^\circ - 43.7^\circ - 47.429^\circ \approx 88.871^\circ$
 $\frac{\sin 88.871^\circ}{a} = \frac{\sin 43.7^\circ}{92.5}$
 $a \approx 133.86$
 Solution 1: $C \approx 47.4^\circ$
 $A \approx 88.9^\circ$
 $a \approx 133.9$
 Case 2: $C \approx 132.571^\circ$
 $A = 180^\circ - 43.7^\circ - 132.571^\circ \approx 3.729^\circ$
 $\frac{\sin 3.729^\circ}{a} = \frac{\sin 43.7^\circ}{92.5}$
 $a \approx 133.60$
 Solution 2: $C \approx 132.6^\circ$
 $A \approx 3.7^\circ$
 $a \approx 8.7$
 27. Angle $A = 90^\circ - 58^\circ = 32^\circ$.
 $\frac{\sin 32^\circ}{8.4} = \frac{\sin B}{12.6}$, $\sin B = \frac{12.6 \sin 32^\circ}{8.4}$,
 so $B' \approx 52.64^\circ$. Thus,
 $C \approx 180^\circ - 32^\circ - 52.64^\circ \approx 95.36^\circ$.
 $\frac{\sin 32^\circ}{8.4} = \frac{\sin 95.36^\circ}{S}$; $S \approx 15.78$.
 Thus, $S \approx 15.8$ knots.



Exercise 6-2

Answers to odd-numbered problems

1. $5\sqrt{5}$ 3. $\sqrt{226}$ 5. $9\sqrt{2}$
 7. $c = 4.0$, $A = 30.7^\circ$, $B = 109.9^\circ$
 9. $a = 77.2$, $C = 14.9^\circ$, $B = 41.1^\circ$
 11. $b = 38.3$, $C = 25.9^\circ$, $A = 53.8^\circ$
 13. $C = 109.0^\circ$, $B = 31.6^\circ$, $A = 39.4^\circ$
 15. $C = 105.3^\circ$, $A = 24.4^\circ$, $B = 50.3^\circ$
 17. $c = 18.1$, $A = 28.3^\circ$, $B = 12.3^\circ$
 19. $a = 28.1$, $C = 40.5^\circ$, $B = 115.0^\circ$
 21. $b = 41.8$, $C = 26.3^\circ$, $A = 41.7^\circ$
 23. $C = 110.9^\circ$, $B = 30.3^\circ$, $A = 38.3^\circ$
 25. $c = 0.28$, $A = 1.12^\circ$, $B = 177.38^\circ$
 27. 326.9 feet 29. 82.4° 31. 125.5°
 33. 102.9° 35. 40 miles 37. 75.0°

Exercise 6–3

Answers to odd-numbered problems

1. $(20\sqrt{3}, 20)$ 3. $(-53.4, 84.5)$
 5. $(-9.4, -3.4)$ 7. $\left(12\frac{1}{2}, -\frac{25\sqrt{3}}{2}\right)$
 9. $(3\sqrt{2}, -3\sqrt{2})$ 11. $(-0.8, 7.8)$
 13. $(5, 53.1^\circ)$ 15. $(6.0, 120.0^\circ)$
 17. $(2.6, -49.1^\circ)$ 19. $(11.2, -63.4^\circ)$
 21. $(7.6, 153.4^\circ)$ 23. $(3.16, -116.6^\circ)$
 25. $(-1, 20)$ 27. $(8\sqrt{2}, 0)$
 29. $(45.2, 31.2^\circ)$ 31. $(36.5, -122.1^\circ)$
 33. $(9.0, -54.2^\circ)$ 35. $(47.1, 139.7^\circ)$
 37. $(6.3, 89.3^\circ)$ 39. $(20.3, 83.9^\circ)$
 41. $(12.9, 21.0^\circ)$ 43. The aircraft is moving west at 136 knots and north at 63 knots. 45. The aircraft is flying east at 193 knots and north at 52 knots.
 47. -228 knots = east-west component
 395 knots = north-south component
 49. a. 24 nm b. 38 nm 51. No; the force vector is $(2, 250, 33^\circ)$. The horizontal component f is $2,250 \cos 33^\circ \approx 1,887$ pounds. This is not enough to move the sled.

53. Force V	Horizontal component V_x	Vertical component V_y
$(1,000, 15^\circ)$	966	259
$(2,000, 15^\circ)$	1,932	518
$(1,000, 30^\circ)$	866	500

Part (a).

Part (b); yes, the components double in value.

Part (c); no, the components do not double.

55. The plane is 40 miles from Orlando, in a direction 59° south of east.
 57. $(15.4, -76.8^\circ)$ 59. $(270.7, 87.5^\circ)$
 61. The ship is about 63 knots from its starting position, at an angle of 40° south of west. 63. The plane's ground speed is 99 knots, in the direction 47° north of west.
 65. The ship is traveling at 26.5 knots in a direction 25.7° north of east.
 67. Magnitude is 266 volts at 341° .
 69. $(87, 98^\circ)$ The heading of the aircraft is 8° west of north, and its airspeed is 87 knots. 71. $(17.7, 76.3^\circ)$ The ships heading is 76.3° , and its speed is 17.7 knots. 73. $(320, 130^\circ)$ The tension in the second cable is 320 pounds, and it makes an angle θ of 50° ($180^\circ - 130^\circ$) with the horizontal.

75. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$ be three vectors in rectangular form. Then,

$$(A + B) + C$$

$$= [(a_1, a_2) + (b_1, b_2)] + (c_1, c_2)$$

The parentheses indicate we add A and B first.

$$= (a_1 + b_1, a_2 + b_2) + (c_1, c_2)$$

Two vectors; one is $A + B$, and the other is C .

$$= ((a_1 + b_1) + c_1, (a_2 + b_2) + c_2)$$

One vector: $(A + B) + C$.

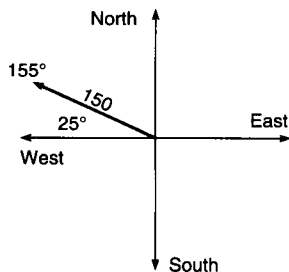
$$= (a_1 + (b_1 + c_1), a_2 + (b_2 + c_2))$$

We can rearrange because real numbers are associative, and the components are real numbers.

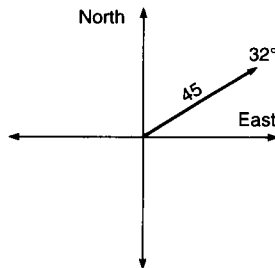
$$= A + (B + C)$$

Solutions to trial exercise problems

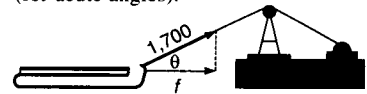
3. $(100.0, 122.3^\circ) = (100 \cos 122.3^\circ, 100 \sin 122.3^\circ)$
 $\approx (-53.4, 84.5)$
 23. $(-\sqrt{2}, -\sqrt{8})$ $|A| = \sqrt{(-\sqrt{2})^2 + (-\sqrt{8})^2} \approx \sqrt{10} \approx 3.16$,
 $\theta' \approx 63.43^\circ$; $\theta = 63.43^\circ - 180^\circ \approx -116.57^\circ$ ($3.16, -116.6^\circ$)
 39. $(15.3, 311^\circ) = (15.3 \cos 311^\circ, 15.3 \sin 311^\circ) = (10.038, -11.547)$
 $(20.9, 117^\circ) = (20.9 \cos 117^\circ, 20.9 \sin 117^\circ) = (-9.488, 18.622)$
 $(13.2, 83^\circ) = (13.2 \cos 83^\circ, 13.2 \sin 83^\circ) = (1.609, 13.102)$
 $= (2.158, 20.177) = (20.3, 83.9^\circ)$
 43. $V = (150, 155^\circ)$; $V_x = 150 \cos 155^\circ \approx -136$ (136 knots due west); $V_y = 150 \sin 155^\circ \approx 63$ (63 knots due north). The aircraft is moving west at 136 knots and north at 63 knots.



49. 18 knots (18 nautical miles per hour) $\times 2.5$ hours = 45 nm (nautical miles);
 $V = (45, 32^\circ)$; $V_x = 45 \cos 32^\circ \approx 38$ nm; distance east of the harbor (part b); $V_y = 45 \sin 32^\circ \approx 24$ nm; distance north of the harbor (part a).



52. $f = 1,700 \cos \theta$. We require $f \geq 1,200$, so $1,200 \geq 1,700 \cos \theta$, or $\frac{12}{17} \geq \cos \theta$. $\cos^{-1} \frac{12}{17} \approx 45.1^\circ$. Thus, $\theta \leq 45.1^\circ$ will move the sled. Note that $\theta \leq 45.1^\circ$ is correct, and not $\theta \geq 45.1^\circ$. This can be seen in the figure. If θ increases, f clearly decreases. Mathematically, the value of $\cos \theta$ increases as θ decreases (for acute angles).

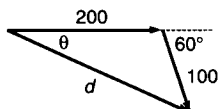


54. At 200 mph, the first leg of the trip is 200 miles. The second is $200 \text{ mph} \times 0.5 \text{ hour} = 100 \text{ miles}$. To find d and θ , we add the vectors $(200, 0^\circ)$ and $(100, -60^\circ)$.

$$\begin{aligned}(200, 0^\circ) &= (200 \cos 0^\circ, 200 \sin 0^\circ) = (200, 0) \\ (100, -60^\circ) &= (200 \cos(-60^\circ), 200 \sin(-60^\circ)) \approx (50, -86.60) \\ (250, -86.60) &\approx (265, 341^\circ)\end{aligned}$$

Thus, $d \approx 265$ miles, and $\theta \approx 341^\circ$.

The aircraft is 265 miles from Minneapolis. $360^\circ - 341^\circ = 19^\circ$, so the aircraft is in a direction 19° south of east, relative to the city.



59. $(199, 19.0^\circ) = (199 \cos 19^\circ, 199 \sin 19^\circ) \approx (188.15, 64.79)$
 $(175, 131^\circ) = (175 \cos 131^\circ, 175 \sin 131^\circ) \approx (-114.81, 132.07)$
 $(96, 130^\circ) = (96 \cos 130^\circ, 96 \sin 130^\circ) \approx (-61.71, 73.54)$
 $(11.64, 270.40) \approx (270.7, 87.5^\circ)$
64. $(19.6, 170^\circ) = (19.6 \cos 170^\circ, 19.6 \sin 170^\circ) \approx (-19.3, 3.4)$
 $(7.2, 95^\circ) = (7.2 \cos 95^\circ, 7.2 \sin 95^\circ) \approx (-0.63, 7.17)$
 $(-19.93, 10.58) \approx (22.6, 152.0^\circ)$

Thus, its true course is $180^\circ - 152^\circ = 28^\circ$ north of west, at a speed of 22.6 knots.



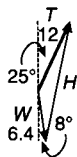
66. $(122, 30^\circ) = (122 \cos 30^\circ, 122 \sin 30^\circ) \approx (105.66, 61.00)$
 $(86, 21^\circ) = (86 \cos 21^\circ, 86 \sin 21^\circ) \approx (80.29, 30.82)$
 $(185.94, 91.82) \approx (207, 26^\circ)$

Magnitude is 207 volts, phase angle is 26° .

71. We let W represent the water current vector.

$$\begin{aligned}H + W &= T \\ H &= T - W \\ &= (12, 65^\circ) - (6.4, -82^\circ) \\ &= (12, 65^\circ) + (6.4, -82^\circ + 180^\circ) \\ &= (12, 65^\circ) + (6.4, 98^\circ) \\ &= (5.07, 10.88) + (-0.89, 6.34) \\ &= (4.18, 17.21) \approx (17.7, 76.3^\circ)\end{aligned}$$

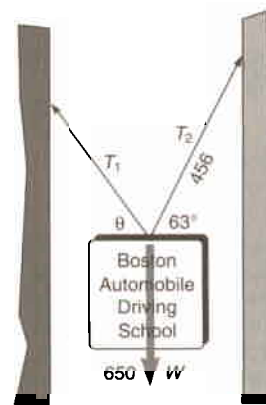
Thus, the ship's heading is 76.3° , and its speed is 17.7 knots.



73. The sign is stationary, so the forces acting on it are balanced (they add to zero).

$$\begin{aligned}T_1 + T_2 + W &= 0 \\ T_1 &= -T_2 - W \\ &= -(456, 63^\circ) - (650, 270^\circ) \\ &= (456, 63^\circ + 180^\circ) + (650, 270^\circ - 180^\circ) \\ &\quad \text{To negate a vector, add or subtract } 180^\circ \text{ from its direction angle.} \\ &= (456, 243^\circ) + (650, 90^\circ) \\ &\approx (-207.02, -406.3) + (0, 650) \\ &\quad \text{Convert to rectangular form.} \\ &\approx (-207.02, 243.7) \\ &\quad \text{Convert back to polar form.} \\ &\approx (320, 130^\circ)\end{aligned}$$

Thus, the tension in the second cable is 320 pounds, and it makes an angle θ of 50° ($180^\circ - 130^\circ$) with the horizontal.



74. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be two vectors in rectangular form.

$$\begin{aligned}\text{Then } A + B &= (a_1, a_2) + (b_1, b_2) \\ &= (a_1 + b_1, a_2 + b_2) \\ &\quad \text{Definition of vector addition.} \\ &= (b_1 + a_1, b_2 + a_2) \\ &\quad a_1, a_2, b_1, b_2 \text{ are real numbers, so their indicated sum commutes.} \\ &= B + A \\ &\quad \text{Definition of vector addition.}\end{aligned}$$

76. The following is for the TI-81:

PRGM EDIT 2 Choose a free location to enter the program. Say 2, by way of example.
A D D V C T R S Enter these characters as the name of the program.

```
:0→A
:0→B
:Lbl 1
:Input R
:If R=0
:Goto 2
:Input θ
:R→P(R,θ)
:A+X→A
:B+Y→B
:Goto 1
:Lbl 2
:A→X
:B→Y
:R→P(X,Y)
:Disp "X,Y"
:Disp X
:Disp Y
:Disp "R,θ"
:Disp R
:Disp θ
```

When running the program one inputs R, then θ , for each vector in polar form. When all vectors have been entered, enter zero (0) for R. The program converts each vector into rectangular form as it is entered, and accumulates the value (x,y) in variables A and B. When all vectors are entered, the accumulated values in A and B are converted to polar form.

Chapter 6 review

- $C = 121.8^\circ$, $b = 2.6$, $c = 12.1$
- $A = 151^\circ$, $a = 4.3$, $c = 0.8$
- $A = 49^\circ$, $C = 52^\circ$, $c = 10.4$
- $A = 76.5^\circ$, $B = 70.8^\circ$, $b = 12.2$;
or $A = 103.5^\circ$, $B = 43.8^\circ$, $b = 9.0$
- 48.2 miles
- $c = 3.8$, $A = 31.9^\circ$, $B = 118.7^\circ$
- $a = 63.9$, $C = 18.2^\circ$, $B = 69.7^\circ$
- $b = 40.2$, $A = 29.5^\circ$, $C = 38.5^\circ$
- $A = 106.3^\circ$, $B = 23.1^\circ$, $C = 50.5^\circ$
- $C = 135.5^\circ$, $A = 12.2^\circ$, $B = 32.3^\circ$
- $c = 2\sqrt{10}$, $b = 7\sqrt{2}$, $a = \sqrt{82}$, $B = 77.9^\circ$, $C = 38.7^\circ$, $A = 63.4^\circ$
- 29.2 km
- (23.8, 13.2)

- (36.3, 57.3°)
- $V_x = 370$ knots, $V_y = 256$ knots
- $V_x = 249$ lb, $V_y = 60$ lb
- (47.6, 20.6°)
- (10.8, 89.5°)
- (7.2, 69.9°)
- (8.1, 187.6°)
- 205 lb, 273°
- 55 knots, 41° south of east
- 183 knots, 62° north of west
- 247 knots, 55° north of west

Chapter 6 test

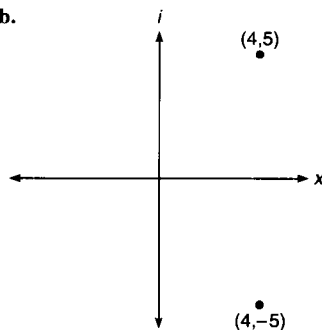
- $B = 84.4^\circ$, $a = 5.3$, $c = 22.5$
- $B = 56.3^\circ$, $A = 61.6^\circ$, $a = 23.9$
- $b = 32.8$, $A = 50.9^\circ$, $C = 29.1^\circ$
- $C = 89.1^\circ$, $A = 39.6^\circ$, $B = 51.3^\circ$
- 59 yards
- 53.1°
- $(\sqrt{3}, 1)$
- (6.4, 51.3°)
- (8.5, 84.9°)
- 584 lb, 65°

Chapter 7

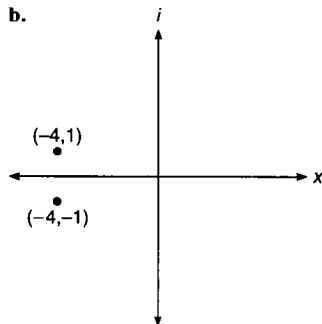
Exercise 7-1

Answers to odd-numbered problems

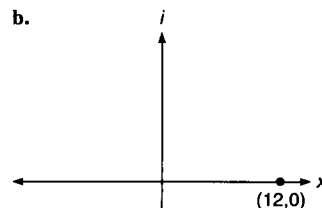
- real, 4; imaginary, -5
- real, -4; imaginary, 1
- real, 12; imaginary, 0
- real, -10; imaginary, 2
- a. $4 + 5i$



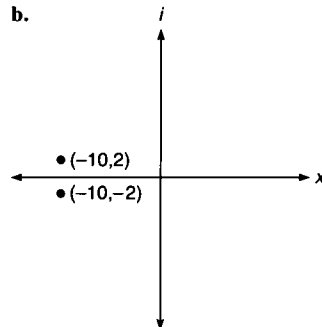
- a. $-4 - i$
- b.



- a. 12
- b.

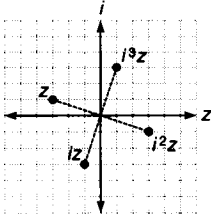


- a. $-10 - 2i$
- b.

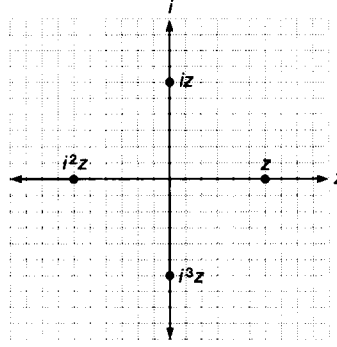


- $2 + 6i$
- $-9 - 7i$
- $9 - 4i$
- $-3 + 192i$
- $-20 - 30i$
- $-46 - 9i$
- $\frac{33}{29} + \frac{10}{29}i$
- $\frac{-12}{5} - \frac{9}{5}i$
- $5.4 \text{ cis } (-21.8^\circ)$
- $3.2 \text{ cis } 108.4^\circ$
- $5 \text{ cis } 126.9^\circ$
- $2 \text{ cis } 30^\circ$
- $3\sqrt{2} \text{ cis } 45^\circ$
- $\sqrt{2} \text{ cis } 225^\circ$
- $5 \text{ cis } 90^\circ$
- $2.9 + 0.8i$
- $3.7 + 2.6i$
- $1 - i$
- $12.3 - 5.7i$
- $\frac{3}{2} + \frac{\sqrt{3}}{2}i$
- $5 - 5\sqrt{3}i$
- $-\sqrt{10}$
- $2 - 2i$
- $15 \text{ cis } 75^\circ$
- $10.8 \text{ cis } (-120^\circ)$
- $4 \text{ cis } 80^\circ$
- $\frac{20}{9} \text{ cis } (-80^\circ)$
- $512 \text{ cis } (-60^\circ)$
- $27 \text{ cis } (-120^\circ)$
- $-8.2 - 0.1i$
- $2, -1 + \sqrt{3}i, -1 - \sqrt{3}i$
- $3, 3i, -3, -3i$
- $-1.1 + 4.9i, -3.7 - 3.4i, 4.8 - 1.5i$
- $2.5 \text{ cis } (-20^\circ)$
- $50 \text{ cis } 45^\circ$
- $2.04 \text{ cis } 6.19^\circ$
- $0.75 + 0.86i$
- Is the following an identity: $a \text{ cis } (-\theta)$
 $= -a \text{ cis } \theta$?
 No. $a \text{ cis } (-\theta)$
 $= a \cos(-\theta) + a \sin(-\theta)i$
 $= a \cos \theta - a \sin \theta i$
 but $-a \text{ cis } \theta = -a \cos \theta - a \sin \theta i$.

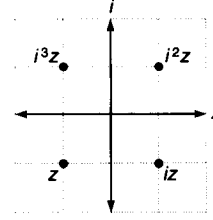
93. a. $-3 + i$ b. $-1 - 3i$
c. $3 - i$ d. $1 + 3i$



95. a. 6 b. $6i$ c. -6
d. $-6i$



97. a. $-1 - i$ b. $1 - i$
c. $1 + i$ d. $-1 + i$



99. a. $\frac{\sqrt{3}}{2} + \frac{1}{2}i$ b. $0.37 + 1.37i$
c. $0.37 - 1.37i$

101. $r^{\frac{1}{n}} \operatorname{cis}\left(\frac{\theta}{n} + \frac{k \cdot 360^\circ}{n}\right)$

$$= r^{\frac{1}{n}} \operatorname{cis}\left[\frac{\theta}{n} + \frac{(an + b) \cdot 360^\circ}{n}\right] \quad \text{Replace } k \text{ by } an + b.$$

$$\begin{aligned} \frac{\theta}{n} + \frac{(an + b) \cdot 360^\circ}{n} &= \frac{\theta}{n} + \frac{an \cdot 360^\circ}{n} + \frac{b \cdot 360^\circ}{n} \\ &= \frac{\theta}{n} + a \cdot 360^\circ + \frac{b \cdot 360^\circ}{n} \\ &= \left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n}\right) + a \cdot 360^\circ \end{aligned}$$

$$= r^{\frac{1}{n}} \operatorname{cis}\left[\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n}\right) + a \cdot 360^\circ\right]$$

$$= r^{\frac{1}{n}} \cos\left[\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n}\right) + a \cdot 360^\circ\right] + ir^{\frac{1}{n}} \sin\left[\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n}\right) + a \cdot 360^\circ\right]$$

$$\cos\left[\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n}\right) + a \cdot 360^\circ\right] = \cos\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n}\right) \cos(a \cdot 360^\circ) - \sin\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n}\right) \sin(a \cdot 360^\circ)$$

$$= \cos\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n}\right), \text{ because } \cos(a \cdot 360^\circ) = 1 \text{ and } \sin(a \cdot 360^\circ) = 0, \text{ when } a \text{ is an integer.}$$

$$\sin\left[\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n}\right) + a \cdot 360^\circ\right] = \sin\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n}\right) \cos(a \cdot 360^\circ) + \cos\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n}\right) \sin(a \cdot 360^\circ)$$

$$= \sin\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n}\right), \text{ because } \cos(a \cdot 360^\circ) = 1 \text{ and } \sin(a \cdot 360^\circ) = 0,$$

when a is an integer.

Thus,

$$r^{\frac{1}{n}} \cos\left[\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n}\right) + a \cdot 360^\circ\right] + ir^{\frac{1}{n}} \sin\left[\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n}\right) + a \cdot 360^\circ\right]$$

$$= r^{\frac{1}{n}} \cos\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n}\right) + ir^{\frac{1}{n}} \sin\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n}\right)$$

$$= r^{\frac{1}{n}} \operatorname{cis}\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n}\right), \text{ where } b < n.$$

This last expression is one of the previous roots.

Solutions to trial exercise problems

23. $(15 + 4i)(3 + 12i)$

$$15(3) + 15(12i) + 4i(3) + 4i(12i)$$

$$45 + 180i + 12i + 48i^2 \quad (i^2 = -1)$$

$$45 + 192i - 48$$

$$-3 + 192i$$

27. $(2 - 3i)^3$

$$(2 - 3i)(2 - 3i)(2 - 3i)$$

$$[(2 - 3i)(2 - 3i)](2 - 3i)$$

$$[4 - 12i + 9i^2](2 - 3i)$$

$$[-5 - 12i](2 - 3i)$$

$$-10 + 15i - 24i + 36i^2$$

$$-10 - 36 - 9i$$

$$-46 - 9i$$

31. $\frac{3 + 6i}{-2 - i}$

$$\frac{3 + 6i}{-2 - i} \cdot \frac{-2 + i}{-2 + i}$$

Multiply by the conjugate of the denominator.

$$\frac{-6 + 3i - 12i + 6i^2}{4 - 2i + 2i - i^2} = \frac{-6 - 9i - 6}{4 + 1}$$

$$\frac{-12 - 9i}{5} = -\frac{12}{5} - \frac{9}{5}i$$

36. $\sqrt{3} - 2i$

$$r = \sqrt{(\sqrt{3})^2 + (-2)^2} = \sqrt{7} \approx 2.6.$$

$$\theta' = \tan^{-1} \frac{-2}{\sqrt{3}} \approx -49.1^\circ; a > 0 \text{ to } \theta = \theta'.$$

The point is $2.6 \operatorname{cis}(-49.1^\circ)$.

39. $\sqrt{3} + i$

The modulus is $\sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$.

$$\tan \theta' = \frac{1}{\sqrt{3}}, \text{ so } \theta' = 30^\circ \text{ (exactly).}$$

$\theta = \theta' = 30^\circ$, since θ is in the first quadrant. Thus, the polar form is $2 \operatorname{cis} 30^\circ$.

47. $3 \operatorname{cis} 15^\circ$

$$3 \cos 15^\circ + (3 \sin 15^\circ)i = 2.9 + 0.8i$$

58. $6 \operatorname{cis} 135^\circ$

$$6(\cos 135^\circ + i \sin 135^\circ)$$

$$6\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$

$$-3\sqrt{2} + 3\sqrt{2}i$$

73. $(3 \operatorname{cis} 200^\circ)^3 = 3^3 \operatorname{cis} 3(200^\circ)$

$$= 27 \operatorname{cis} 600^\circ$$

$$= 27 \operatorname{cis}(600^\circ - 2 \cdot 360^\circ)$$

$$= 27 \operatorname{cis}(-120^\circ)$$

76. $(0.8 + 0.6i)^{10}$

We first put the complex number in polar form. The modulus is $\sqrt{0.8^2 + 0.6^2} = \sqrt{1} = 1$.

$$\tan \theta' = \frac{0.6}{0.8} = \frac{3}{4}, \text{ so } \theta' = 36.8699^\circ.$$

Since θ terminates in quadrant I, $\theta = 36.8699^\circ$. We now compute:

$$(1 \operatorname{cis} 36.8699^\circ)^{10} = 1^{10} \operatorname{cis}$$

$$10(36.8699^\circ) = 1 \operatorname{cis} 368.699^\circ = 1 \operatorname{cis}$$

8.699° . Since the original complex

number was in its rectangular form, we

will put this result in rectangular form.

$$1 \operatorname{cis} 8.699^\circ = \cos 8.699^\circ + i \sin 8.699^\circ$$

$$= 0.99 + 0.15i = 1.0 + 0.2i$$

82. Find the 4 fourth roots of $\sqrt{3} + 3i$ to the nearest tenth.

$$\sqrt{3} + 3i = 2\sqrt{3} \operatorname{cis} 60^\circ, \text{ so}$$

$$(\sqrt{3} + 3i)^{1/4} = (2\sqrt{3} \operatorname{cis} 60^\circ)^{1/4}$$

$$\text{Evaluate } (2\sqrt{3})^{1/4} \operatorname{cis}\left(\frac{60^\circ}{4} + \frac{k \cdot 360^\circ}{4}\right)$$

$$\approx 1.364 \operatorname{cis}(15^\circ + k \cdot 90^\circ) \text{ for } k = 0, 1, 2, 3.$$

$$k = 0: 1.364 \operatorname{cis}(15^\circ)$$

$$= 1.364 \cos 15^\circ + 1.364 \sin 15^\circ i$$

$$\approx 1.3 + 0.4i$$

$$k = 1: 1.364 \operatorname{cis}(15^\circ + 90^\circ)$$

$$= 1.364 \cos 105^\circ + 1.364 \sin 105^\circ i$$

$$\approx -0.4 + 1.3i$$

$$k = 2: 1.364 \operatorname{cis}(15^\circ + 180^\circ)$$

$$= 1.364 \cos 195^\circ + 1.364 \sin 195^\circ i$$

$$\approx -1.3 - 0.4i$$

$$k = 3: 1.364 \operatorname{cis}(15^\circ + 270^\circ)$$

$$= 1.364 \cos 285^\circ + 1.364 \sin 285^\circ i$$

$$\approx 0.4 - 1.3i$$

87. $Z_T = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(2 + i)(3 - 5i)}{(2 + i) + (3 - 5i)}$

$$= \frac{(2 + i)(3 - 5i)}{5 - 4i} = \frac{6 - 7i + 5}{5 - 4i}$$

$$= \frac{11 - 7i}{5 - 4i} = \frac{13.038 \operatorname{cis} 327.529^\circ}{6.403 \operatorname{cis} 321.340^\circ}$$

$$= \frac{13.038}{6.403} \operatorname{cis}(327.529^\circ - 321.340^\circ)$$

$$= 2.04 \operatorname{cis} 6.19^\circ$$

93. a. $-3 + i$

b. $i(-3 + i) = -3i + i^2$

$$= -3i - 1$$

$$= -1 - 3i$$

c. $i^2(-3 + i) = -1(-3 + i)$

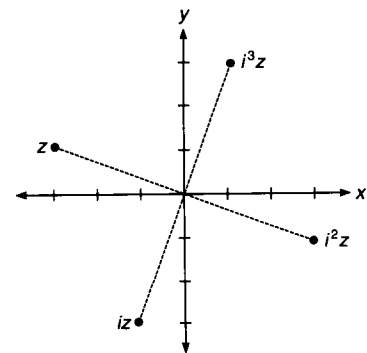
$$= 3 - i$$

d. $i^3 z = i i^2 z = i(-1)z$

$$= -iz = -i(-3 + i)$$

$$= 3i - i^2 = 3i - (-1)$$

$$= 1 + 3i$$



100. $\frac{r_1 \operatorname{cis} \theta_1}{r_2 \operatorname{cis} \theta_2} = \frac{r_1 \cos \theta_1 + i r_1 \sin \theta_1}{r_2 \cos \theta_2 + i r_2 \sin \theta_2}$

$$= \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)}$$

$$= \frac{r_1}{r_2} \cdot \frac{\cos \theta_1 + i \sin \theta_1}{\cos \theta_2 + i \sin \theta_2}$$

$$= \frac{r_1}{r_2} \cdot \frac{\cos \theta_1 + i \sin \theta_1}{\cos \theta_2 + i \sin \theta_2} \cdot \frac{\cos \theta_2 - i \sin \theta_2}{\cos \theta_2 - i \sin \theta_2}$$

$$= \frac{r_1}{r_2} \cdot \frac{\cos \theta_1 \cos \theta_2 - i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 - i^2 \sin \theta_1 \sin \theta_2}{\cos^2 \theta_2 - i^2 \sin^2 \theta_2}$$

$$= \frac{r_1}{r_2} \cdot \frac{\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 + i(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)}{\cos^2 \theta_2 + \sin^2 \theta_2}$$

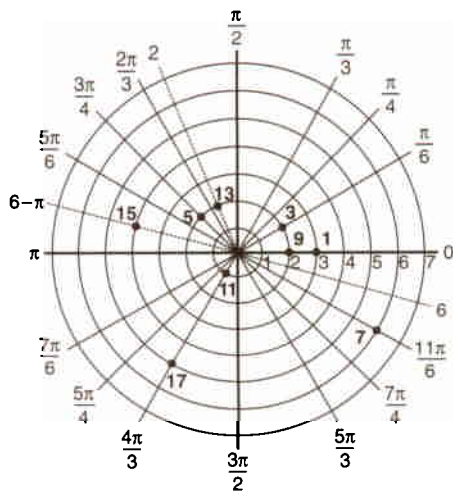
$$= \frac{r_1}{r_2} \cdot \frac{\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)}{1}$$

$$= \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

Exercise 7-2

Answers to odd-numbered problems

The figure shows the answers to odd-numbered problems 1 through 17.



19. $\left(2, \frac{13\pi}{6}\right) \left(2, \frac{25\pi}{6}\right) \left(-2, \frac{7\pi}{6}\right)$
 21. $\left(6, \frac{23\pi}{6}\right) \left(6, -\frac{\pi}{6}\right) \left(-6, \frac{5\pi}{6}\right)$
 23. $(2, 2 + 2\pi) (2, 2 + 4\pi) (-2, 2 + \pi)$
 25. $(0, 4)$ 27. $\left(-\frac{5\sqrt{3}}{2}, \frac{5}{2}\right)$
 29. $(-2, -2\sqrt{3})$ 31. $(1.08, 1.68)$
 33. $(2.05, 2.19)$ 35. $(-2.61, -3.03)$
 37. $\left(4, -\frac{5\pi}{6}\right)$ 39. $(2, \pi)$
 41. $\left(4\sqrt{2}, -\frac{3\pi}{4}\right)$ 43. $(3.61, 0.98)$
 45. $(4.12, -1.33)$ 47. $(6.40, 0.67)$
 49. $\tan \theta = 4$ 51. $r = \frac{2}{\sin \theta + 3 \cos \theta}$
 53. $r = \frac{b}{\sin \theta - m \cos \theta}$
 55. $r^2 = \frac{5}{\sin^2 \theta - 2 \cos^2 \theta}$
 57. $r^2 = \frac{1}{\cos^2 \theta + 2}$ 59. $x^2 + y^2 = y$
 61. $x = 2$ 63. $(x^2 + y^2)^3 = 36x^2y^2$
 65. $x^4 + 2x^2y^2 + y^4 = 2xy$
 67. $x^3 + xy^2 = y$

69. $x^2 + y^2 = 4y^2 + 12y + 9$

71. $2(r \cos \theta r \sin \theta) = 5, r^2 \sin 2\theta = 5,$

$r^2 = \frac{5}{\sin 2\theta} = 5 \csc 2\theta$

73. Consider a point $P = (r, \theta)$, where $r < 0$. Then $P = (-r, \theta + \pi)$, where $-r > 0$.

Therefore, since $-r > 0$, $y = -r \sin(\theta + \pi)$ is true,

$y = -r \sin(\theta + \pi)$

A true statement.

$= -r(\sin \theta \cos \pi + \cos \theta \sin \pi)$

$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$= -r(\sin \theta(-1) + \cos \theta(0))$

$= -r(-\sin \theta)$

$= r \sin \theta$

Thus, $y = r \sin \theta$, even if $r < 0$.

75. $(x^2 + y^2 + 2y)^2 = x^2 + y^2$

77. $(x^2 + y^2)^3 = (x^2 + y^2 + 4xy)^2$

79. $(x^2 + y^2 + 3x)^2 = x^2 + y^2$

Solutions to trial exercise problems

15. $(-4, 6)$ represents a point with $r = -4$ and $\theta = 6$ (radians). We can convert r to a positive value if we add or subtract π from θ . Since $6 > \pi$, subtract:

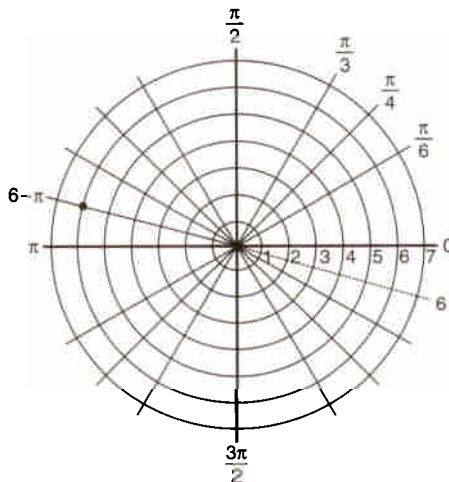
$(-4, 6) = (4, 6 - \pi)$

$= (4, 2.9)$ (approximately).

(See the diagram.)

To construct an angle of 2.9 radians

convert it to $\approx 166^\circ$.



21. $\left(6, \frac{11\pi}{6}\right)$

Add 2π to θ :

$\left(6, \frac{11\pi}{6}\right) = \left(6, \frac{11\pi}{6} + \frac{12\pi}{6}\right)$

$= \left(6, \frac{23\pi}{6}\right)$

Subtract 2π from θ :

$\left(6, \frac{11\pi}{6}\right) = \left(6, \frac{11\pi}{6} - \frac{12\pi}{6}\right)$

$= \left(6, -\frac{\pi}{6}\right)$

To obtain $r < 0$, add or subtract an odd multiple of π . We subtract π .

$\left(6, \frac{11\pi}{6}\right) = \left(-6, \frac{11\pi}{6} - \frac{6\pi}{6}\right)$

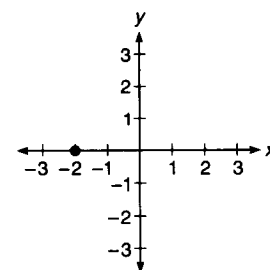
$= \left(-6, \frac{5\pi}{6}\right)$

31. $(2, 1)$

$(2, 1) = (2 \cos 1, 2 \sin 1)$

$= (1.08, 1.68)$

39. $(-2, 0)$ is plotted in the diagram. We can see that $\theta = 180^\circ$ or π (radians), and that $r = 2$. Thus, $(-2, 0)$ (rectangular) $= (2, \pi)$ (polar).



45. $(1, -4)$

$r = \sqrt{1^2 + 4^2} = \sqrt{17} \approx 4.12$

$\theta' = \tan^{-1}(-4) \approx -1.326$

$x > 0$ so $\theta = \theta'$.

$(4.12, -1.33)$

53. $y = mx + b, b \neq 0$

Use $x = r \cos \theta, y = r \sin \theta$.

$$y = mx + b$$

$$r \sin \theta = m(r \cos \theta) + b$$

Solve for r when possible.

$$r \sin \theta - m r \cos \theta = b$$

$$r(\sin \theta - m \cos \theta) = b$$

$$r = \frac{b}{\sin \theta - m \cos \theta}$$

The quotient is defined when

$$\sin \theta - m \cos \theta \neq 0$$

$$\sin \theta \neq m \cos \theta$$

$$\frac{\sin \theta}{\cos \theta} \neq m, \cos \theta \neq 0$$

$$\tan \theta \neq m, \cos \theta \neq 0$$

$\cos \theta = 0$ implies that θ is an odd

multiple of $\frac{\pi}{2}$, representing a vertical

line. However, $y = mx + b$ can only represent a nonvertical line, so the condition $\cos \theta \neq 0$ is satisfied.

If $\tan \theta = m$, since $\tan \theta = \frac{y}{x}$ if θ is

not an odd multiple of $\frac{\pi}{2}$, we have $\frac{y}{x}$

$= m$, or $y = mx$. This would be a line parallel to $y = mx + b$ with $b = 0$, but the problem specifies that $b \neq 0$. Thus, we are also guaranteed that $\tan \theta \neq m$.

Therefore, under all conditions for which $y = mx + b, b \neq 0$ are satisfied, the polar equation is $r =$

$$\frac{b}{\sin \theta - m \cos \theta}$$

55. $y^2 - 2x^2 = 5$

$$(r \sin \theta)^2 - 2(r \cos \theta)^2 = 5$$

$$r^2 \sin^2 \theta - 2r^2 \cos^2 \theta = 5$$

$$r^2(\sin^2 \theta - 2 \cos^2 \theta) = 5$$

$$r^2 = \frac{5}{\sin^2 \theta - 2 \cos^2 \theta}$$

63. $r = 3 \sin \theta$

$$r = 3 \sin \theta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$= 3(2 \sin \theta \cos \theta)$$

$$= 6 \frac{y}{r} \cdot \frac{x}{r}$$

$$r^3 = 6xy$$

$$r = \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2}$$

$$[(x^2 + y^2)^{1/2}]^3 = 6xy$$

$$(x^2 + y^2)^{3/2} = 6xy, \text{ or } (x^2 + y^2)^3 = 36x^2y^2.$$

69. $r = \frac{3}{1 - 2 \sin \theta}$ (Note $r \neq 0$ because the quotient cannot be 0.)

$$r = \frac{3}{1 - 2 \frac{y}{r}}$$

$$= \frac{3}{1 - 2 \frac{y}{r}} \cdot \frac{r}{r} = \frac{3r}{r - 2y}$$

$$r(r - 2y) = 3r; r^2 - 2ry = 3r;$$

$$r^2 = 3r + 2ry$$

$$r^2 = r(3 + 2y)$$

Divide by r , since $r \neq 0$.

$$r = 3 + 2y$$

$$(x^2 + y^2)^{1/2} = 3 + 2y$$

$$x^2 + y^2 = (3 + 2y)^2$$

$$x^2 + y^2 = 4y^2 + 12y + 9$$

78. Problem 79 in section 5-3 shows that

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta.$$

$$r = 2 \cos 3\theta$$

$$r = 2(4 \cos^3 \theta - 3 \cos \theta)$$

$$r = 8 \cos^3 \theta - 6 \cos \theta$$

$$r = 8 \left(\frac{x}{r} \right)^3 - 6 \frac{x}{r}$$

$$r = \frac{8x^3}{r^3} - \frac{6x}{r}$$

$$r^4 = 8x^3 - 6xr^2$$

Multiply each member by r^3 .

$$(x^2 + y^2)^2 = 8x^3 - 6x(x^2 + y^2)$$

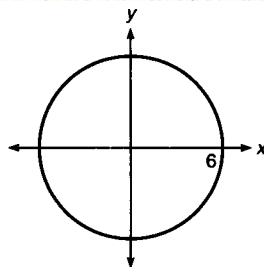
$$x^4 + 2x^2y^2 + y^4 = 8x^3 - 6x^3 - 6xy^2$$

$$x^4 - 2x^3 + 2x^2y^2 + 6xy^2 + y^4 = 0$$

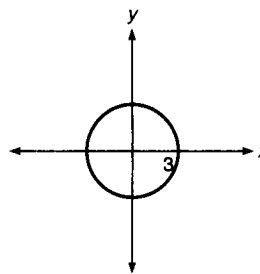
Exercise 7-3

Answers to odd-numbered problems

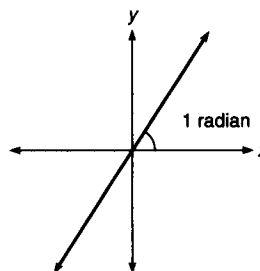
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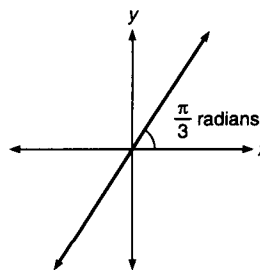
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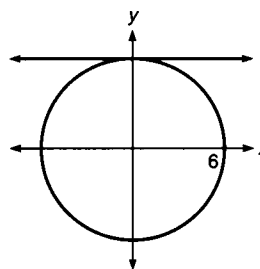
5.



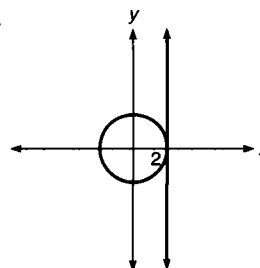
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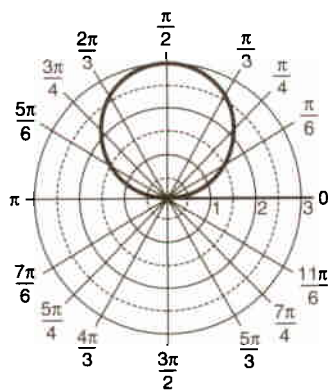
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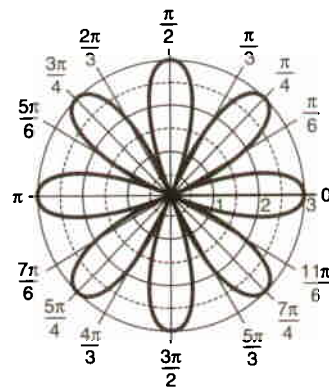
11.



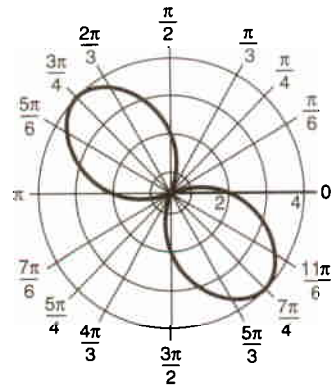
13.



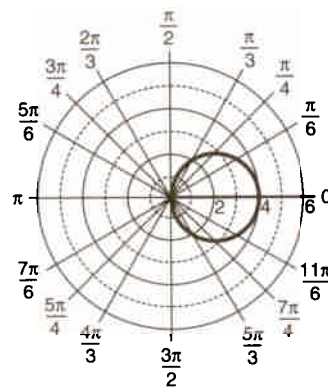
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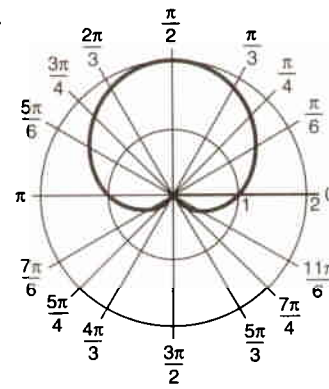
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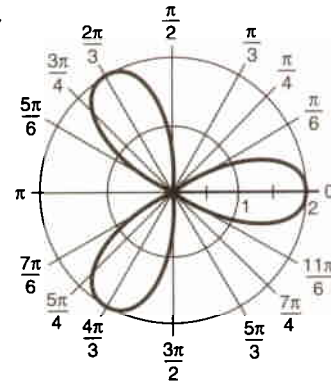
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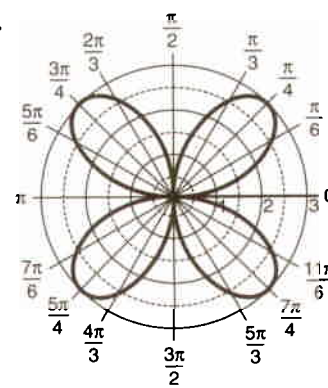
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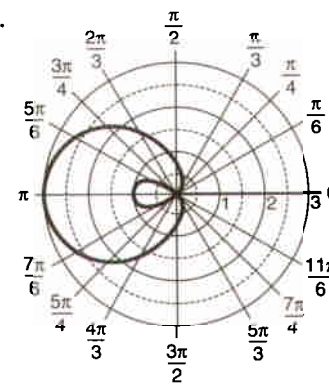
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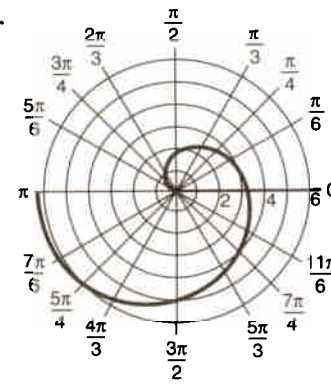
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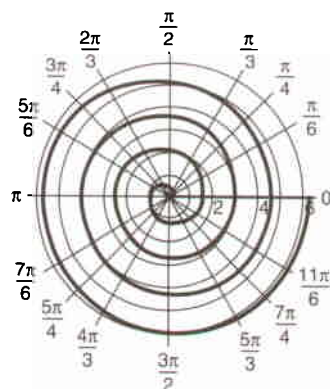
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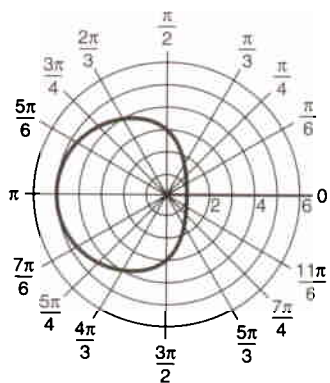
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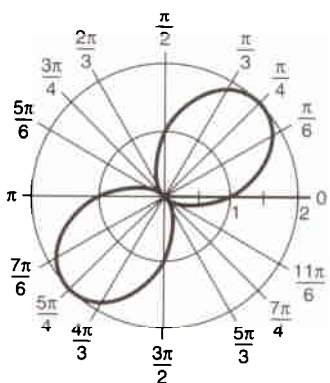
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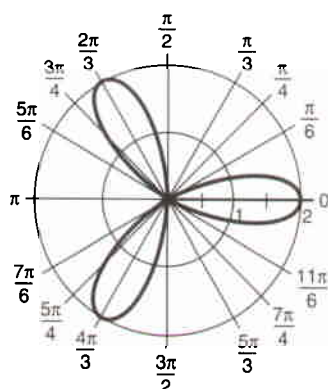
33.



35.



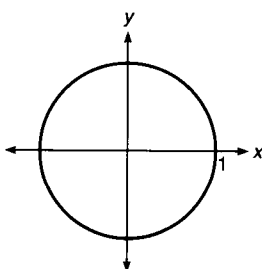
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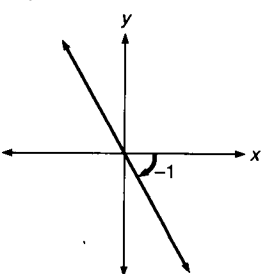
Solutions to trial exercise problems

4. $r = -1$

Since any point $(-1, \theta)$ is equivalent to the point $(1, \theta + \pi)$, and since θ can be any value (since it is not specified), the graph of this equation is equivalent to the graph of $r = 1$, which is a circle with radius 1, centered at the pole.


8. $\theta = -1$

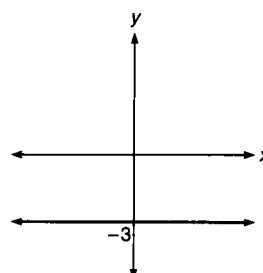
The radius r can be any value, but the angle must be -1 (radians). This produces points at all distances from the pole, along the line in which the angle is -1 (radians) or about -57° .


12. $r \sin \theta = -3$

$$y = -3$$

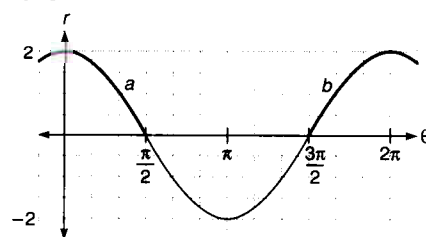
Replace $r \sin \theta$ by y .

This is the horizontal line $y = -3$.

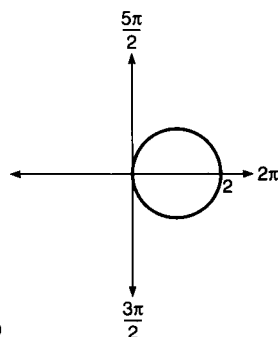
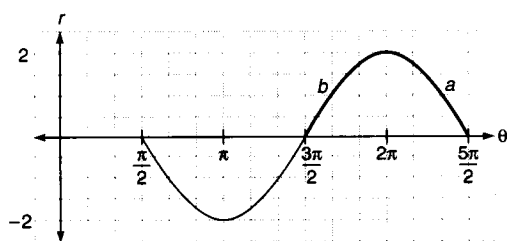

14. $r = 2 \cos \theta$

The following graph shows the rectangular coordinate graph of $r = 2 \cos \theta$. If we move the "negative lobe" between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ by adding π to all x values, it becomes a positive lobe between $\frac{3\pi}{2}$ and $\frac{5\pi}{2}$. This just repeats

the positive lobe that starts at $\frac{3\pi}{2}$ in the graph. Thus, we can ignore the negative lobe for finding the polar graph.



It is easier to imagine what is happening if we move the lobe that starts at 0 to start at 2π . We then see that we have a lobe between $\frac{3\pi}{2}$ and $\frac{5\pi}{2}$, with its maximum value at 2π .

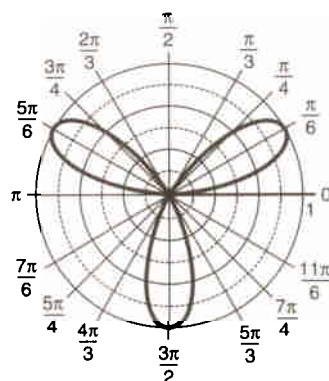
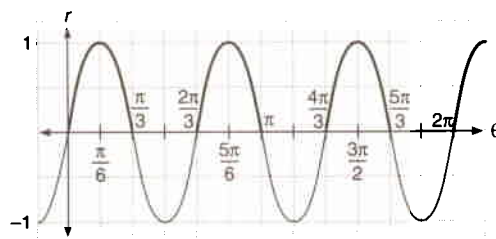


This positive lobe produces a circle between $\frac{3\pi}{2}$ and $\frac{5\pi}{2}$. It takes some experience or a lot of point plotting to know that this lobe is a circle. Remember, however, that graphs of the form $r = k \sin \theta$ or $r = k \cos \theta$ produce circles.

20. $r = \sin 3\theta$

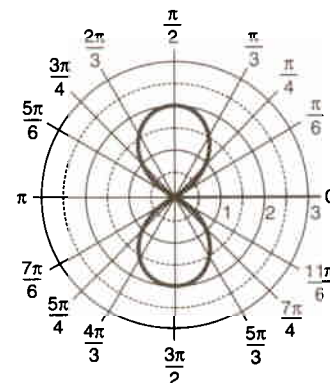
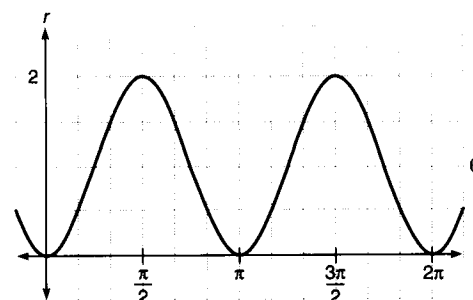
The rectangular coordinate graph of $r = \sin 3\theta$ is shown below. The negative lobes can be ignored because shifting them by π units and flipping them about the θ axis causes them to overlap the positive lobes. We see there are lobes between 0 and $\frac{\pi}{3}$, with

maximum at $\frac{\pi}{6}$, $\frac{2\pi}{3}$, and π , with maximum at $\frac{5\pi}{6}$, and $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$, with maximum at $\frac{3\pi}{2}$.



These lobes produce the polar graph shown.

26. $r = 1 - \cos 2\theta$



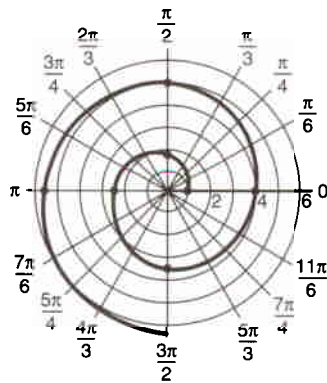
The rectangular graph shows two lobes, one between 0 and π , with maximum at $\frac{\pi}{2}$, and one between π

and 2π , with maximum at $\frac{3\pi}{2}$. The two positive lobes produce the polar graph shown. The two lobes are not circles, which would require algebra beyond the scope of this text to show. Of the problems in this text, only those equations categorized in the text as producing circles will in fact produce circles.

32. $r = 1 + \frac{\theta}{2}$

This graph is not periodic, and thus an analysis of its rectangular graph is less helpful than in cases where the trigonometric functions are involved. We simply plot points to obtain the graph. It is clear that as the value of θ increases, r increases. This property produces a spiral, which starts at $r = 1$. A table of values and the graph are shown.

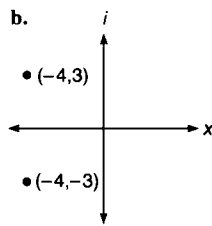
θ	r
0	1
$\frac{\pi}{2}$	1.8
π	2.6
$\frac{3\pi}{2}$	3.4
2π	4.1
$\frac{5\pi}{2}$	4.9
3π	5.7



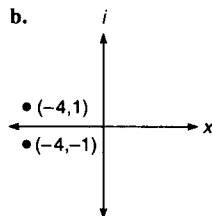
Chapter 7 review

1. real, 3; imaginary, -1 2. real, 0; imaginary, 3 3. real, 12; imaginary, 0
 4. real, 0; imaginary, -1 5. $-1 - 2i$
 6. $6 + i$ 7. $63 + 48i$ 8. $24 + 27i$
 9. $12 + 8i$ 10. $53 - 60i$
 11. $\frac{42}{29} - \frac{40}{29}i$ 12. $\frac{-2}{5} - \frac{3}{10}i$
 13. $\frac{3}{7} - \frac{4}{7}i$ 14. $1 + 4i$
 15. $-2 + 4i$ 16. $\frac{-4}{5} - \frac{7}{5}i$
 17. $\frac{-25}{26} + \frac{5}{26}i$

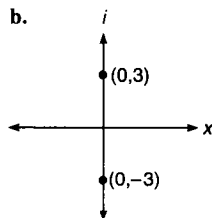
18. a. $-4 - 3i$



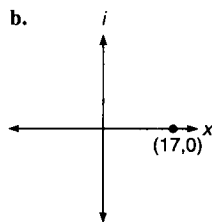
19. a. $-4 - i$



20. a. $-3i$



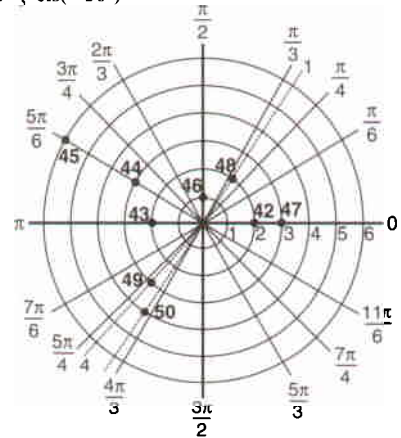
21. a. 17



22. $3.6 \operatorname{cis}(-33.7^\circ)$ 23. $3.5 \operatorname{cis} 60^\circ$
 24. $2.2 \operatorname{cis}(-116.6^\circ)$ 25. $2 \operatorname{cis}(-30^\circ)$
 26. $3\sqrt{2} \operatorname{cis} 45^\circ$ 27. $4 \operatorname{cis}(-90^\circ)$
 28. $2.5 + 1.7i$ 29. $-1.4 - 2.7i$
 30. $-\frac{3}{2} - \frac{3\sqrt{3}}{2}i$ 31. $5\sqrt{3} - 5i$
 32. $6 \operatorname{cis} 70^\circ$ 33. $13 \operatorname{cis} 140^\circ$
 34. $8 \operatorname{cis} 100^\circ$ 35. $\frac{1}{2} \operatorname{cis} 36^\circ$
 36. $8 \operatorname{cis} 30^\circ$ 37. $16 \operatorname{cis}(-120^\circ)$
 38. $0.4 - 0.9i$ 39. $2, 2i, -2, -2i$

40. $\frac{3}{2} + \frac{3\sqrt{3}}{2}i, -3, \frac{3}{2} - \frac{3\sqrt{3}}{2}i$

41. $\frac{13}{3} \operatorname{cis}(-50^\circ)$



51. $\left(\frac{3\sqrt{3}}{2}, \frac{-3}{2}\right)$ 52. $(-2, 2\sqrt{3})$

53. $(-1, \sqrt{3})$ 54. $(1.08, 1.68)$

55. $(-2.08, 4.55)$ 56. $(0.88, 0.48)$

57. $(2.24, 0.46)$ 58. $(5.83, 2.60)$

59. $(4.12, -1.82)$ 60. $\tan \theta = -3$

61. $r = \frac{2}{\sin \theta - 4 \cos \theta}$

62. $r^2 = \frac{5}{2 - 3 \cos^2 \theta}$ 63. $r = \frac{3 \cos \theta}{\sin^2 \theta}$

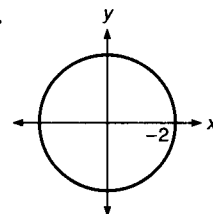
64. $x^2 + y^2 = y$ 65. $x = 2$

66. $x^4 + 2x^2y^2 + y^4 = 2xy$

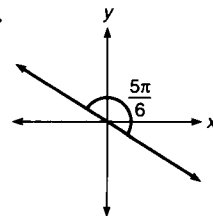
67. $x^3 + xy^2 = y$ 68. $y = 2$

69. $4x^2 + 3y^2 - 6y - 9 = 0$

70.



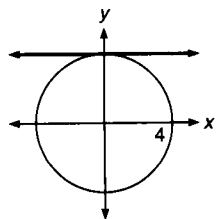
71.



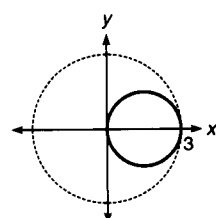
72. $2r \sin \theta = 8$

$r \sin \theta = 4$

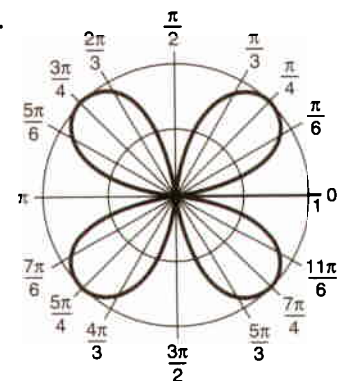
$y = 4$



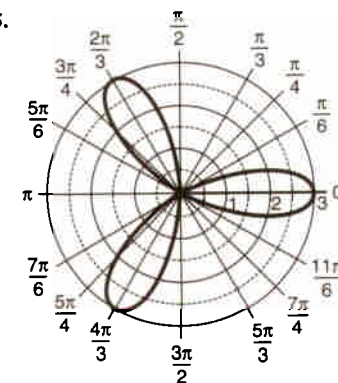
73. $r = 3 \cos \theta$



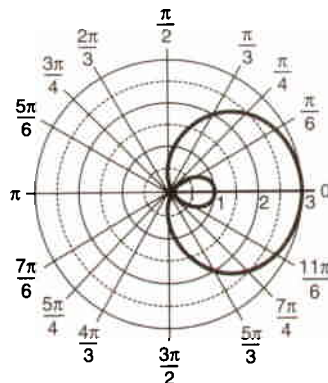
74.



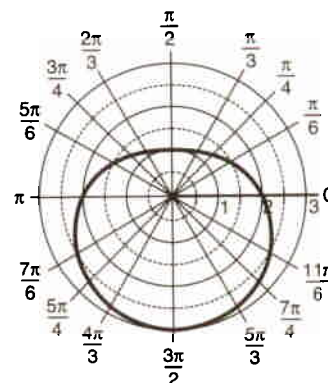
75.



76.



77.



Chapter 7 test

1. $15 - 6i$

2. 34

3. $\frac{19}{13} - \frac{4}{13}i$

4. $-6 + 8i$

5. $6.4 \text{ cis } (-51.3^\circ)$

6. $-1 + \sqrt{3}i$

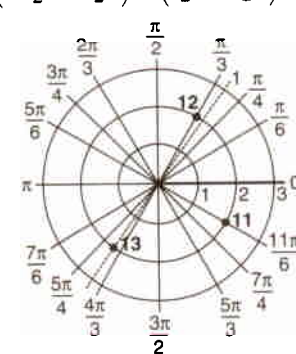
7. $14 \text{ cis } 110^\circ$

8. $3 \text{ cis } 120^\circ$

9. $27 \text{ cis } 90^\circ$

10. $2\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right), 2\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right),$

$2\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right), 2\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$



14. $(2.09, 2.15)$

15. $(2.08, -4.55)$

16. $\left(2, -\frac{5\pi}{6}\right)$

17. $\left(5\sqrt{2}, \frac{3\pi}{4}\right)$

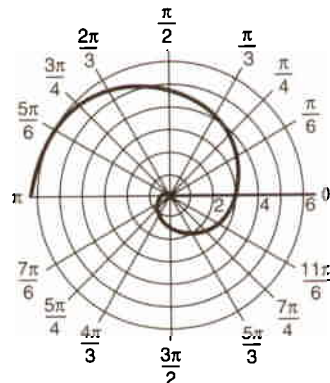
18. $r = \frac{5}{\sin \theta + 3 \cos \theta}$

19. $2r^2 \sin^2 \theta - r \cos \theta = 5$

20. $y = 2$

21. $x^4 + 2x^2y^2 + y^4 = x^2 - y^2$

22.

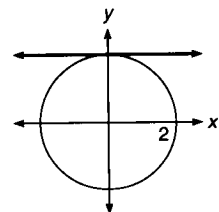


23. $r = 2 \csc \theta$

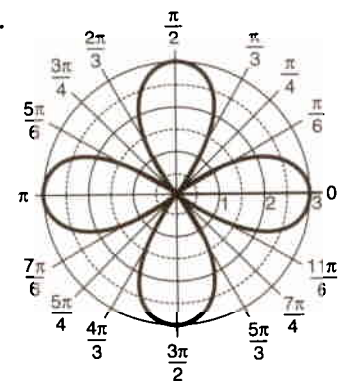
$r = \frac{2}{\sin \theta}$

$r \sin \theta = 2$

$y = 2$



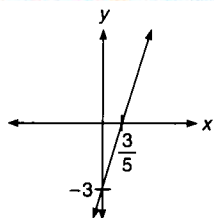
24.



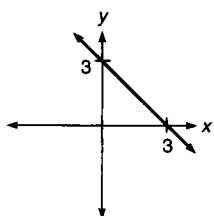
Appendix A

Answers to odd-numbered problems

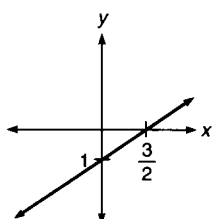
1.



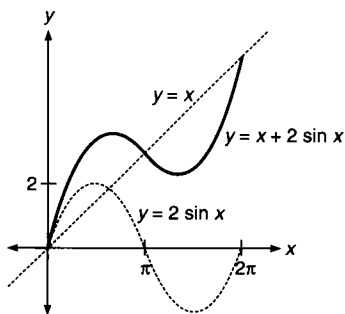
3.



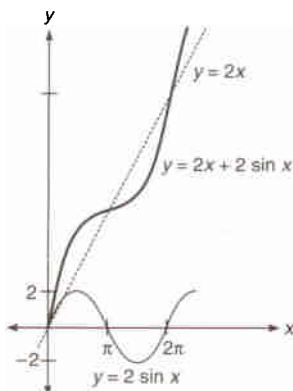
5.



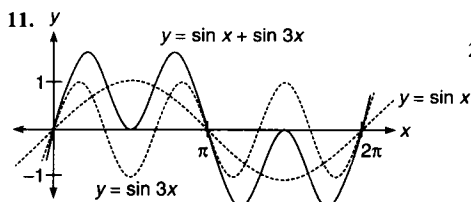
7.



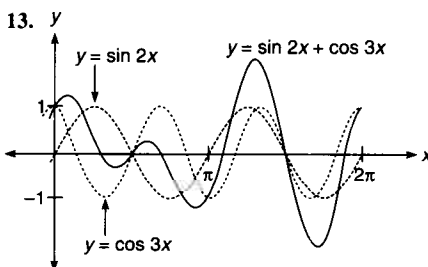
9.



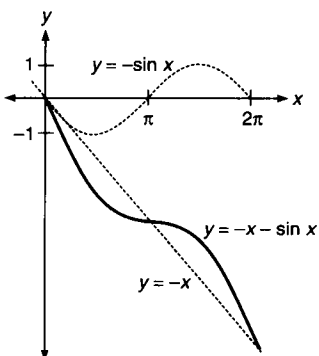
11.



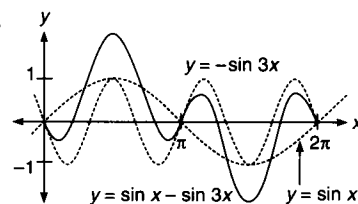
13.



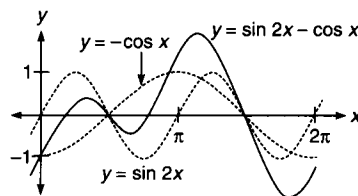
15.



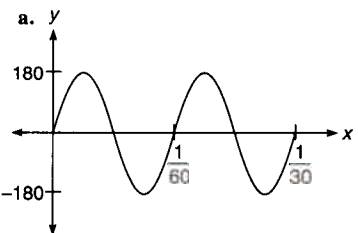
17.



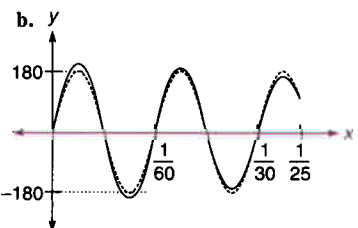
19.



21.

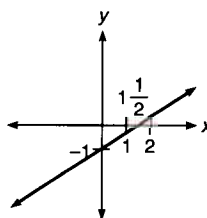


b.



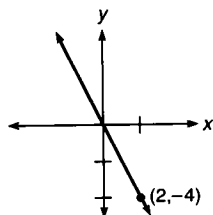
Solutions to trial exercise problems

5. $f(x) = \frac{2}{3}x - 1$; $y = \frac{2}{3}x - 1$
 Let $x = 0$: $y = -1$, so $(0, -1)$ is the y -intercept.
 Let $y = 0$:
 $0 = \frac{2}{3}x - 1$
 $1 = \frac{2}{3}x$ Multiply by $\frac{3}{2}$.
 $\frac{3}{2} = x$, so $(\frac{3}{2}, 0)$ is the x -intercept.
 (See the diagram.)



6. $f(x) = -2x$; $y = -2x$

By letting $x = 0$ and then $y = 0$, we find that the x - and y -intercepts are the same point $(0,0)$. Thus, to graph this function we need another point. To get another point, we can let x (or y) be any value but 0, which we have already used. Let $x = 2$; thus, $y = -4$, and $(2,-4)$ is another point to plot. (See the diagram.)



13. $y = \sin 2x + \cos 3x$

First we graph $y = \sin 2x$:

$$0 \leq 2x \leq 2\pi$$

$$0 \leq x \leq \pi$$

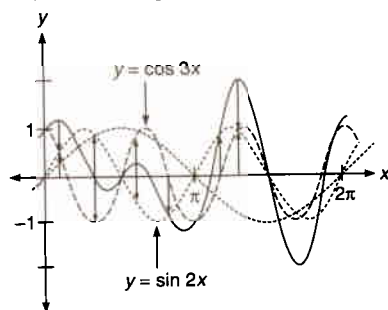
(shown in short dashed lines in the diagram)

and $y = \cos 3x$:

$$0 \leq 3x \leq 2\pi$$

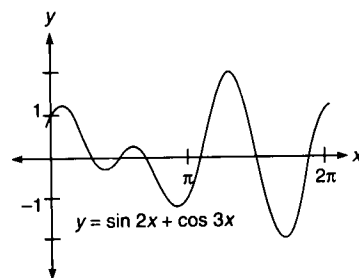
$$0 \leq x \leq \frac{2\pi}{3}$$

(shown in long dashed lines).



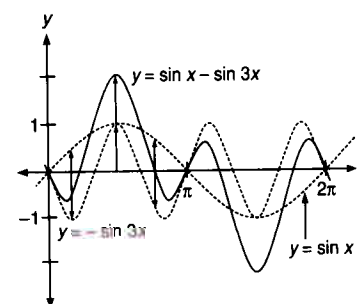
Next we add the ordinates. Remember that wherever one function is zero (crosses the x -axis), the result is the

other function. Wherever the functions have equal absolute values but opposite signs, the result is zero. Wherever both functions cross, the result is twice the height of either function. (See the diagram.)



17. $y = \sin x - \sin 3x$

We rewrite the function as $y = \sin x + (-\sin 3x)$. We now graph $y = \sin x$ and $y = -\sin 3x$, and add the ordinates. (See the diagram.)



20. $y = 7.5 \sin \frac{\pi}{6}x + 10 \sin \frac{\pi}{3}x$

$$0 \leq \frac{\pi}{6}x \leq 2\pi. \text{ Multiply by } \frac{6}{\pi}.$$

$$\frac{6}{\pi} \cdot 0 \leq \frac{6}{\pi} \cdot \frac{\pi}{6}x \leq \frac{6}{\pi} \cdot 2\pi; 0 \leq x \leq 12.$$

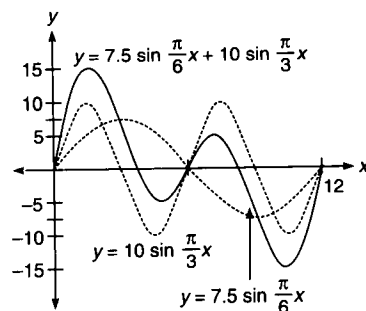
Similarly,

$$0 \leq \frac{\pi}{3}x \leq 2\pi. \text{ Multiply by } \frac{3}{\pi}.$$

$$\frac{3}{\pi} \cdot 0 \leq \frac{3}{\pi} \cdot \frac{\pi}{3}x \leq \frac{3}{\pi} \cdot 2\pi;$$

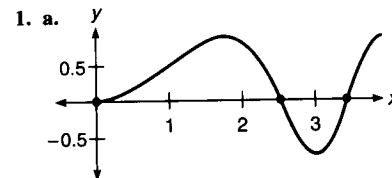
$$0 \leq 2x \leq 12; 0 \leq x \leq 6.$$

(See the diagram.)

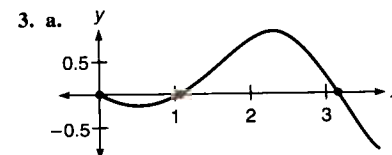


Appendix B

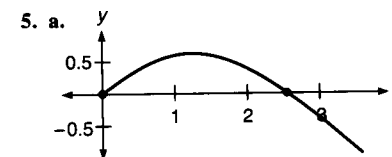
Answers to odd-numbered problems



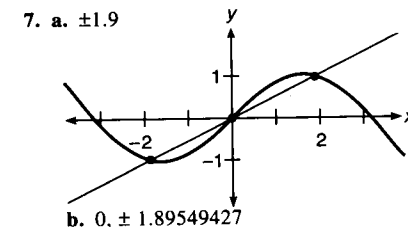
b. 0, 2.5, 3.5 c. 0, 2.50662827, 3.54490770



b. 0, 1.1, 3.2 c. 0, 1.04719755, 3.14159265 (π)

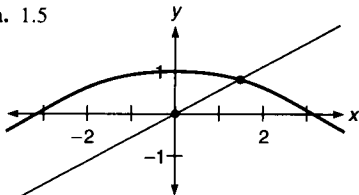


b. 0, 2.5 c. 0, 2.47457679



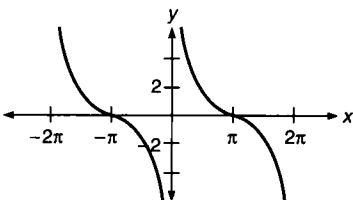
b. 0, ± 1.89549427

9. a. 1.5

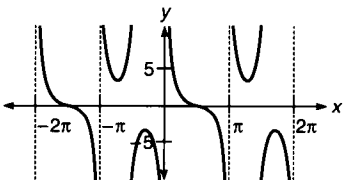


b. 1.47817027

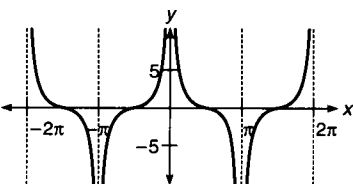
11. The graph of $y = \csc \theta + \cot \theta$, and $y = \frac{1 + \cos \theta}{\sin \theta}$ both look like the following.



13. The graphs of $y = \frac{\csc \theta}{\sec \theta + \tan \theta}$ and of $y = \frac{\cos \theta}{\sin \theta + \sin^2 \theta}$ both look like the following.



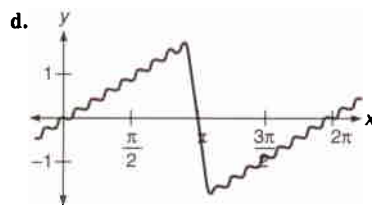
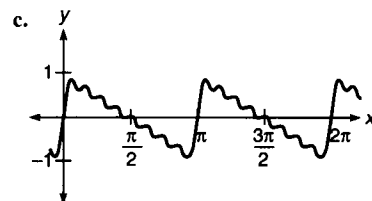
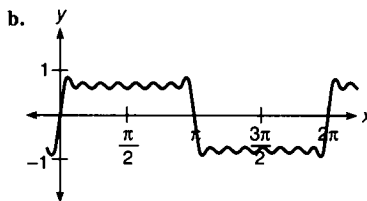
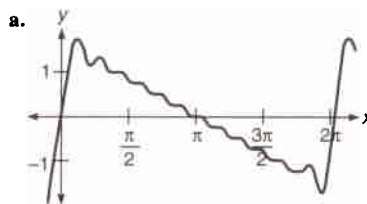
15. The graph of both $y = \frac{1}{\sec \theta - \cos \theta}$ and of $y = \cot \theta \csc \theta$ are as follows.



17. See the answers to appendix A.

19. See the answers to appendix A, problem 21.

21. Each graph done with the first 12 terms.



Solutions to trial exercise problems

22. a. x	$\sin(x)$	$\sin[(4\pi + 1)x]$	$\sin[(8\pi + 1)x]$	$\sin[(12\pi + 1)x]$
0	0.00	0.00	0.00	0.00
0.5	0.48	0.48	0.48	0.48
1	0.84	0.84	0.84	0.84
1.5	1.00	1.00	1.00	1.00
2	0.91	0.91	0.91	0.91
2.5	0.60	0.60	0.60	0.60
3	0.14	0.14	0.14	0.14
3.5	-0.35	-0.35	-0.35	-0.35
4	-0.76	-0.76	-0.76	-0.76
4.5	-0.98	-0.98	-0.98	-0.98
5	-0.96	-0.96	-0.96	-0.96
5.5	-0.71	-0.71	-0.71	-0.71
6	-0.28	-0.28	-0.28	-0.28
6.5	0.22	0.22	0.22	0.22

All functions of the form $y = \sin[(4k\pi + 1)x]$ are equal if x is an integer multiple of $\frac{1}{2}$.

- b. The second column of part b of the next table shows the values that these functions produce. They are the same for the four functions. All functions of the form $y = \sin[(8k\pi + 1)x]$ are equal if x is an integer multiple of $\frac{1}{4}$.

- c. The second column of part c of the table below shows the values that these functions produce. They are the same for the four functions.

All functions of the form $y = \sin[(20k\pi + 1)x]$ are equal if x is an integer multiple of $\frac{1}{10}$.

Part b		Part c	
x	$\sin[(8\pi + 1)x]$	x	$\sin[(20k\pi + 1)x]$
0	0.00	0	0.00
0.25	0.25	0.1	0.10
0.5	0.48	0.2	0.20
0.75	0.68	0.3	0.30
1	0.84	0.4	0.39
1.25	0.95	0.5	0.48
1.5	1.00	0.6	0.56
1.75	0.98	0.7	0.64
2	0.91	0.8	0.72
2.25	0.78	0.9	0.78
2.5	0.60	1	0.84
2.75	0.38	1.1	0.89
3	0.14	1.2	0.93
3.25	-0.11	1.3	0.96

- d. In parts a, b, c, we notice that $\frac{1}{2} \cdot 4k\pi = 2k\pi$, $\frac{1}{4} \cdot 8k\pi = 2k\pi$, $\frac{1}{10} \cdot 20k\pi = 2k\pi$, so we know that for $\frac{1}{100}$ we need $\frac{1}{100} \cdot \beta k\pi = 2k\pi$, so $\beta = 200$. Thus, all functions of the form $y = \sin[(200k\pi + 1)x]$ are equal if x is an integer multiple of $\frac{1}{100}$.

- e. To see why the generalization of part a is true we consider the following:

If x is an integer multiple of $\frac{1}{2}$ we can describe it as

$$x = \frac{n}{2} \text{ for some integer } n. \text{ Then,}$$

$$\begin{aligned} \sin[(4k\pi + 1)x] &= \sin\left[\left(4k\pi + 1\right) \cdot \frac{n}{2}\right] = \sin\left(2nk\pi + \frac{n}{2}\right) \\ &= \sin 2nk\pi \cos \frac{n}{2} + \cos 2nk\pi \sin \frac{n}{2} \\ &= 0 \cdot \cos \frac{n}{2} + 1 \cdot \sin \frac{n}{2} \\ &= \sin \frac{n}{2} = \sin x \end{aligned}$$

Thus, as long as x is an integer multiple of $\frac{1}{2}$, $\sin[(4k\pi + 1)x] = \sin x$.

Similar reasoning shows why the generalizations of parts b, c, and d are true.

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Student Solutions Manual to Accompany Trigonometry with Applications

Introduction to the Student

When you study mathematics you are doing something which will help you for the rest of your life. Although most students do not realize it mathematics is used in almost every discipline which a student is likely to enter, from nuclear physics to music. In fact, in the age of electronic calculating devices mathematics is more important than ever.

Assumptions upon which the text was created

In writing this text we have assumed that you have completed an intermediate algebra course, and therefore has been introduced to solving equations, factoring, radicals and graphing linear equations. This means that the language of algebra should not be new to you. The following problems and their solutions should not seem entirely foreign to you, even if you have forgotten some of the details.

1. Solve the equation $5x - 3 = 2(x + 7)$

Solution: $5x - 3 = 2x + 14$

$$5x - 2x - 3 = 14$$

Subtract $2x$ from each member

$$3x - 3 = 14$$

$$5x - 2x = 3x$$

$$3x = 17$$

Add 3 to both members

$$x = \frac{17}{3}$$

Divide each member by 3

2. Factor $3x^2 - 2x - 4$

Solution: $3x^2 - 9x - 4$

$$(3x \quad \quad)(x \quad \quad)$$

$$3x \cdot x = 3x^2$$

$$(3x \pm 4)(3x \pm 1)$$

Several ways to get -4 in the third term

$$(3x \pm 2)(3x \pm 2)$$

The choice which gives $-9x$ for the middle term

$$(3x - 4)(3x + 1)$$

3. Simplify $\sqrt{8x^5}$

Solution: $\sqrt{8x^5} = \sqrt{2^2 \cdot 2 \cdot x^4 \cdot x}$
 $= 2x^2\sqrt{2x}$

Factor

$$\sqrt{2^2} = 2; \sqrt{x^4} = x^2$$

Throughout the text we will remind you about the details of this material as needed.

It is also assumed that you own a scientific calculator or graphing calculator, and are familiar with the basic keys for arithmetic computation. Keystrokes for a typical scientific calculator are presented in the text where they go beyond the basic arithmetic operations.



Graphing Calculators/Computers

You may own a graphing calculator instead of a scientific calculator. The text specifically indicates how to use these devices. Examples are presented based on the TEXAS INSTRUMENTS TI-81 graphing calculator. The introductory section *Computer Aided Mathematics* introduces the basic principles involved in using a graphing calculator, using the TI-81 as an example.

Tips for Success

- **Work** Success in a mathematics course requires a lot of work! You must work assigned problems all the time. Much of what you will learn is complicated and needs constant practice.
- **Regular study time** Try to set aside some time every day in which you will do some mathematics. If you have already finished all assigned problems then go back to previous sections of the text and do some review problems. Certain sections will have seemed harder than others. Go back to these harder sections! With enough work you will understand everything. Remember what Thomas Edison said. “Genius is 10% inspiration and 90% perspiration.”
- **You can do it!** It is true that some people have a “knack” for mathematics, just as some do for tennis, basketball, music, drawing, or English composition. However, this does not mean that you cannot do these things if you don’t have the knack – it just means that you have to work at it. No one gets good at anything without working at it. If they don’t seem to work at something it’s just because they love it and don’t consider what they do to be work.
- **How to study** Read the text seated at a desk. Have a pencil and paper at hand. When you come to an example, copy the steps onto your paper as you read. This will help you understand better for several reasons.
 - Mathematics looks different when it is written by hand than when it is printed. You should get used to what it looks like when you write it!
 - Some ways are better than others to organize mathematics problems. We try to show you a good way to do this in the text, and you want to get used to it.
 - It has been shown that one learns best when more than one sense is used. Writing uses motor skills as well as sight.
- **Do the homework** It has been said that mathematics is not a spectator sport. You could watch someone play the piano for years, but this would not help you learn to play the piano. You must do it yourself.
- **Mathematics Ability is not in the Genes** – *Anyone can do mathematics.* Many people tend to feel that difficulties in learning mathematics indicate lack of ability, and that mathematics should therefore be avoided by an individual encountering problems. The answer to difficulties is to work harder, not to give up. Mathematics is very important throughout most professions, and it just won’t go away!

Features of the text to help you learn

Problem Sets

- The drill portion of the problem sets are similar to the examples of that section.
- The *answers to all problems* appear at the end of the text, except for even-numbered exercises. The *complete solutions to selected problems*, indicated by boxing the problem number, also appear at the end of the text. If you have trouble with a particular problem you should be able to find the solution to a similar problem either in the examples in the text, the selected (box numbered) problems, or in this manual.
- The exercises progress from straightforward application of the material covered in the exposition to problem solving via more difficult application problems and then to problems which require some ingenuity and creativity. Those problems requiring exceptional ingenuity or amount of work are marked with the symbol . If you try the complete range of problems you will have a good introduction to the way in which this material is used in a variety of disciplines.

Chapter Summary, Review, Test at the end of each chapter

- Chapter Summary – This reviews the highlights and key points of the chapter. Read this over after the chapter is completed. Think about each item. If something seems unfamiliar go and find it in the text and review it.
- Chapter Review – This presents review problems from the chapter, keyed to sections. You should work this after the chapter is completed.
- Chapter Test – This is designed to help you practice the material as it might appear on a test, out of the context of each section. The chapter test may well be longer than what would be given in class. The test provides material out of the context provided by knowing what section it is from, and by being surrounded by similar problems. In the homework sets there are inevitably many clues to the method of solution, including nearby problems and temporal and physical proximity to explanations. The chapter test is an aid to make that last link in learning – recognition of problem type, with attending method of solution.

Applications — Applications problems are put in the text to show you where mathematics is used in the various disciplines. Do not be afraid of them. You will see that you don't have to know, for example, electronics to do those applications which apply to electronics. The problems always make clear what mathematical operations and principles are involved.

Work hard and you will enjoy your education
and succeed at it.

Chapter 1

Exercise 1-1

1. $13^\circ 25' = \left(13 + \frac{25}{60}\right)^\circ \approx 13.417^\circ$;

acute

5. $25^\circ 33' 19'' =$

$\left(25 + \frac{33}{60} + \frac{19}{3600}\right)^\circ \approx 25.555^\circ$; acute

29. $c^2 = a^2 + b^2$

$10^2 = a^2 + 8^2$

$100 = a^2 + 64$

$36 = a^2$

$a = \sqrt{36} = 6$

33. $c^2 = a^2 + b^2$

$c^2 = (\sqrt{5})^2 + 3^2$

$c^2 = 5 + 9$

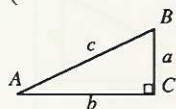
57. $d^2 = 15^2 + 10.5^2$, so $d \approx 18.3$ cm, and

$\frac{18.3 \text{ cm}}{0.8 \text{ cm/min}} = 23$ minutes to make the cut.

9. $33^\circ 5' 55'' =$

$\left(33 + \frac{5}{60} + \frac{55}{3600}\right)^\circ \approx 33.099^\circ$; acute

13.



$c^2 = 14$

$c = \sqrt{14} \approx 3.7 \approx 4$

37. $c^2 = a^2 + b^2$

$c^2 = (\sqrt{7})^2 + 3^2$

$c^2 = 7 + 9$

$c^2 = 16$

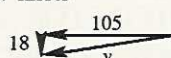
$c = 4$

41. $c^2 = a^2 + b^2$

$c^2 = (3\sqrt{2})^2 + (4\sqrt{5})^2$

$61. v^2 = 18^2 + 105^2 = 11349$

$v \approx 107$ knots



$c^2 = 9 \cdot 2 + 16 \cdot 5$

$c^2 = 98$

$c = \sqrt{98}$

$c = 7\sqrt{2} \approx 9.9$

45. $c^2 = a^2 + b^2$

$c^2 = 1^2 + 1^2$

$c^2 = 2$

$c = \sqrt{2} \approx 1.4 \approx 1$

17. $90^\circ - 18^\circ 12' = 89^\circ 60' - 18^\circ 12' = 71^\circ 48'$

21. $.3^2 + .4^2 = .5^2$ so it is a right triangle.

Hypotenuse is 0.5

25. $0.32^2 + 2.55^2 = 2.57^2$, so this is a right triangle with hypotenuse 2.57.

49. $213^2 = 193^2 + w^2$

$8120 = w^2$

$90.1 \text{ feet} \approx w$

53. $Z^2 = R^2 + X_L^2$

$4340^2 = R^2 + 2150^2$

$14213100 = R^2$

$3770 \text{ ohms} \approx R$

65. $B^2 = a^2 + b^2$

$C^2 = B^2 + c^2 = a^2 + b^2 + c^2$

$D^2 = C^2 + d^2$

$= a^2 + b^2 + c^2 + d^2$

$E^2 = D^2 + e^2$

$= a^2 + b^2 + c^2 + d^2 + e^2$

$E^2 = 20^2 + 5^2 + 12^2 + 8^2 + 10^2$

$E^2 = 733$

$E \approx 27$

Exercise 1-2

1. $\sin A = \frac{a}{c}$

$\csc A = \frac{c}{a}$

$\cos A = \frac{b}{c}$

$\tan A = \frac{a}{b}$

$\sin B = \frac{b}{c}$

$\cos B = \frac{a}{c}$

$\tan B = \frac{b}{a}$

5. $c^2 = 1^2 + 3^2$

$c = \sqrt{10}$

$\sin B = \frac{b}{c} = \frac{3}{\sqrt{10}} = \frac{3}{10}\sqrt{10}$

$\csc B = \frac{\sqrt{10}}{3}$

$\cos B = \frac{a}{c} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$

$\sec B = \sqrt{10}$

$\tan B = \frac{b}{a} = \frac{3}{1} = 3$ $\cot B = \frac{1}{3}$

9. $2^2 + b^2 = (\sqrt{5})^2$

$b^2 = 1$

$b = 1$

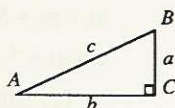
$\sin B = \frac{b}{c} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

$\csc B = \sqrt{5}$

$\cos B = \frac{a}{c} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

$\sec B = \frac{\sqrt{5}}{2}$

$\tan B = \frac{b}{a} = \frac{1}{2}$ $\cot B = \frac{2}{1} = 2$



13. $6^2 + b^2 = 10^2$

$b = 8$

$b = 8$

$\sin A = \frac{a}{c} = \frac{6}{10} = \frac{3}{5}$

$\csc A = \frac{5}{3}$

$\cos A = \frac{b}{c} = \frac{8}{10} = \frac{4}{5}$

$\sec A = \frac{5}{4}$

$\tan A = \frac{a}{b} = \frac{6}{8} = \frac{3}{4}$

$\cot A = \frac{4}{3}$

17. $x^2 + b^2 = z^2$

$b^2 = z^2 - x^2$

$b = \sqrt{z^2 - x^2}$

$\sin B = \frac{b}{c} = \frac{\sqrt{z^2 - x^2}}{z}$

$\csc B = \frac{z}{\sqrt{z^2 - x^2}}$

$\cos B = \frac{a}{c} = \frac{x}{z}$

$\sec B = \frac{c}{a} = \frac{z}{x}$

$\tan B = \frac{b}{a} = \frac{\sqrt{z^2 - x^2}}{x}$

$\cot B = \frac{x}{\sqrt{z^2 - x^2}}$

21. $9^2 + 5^2 = c^2$

$106 = c^2$

$\sqrt{106} = c$

$\sin B = \frac{b}{c} = \frac{5}{\sqrt{106}} \cdot \frac{\sqrt{106}}{\sqrt{106}}$

$= \frac{5\sqrt{106}}{106}$ $\csc B = \frac{\sqrt{106}}{5}$

$\cos B = \frac{a}{c} = \frac{9}{\sqrt{106}} = \frac{9}{106}\sqrt{106}$

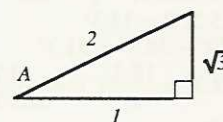
$\sec B = \frac{\sqrt{106}}{9}$

$\tan B = \frac{b}{a} = \frac{5}{9}$ $\cot B = \frac{9}{5}$

25. $\cos A = 0.5 = \frac{1}{2}$ $\sec A = 2$

$\sin A = \frac{\sqrt{3}}{2}$ $\csc A = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$

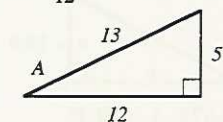
$\tan A = \sqrt{3}$ $\cot A = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$



29. $\sin A = \frac{5}{13}$ $\csc A = \frac{13}{5}$

$\cos A = \frac{12}{13}$ $\sec A = \frac{13}{12}$

$\tan A = \frac{5}{12}$ $\cot A = \frac{12}{5}$



33. $\sin^2 A + \cos^2 A$

$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2$

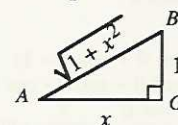
$\frac{a^2}{c^2} + \frac{b^2}{c^2}$

$\frac{a^2 + b^2}{c^2}$

$\frac{c^2}{c^2}$

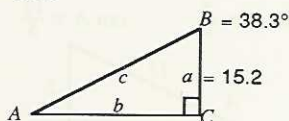
1

37. $\tan B = x = \frac{x}{1}$ $\tan A = \frac{\text{opp}}{\text{adj}} = \frac{1}{x}$

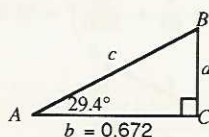


Exercise 1-3

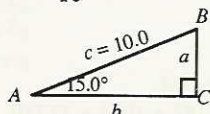
1. $\sin 31.28^\circ \approx 0.5192$
5. $\cot 28.87^\circ \approx 1.8137$
 $28.87 \text{ [tan] } [1/x]$
 $\text{TI-81 [(] [TAN] 28.87 [)] }$
 $[x^{-1}] \text{ [ENTER]}$
9. $\sec 66.47^\circ \approx 2.5048$
13. $\sin 78^\circ 33' \approx 0.9801$
 $78 \text{ [] } 33 \text{ [] } 60 \text{ [] } \text{[sin]}$
 $\text{TI-81 [SIN] [(] 78 [] 33 [] 60 []] }$
 [)] [ENTER]
17. $\tan 35^\circ 8' \approx 0.7037$
21. $\sin 48^\circ 8' \approx 0.7447$
25. $R = \frac{611.1}{2 \sin 18^\circ 20'} \approx 971.40 \text{ sq. meters}$
29. $P = 42 \cdot 25 \cdot \cos 45^\circ \approx 742.5 \text{ watts}$
33. Using the figure, we find $c = \sqrt{2}$. Then
 $\sin 45^\circ = \sin A = \frac{a}{c} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
 $\cos 45^\circ = \cos A = \frac{b}{c} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
 $\tan 45^\circ = \tan A = \frac{a}{b} = \frac{1}{1} = 1$
37. $\tan \theta = 1.8807 \quad \theta \approx 62.0^\circ$
41. $\sin \theta = \frac{35.9}{68.3} \quad \theta \approx 31.7^\circ$
45. $\sec \theta = 4.8097 \quad \theta \approx 78.0^\circ$
49. $a = 15.2, B = 38.3^\circ$
 $A = 90^\circ - 38.3^\circ = 51.7^\circ$
 $\cos 38.3^\circ = \frac{15.2}{c}; c = \frac{15.2}{\cos 38.3^\circ} \approx 19.4$
 $\tan 38.3^\circ = \frac{b}{15.2}; b = 15.2 \tan 38.3^\circ \approx 12.0$



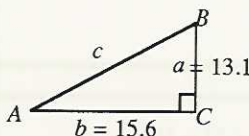
53. $b = 0.672, A = 29.4^\circ$
 $B = 90^\circ - 29.4^\circ = 60.6^\circ$
 $\tan 29.4^\circ = \frac{a}{0.672}; a = 0.672 \tan 29.4^\circ \approx 0.379$
 $\cos 29.4^\circ = \frac{0.672}{c}; c = \frac{0.672}{\cos 29.4^\circ} \approx 0.771$



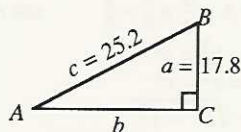
57. $c = 10.0, A = 15.0^\circ$
 $B = 90^\circ - 15.0^\circ = 75.0^\circ$
 $\cos 15^\circ = \frac{b}{10}; b = 10 \cos 15^\circ \approx 9.7$
 $\sin 15^\circ = \frac{a}{10}; a = 10 \sin 15^\circ \approx 2.6$



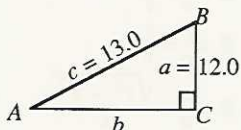
61. $a = 13.1 \quad b = 15.6$
 $c^2 = \sqrt{13.1^2 + 15.6^2}; c \approx 20.4$
 $\tan A = \frac{13.1}{15.6}; A \approx 40.0^\circ$
 $\tan B = \frac{15.6}{13.1}; B \approx 50.0^\circ$



65. $a = 17.8 \quad c = 25.2$
 $17.8^2 + b^2 = 25.2^2$
 $b = \sqrt{25.2^2 - 17.8^2} \approx 17.8$
 $\sin A = \frac{17.8}{25.2}; A \approx 44.9^\circ$
 $\cos B = \frac{17.8}{25.2}; B \approx 45.1^\circ$

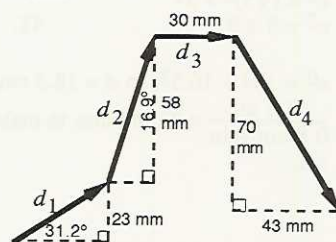


69. $a = 12.0 \quad c = 13.0$
 $b = \sqrt{13^2 - 12^2} \approx 5.0$
 $\sin A = \frac{12}{13}; A \approx 67.4^\circ$
 $\cos B = \frac{12}{13}; B \approx 22.6^\circ$

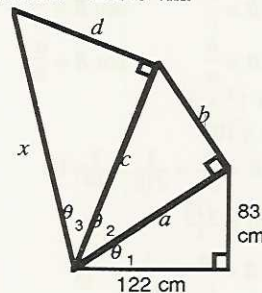


73. $\sin \theta = \frac{X_L}{Z}; \sin 24.2^\circ = \frac{22.6}{Z}$
 $Z = \frac{22.6}{\sin 24.2^\circ} \approx 55.1 \text{ ohms}$

77. $E = \frac{45}{2.5 \cos 15^\circ} \approx 18.6 \text{ volts}$
81. $\sin 31.2^\circ = \frac{23}{d_1}; d_1 = \frac{23}{\sin 31.2^\circ} \approx 44.399 \text{ mm}$
 $\cos 16.9^\circ = \frac{58}{d_2}; d_2 = \frac{58}{\cos 16.9^\circ} \approx 60.618 \text{ mm}$
 $d_3 = 30 \text{ mm}$
 $d_4 = \sqrt{70^2 + 43^2} \approx 82.152 \text{ mm}$
 $\text{distance} = d_1 + d_2 + d_3 + d_4 = 217.2 \text{ mm}$



85. $a = \sqrt{122^2 + 83^2} \approx 147.5567687 \text{ cm}$
 $\tan \theta_1 = \frac{83}{122}; \theta_1 \approx 34.22854584^\circ$
 $\theta_2 = \theta_1 + 5^\circ \approx 39.22854584^\circ$
 $\theta_3 = \theta_2 + 5^\circ \approx 44.22854584^\circ$
 $\cos \theta_2 = \frac{a}{c}; c = \frac{a}{\cos \theta_2} = \frac{147.5567687}{\cos 39.22854584^\circ} \approx 190.4868947 \text{ cm}$
 $\cos \theta_3 = \frac{c}{x}; x = \frac{c}{\cos \theta_3} = \frac{190.4868947}{\cos 44.22854584^\circ} \approx 265.8340543 \text{ cm}$
 $\text{Thus } x \approx 265.8 \text{ cm.}$



Exercise 1-4

1. $\tan \theta \cot \theta$
 $\frac{1}{\cot \theta} \cdot \cot \theta$
 1
5. $\sec \theta (\cot \theta + \cos \theta - 1)$
 $\sec \theta \cdot \cot \theta + \sec \theta \cdot \cos \theta - \sec \theta$
 $\frac{1}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta} + \frac{1}{\cos \theta} \cdot \cos \theta - \sec \theta$

9. $\frac{1}{\sin \theta} + 1 - \sec \theta$
 $\csc \theta + 1 - \sec \theta$
 $1 - \cos^2 \theta$
 $(\sin^2 \theta + \cos^2 \theta) - \cos^2 \theta$
 $\sin^2 \theta$

13. $(\cos \theta + \sin \theta)(\cos \theta - \sin \theta) + 2 \sin^2 \theta$
 $\cos^2 \theta - \cos \theta \cdot \sin \theta + \sin \theta \cos \theta - \sin^2 \theta + 2 \sin^2 \theta$
 $\cos^2 \theta + \sin^2 \theta$
 1
17. $\tan 32^\circ 40' \left| \frac{\sin 32^\circ 40'}{\cos 32^\circ 40'} \right|$
 $0.64117 \left| \frac{0.53975}{0.84182} \right| \approx 0.64117$

21. $\sin \beta (\cot \beta - \csc \beta + \sin \beta)$
 $\sin \beta \cdot \cot \beta - \sin \beta \cdot \csc \beta + \sin \beta \cdot \sin \beta$
 $\sin \beta$
 $2 \cos x = 1$
 $\cos x = \frac{1}{2}$
 $x = \cos^{-1} \frac{1}{2} = 60^\circ$
25. $5 \sin x = 1$
 $\sin x = \frac{1}{5}$
 $x = \sin^{-1} \frac{1}{5} \approx 11.5^\circ$
29. $\frac{\sin x}{\cos x} = \frac{2}{11}$
 $\sin x = \frac{2}{11}$
 $x = \sin^{-1} \frac{2}{11} \approx 10.5^\circ$
33. $\frac{\sin x}{3} = \frac{2}{11}$
 $\sin x = \frac{6}{11}$
 $x = \sin^{-1} \frac{6}{11} \approx 33.1^\circ$
37. $\tan 2x = \sqrt{3}$
 $2x = \tan^{-1} \sqrt{3}$
 $2x = 60^\circ$
 $x = 30^\circ$
41. $2 \cos 4x = 1$
 $\cos 4x = \frac{1}{2}$
 $4x = \cos^{-1} \frac{1}{2}$
 $4x = 60^\circ$
 $x = 15^\circ$
45. $\sin B = \frac{b}{c}$, $\cos B = \frac{a}{c}$
 $\frac{\sin B}{\cos B} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{b}{a} \cdot \frac{c}{c} = \frac{b}{a} = \tan B$

Chapter 1 Review

1. $17^\circ 34' 17'' = (17 + \frac{34}{60} + \frac{17}{3600})^\circ \approx 17.57^\circ$; acute
3. $125^\circ 37' = (125 + \frac{37}{60})^\circ \approx 125.62^\circ$; obtuse
5. $180^\circ - (81^\circ 43' 12'' + 38^\circ 19' 56'') = 180^\circ - 120^\circ 3' 8'' = 59^\circ 56' 52''$
7. $c = \sqrt{a^2 + b^2} = \sqrt{8^2 + 12^2} = \sqrt{208}$
 $= \sqrt{16 \cdot 13} = 4\sqrt{13} \approx 14.4$
9. $a = \sqrt{c^2 - b^2}$
 $= \sqrt{13^2 - 8^2}$
 $= \sqrt{105} \approx 10.2$
11. $R = \sqrt{Z^2 - X_L^2} = \sqrt{56.6^2 - 40^2}$
 $\approx 40.0 \text{ ohms}$
13. $b = \sqrt{15^2 - 5^2} = \sqrt{200} = 10\sqrt{2}$
 $\sin B = \frac{b}{c} = \frac{10\sqrt{2}}{15} = \frac{2}{3}\sqrt{2}$
 $\csc B = \frac{3}{2\sqrt{2}} = \frac{3}{4}\sqrt{2}$
 $\cos B = \frac{a}{c} = \frac{5}{15} = \frac{1}{3}$ $\sec B = 3$
 $\tan B = \frac{b}{a} = \frac{10\sqrt{2}}{5} = 2\sqrt{2}$
 $\cot B = \frac{1}{2\sqrt{2}} = \frac{1}{4}\sqrt{2}$
15. $\sin B = \frac{\text{opp}}{\text{hyp}} = \frac{3}{5}$
17. $\sec A = 3$ so $\cos A = \frac{1}{3}$
 $\tan B = \frac{\text{opp}}{\text{adj}} = \frac{1}{2\sqrt{2}} = \frac{1}{4}\sqrt{2}$
 $\sin 15.5^\circ \approx 0.2672$
 $\tan 17.9^\circ \approx 0.3230$
 $\tan 16.3^\circ \approx 0.2924$
 $\cot 31^\circ 20' = \frac{1}{\tan 31^\circ 20'} \approx 1.6426$
 $\tan 31^\circ 30' \approx 0.6128$
 $\tan 63^\circ 30' \approx 2.0057$
 $\sin 53.2^\circ \approx 0.8007$
 $\csc 53.2^\circ \approx 1.2489$
 $\cos 53.2^\circ \approx 0.5990$
 $\sec 53.2^\circ \approx 1.6694$
 $\tan 53.2^\circ \approx 1.3367$
 $\cot 53.2^\circ \approx 0.7481$
 $\sin 53^\circ 20' \approx 0.8021$
 $\csc 53^\circ 20' \approx 1.2467$
 $\cos 53^\circ 20' \approx 0.5972$
 $\sec 53^\circ 20' \approx 1.6746$
 $\tan 53^\circ 20' \approx 1.3432$
 $\cot 53^\circ 20' \approx 0.7445$
 $\theta = \sin^{-1} 0.6314 \approx 39.2^\circ$
 $\theta = \tan^{-1} (\frac{35.9}{50.0}) \approx 35.7^\circ$
 $\sin \theta = \frac{1}{1.425}$; $\theta = \sin^{-1} \frac{1}{1.425} \approx 44.6^\circ$
 $B = 90^\circ - 56.1^\circ = 33.9^\circ$
 $\sin 56.1^\circ = \frac{23.3}{c}$; $c = \frac{23.3}{\sin 56.1^\circ} \approx 28.1$
 $\tan 56.1^\circ = \frac{23.3}{b}$; $b = \frac{23.3}{\tan 56.1^\circ} \approx 15.7$
41. $c = \sqrt{4^2 + 8.55^2} \approx 9.44$
 $\tan A = \frac{4}{8.55}$; $A = \tan^{-1} \frac{4}{8.55} \approx 25.07^\circ$
 $B \approx 90^\circ - 25.07^\circ = 64.93^\circ$
43. $\sin 11.2^\circ = \frac{h}{22.6}$; $h = 22.6 \sin 11.2^\circ \approx 4.39 \text{ kilometers}$
45. $\sin^2 45^\circ + \cos^2 45^\circ = (\frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2 = \frac{2}{4} + \frac{2}{4} = 1$
47. $\csc \theta (\sin \theta - \tan \theta)$
 $\frac{1}{\sin \theta} \cdot \sin \theta - \frac{1}{\sin \theta} \cdot \frac{\sin \theta}{\cos \theta}$
 $1 - \frac{1}{\cos \theta}$
 $1 - \sec \theta$
49. $\frac{\cos \alpha + 2 - \cot \alpha}{\cos \alpha}$
 $\frac{\cos \alpha}{\cos \alpha} + \frac{2}{\cos \alpha} - \frac{\cot \alpha}{\cos \alpha}$
 $1 + 2 \sec \alpha - \frac{\cos \alpha}{\sin \alpha} \cdot \frac{1}{\cos \alpha}$
 $1 + 2 \sec \alpha - \frac{1}{\sin \alpha}$
 $1 + 2 \sec \alpha - \csc \alpha$
51. $3 \sin 2x = 2$
 $\sin 2x = \frac{2}{3}$
 $2x = \sin^{-1} \frac{2}{3}$
 $x = \frac{1}{2} \sin^{-1} \frac{2}{3} \approx 20.9^\circ$

Chapter 1 Test

1. $(26 + \frac{27}{60} + \frac{43}{3600})^\circ \approx 26.462^\circ$
3. $b = \sqrt{12^2 - 6^2} = \sqrt{108} = \sqrt{36 \cdot 3} = 6\sqrt{3}$
5. $\sec A = \frac{5}{3}$, so $\cos A = \frac{3}{5}$; $\tan B = \frac{3}{4}$
7. $\sec 78.3^\circ = \frac{1}{\cos 78.3^\circ} \approx 4.9313$
9. $P = 220 \cdot 4.1 \cos 48^\circ \approx 603.6 \text{ watts}$
11. $\cos \theta = \frac{1}{4.121}$; $\theta = \cos^{-1} \frac{1}{4.121} \approx 76.0^\circ$
13. $c = \sqrt{5.2^2 + 7.9^2} \approx 9.5$
 $\tan A = \frac{5.2}{7.9}$; $A = \tan^{-1} \frac{5.2}{7.9} \approx 33.4^\circ$
 $B \approx 90^\circ - 33.4^\circ = 56.6^\circ$
15. $\cos \theta (\sec \theta - \cos \theta)$
 $\cos \theta \cdot \frac{1}{\cos \theta} - \cos^2 \theta$
 $1 - \cos^2 \theta$
 $\sin^2 \theta$

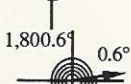
Chapter 2

Exercise 2-1

1. a. Is a function because no first element repeats.
b. domain is first elements: {3,4,6,7}
range is second elements: {5,9,10}
c. Not one to one because a second element (5) repeats.
5. a. Not a function because a first element (1) repeats.
9. a. 9 b. 19 c. 11
d. $f(7)$ is not defined because 7 is not one of the first elements of any of the ordered pairs in f. d. $f(250)$ not defined for the same reasons.
13. $h(x) = (x-3)(x+2)$
a. $h(-2) = (-2-3)(-2+2) = 0(-2,0)$
b. $h(0) = (0-3)(0+2) = -6 \quad (0,-6)$
c. $h(\sqrt{3}) = (\sqrt{3}-3)(\sqrt{3}+2) = 3+2\sqrt{3}-3\sqrt{3}-6 = -3-\sqrt{3}$
 $(\sqrt{3}, -3-\sqrt{3})$
d. $h(\frac{1}{2}) = (\frac{1}{2}-3)(\frac{1}{2}+2) = (-\frac{5}{2})(\frac{5}{2}) = -\frac{25}{4}$
 $(\frac{1}{2}, -\frac{25}{4})$
17. $f(x) = 3x^4 - x^2 + 2$
a. $f(-2) = 3(-2)^4 - (-2)^2 + 2 = 46 \quad (-2, 46)$
b. $f(0) = 3 \cdot 0^4 - 0^2 + 2 = 2 \quad (0, 2)$
c. $f(\sqrt{3}) = 3(\sqrt{3})^4 - (\sqrt{3})^2 + 2 = 26 \quad (\sqrt{3}, 26)$
d. $f(\frac{1}{2}) = 3(\frac{1}{2})^4 - (\frac{1}{2})^2 + 2 = \frac{31}{16} \quad (\frac{1}{2}, \frac{31}{16})$
21. $t(x) = 500 - 2x$
a. $t(3) = 500 - 2(3) = 494^\circ$
b. $t(12) = 500 - 2(12) = 476^\circ$
c. $t(104) = 500 - 2(104) = 292^\circ$

Exercise 2-2

1. 420°
 $420^\circ - 360^\circ = 60^\circ$

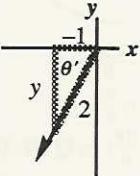
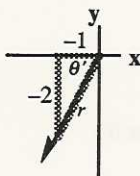
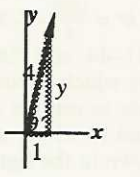
5. $1,800.6^\circ$
 $1,800.6^\circ - 5 \cdot 360^\circ = 0.6^\circ$

25. $(-5, 8)$ $r = \sqrt{(-5)^2 + 8^2} = \sqrt{89}$
 $\sin \theta = \frac{y}{r} = \frac{8}{\sqrt{89}} = \frac{8\sqrt{89}}{89}$ $\csc \theta = \frac{1}{\sin \theta} = \frac{\sqrt{89}}{8}$
 $\cos \theta = \frac{x}{r} = \frac{-5}{\sqrt{89}} = -\frac{5\sqrt{89}}{89}$ $\sec \theta = \frac{1}{\cos \theta} = -\frac{\sqrt{89}}{5}$
 $\tan \theta = \frac{y}{x} = \frac{8}{-5} = -1\frac{3}{5}$ $\cot \theta = \frac{1}{\tan \theta} = -\frac{5}{8}$
29. $(-1, 4)$ $r = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$
 $\sin \theta = \frac{y}{r} = \frac{4}{\sqrt{17}} = \frac{4\sqrt{17}}{17}$ $\csc \theta = \frac{1}{\sin \theta} = \frac{\sqrt{17}}{4}$
 $\cos \theta = \frac{x}{r} = \frac{-1}{\sqrt{17}} = -\frac{\sqrt{17}}{17}$ $\sec \theta = \frac{1}{\cos \theta} = -\sqrt{17}$
 $\tan \theta = \frac{y}{x} = \frac{4}{-1} = -4$ $\cot \theta = \frac{1}{\tan \theta} = -\frac{1}{4}$
33. $(-\sqrt{2}, 6)$ $r = \sqrt{(-\sqrt{2})^2 + 6^2} = \sqrt{38}$
 $\sin \theta = \frac{y}{r} = \frac{6}{\sqrt{38}} = \frac{6\sqrt{38}}{38} = \frac{3\sqrt{38}}{19}$ $\csc \theta = \frac{1}{\sin \theta} = \frac{\sqrt{38}}{6}$
 $\cos \theta = \frac{x}{r} = \frac{-\sqrt{2}}{\sqrt{38}} = -\frac{\sqrt{76}}{38} = -\frac{2\sqrt{19}}{38} = -\frac{\sqrt{19}}{19}$ $\sec \theta = \frac{1}{\cos \theta} = -\frac{\sqrt{38}}{\sqrt{2}} = -\frac{\sqrt{76}}{2} = -\frac{2\sqrt{19}}{2} = -\sqrt{19}$
 $\tan \theta = \frac{y}{x} = \frac{6}{-\sqrt{2}} = -\frac{6\sqrt{2}}{2} = -3\sqrt{2}$ $\cot \theta = \frac{1}{\tan \theta} = -\frac{\sqrt{2}}{6}$
37. $(\sqrt{6}, -\sqrt{10})$ $r = \sqrt{(\sqrt{6})^2 + (-\sqrt{10})^2} = \sqrt{16} = 4$
 $\sin \theta = \frac{y}{r} = \frac{-\sqrt{10}}{4}$ $\csc \theta = -\frac{4}{\sqrt{10}} = -\frac{4\sqrt{10}}{10} = -\frac{2\sqrt{10}}{5}$
 $\cos \theta = \frac{x}{r} = \frac{\sqrt{6}}{4}$ $\sec \theta = \frac{1}{\cos \theta} = \frac{4}{\sqrt{6}} = \frac{4\sqrt{6}}{6} = \frac{2\sqrt{6}}{3}$
 $\tan \theta = \frac{y}{x} = \frac{-\sqrt{10}}{\sqrt{6}} = -\frac{\sqrt{60}}{6} = -\frac{2\sqrt{15}}{6} = -\frac{\sqrt{15}}{3}$ $\cot \theta = \frac{1}{\tan \theta} = -\frac{3}{\sqrt{15}} = -\frac{3\sqrt{15}}{15} = -\frac{\sqrt{15}}{5}$
41. $(\sqrt{2}b, b)$ $r = \sqrt{(\sqrt{2}b)^2 + b^2} = \sqrt{2b^2 + b^2} = \sqrt{3b^2} = \sqrt{3}b$
 $\sin \theta = \frac{y}{r} = \frac{b}{\sqrt{3}b} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$ $\csc \theta = \sqrt{3}$
 $\cos \theta = \frac{x}{r} = \frac{\sqrt{2}b}{\sqrt{3}b} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{1}{3}\sqrt{6}$ $\sec \theta = \frac{\sqrt{3}}{\sqrt{2}} = \frac{1}{2}\sqrt{6}$
 $\tan \theta = \frac{y}{x} = \frac{b}{\sqrt{2}b} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$ $\cot \theta = \sqrt{2}$
45. $\sec \theta = \frac{r}{x}$ Definition of $\sec \theta$. 49. $\sin \theta = \frac{y}{r} = \frac{1}{\frac{r}{y}} = \frac{1}{\csc \theta}$
 $= \frac{1}{\frac{x}{r}} = \frac{1}{\cos \theta}$

Exercise 2-3

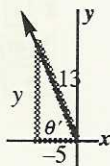
1. $\sin \theta > 0, \cos \theta < 0$
I, II II, III
5. $\tan \theta < 0, \csc \theta < 0$
 $\sin \theta < 0$
II III, IV
9. $\sec \theta > 0, \sin \theta < 0$
 $\cos \theta > 0$
I, IV IV
13. $\theta = 164.2^\circ$ in Quadrant II, so $\theta' = 180^\circ - \theta = 180^\circ - 164.2^\circ = 15.8^\circ$.
17. -255.3° is coterminal with $-255.3^\circ + 360^\circ = 104.7^\circ$. $\theta = 104.7^\circ$ in Quadrant II, so $\theta' = 180^\circ - \theta = 180^\circ - 104.7^\circ = 75.3^\circ$. Thus θ' for -255.3° is 75.3° .

21. -181.0° is coterminal with $-181.0^\circ + 360^\circ = 179.0^\circ$. $\theta = 179.0^\circ$ in Quadrant II, so $\theta' = 180^\circ - \theta = 180^\circ - 179.0^\circ = 1.0^\circ$. Thus θ' for -181.0° is 1.0° .
25. -252° is coterminal with $-252^\circ + 360^\circ = 108^\circ$. $\theta = 108^\circ$ in Quadrant II, so $\theta' = 180^\circ - \theta = 180^\circ - 108^\circ = 72^\circ$. Thus θ' for -252° is 72° .
29. $\theta' = 30^\circ$; $\sin 30^\circ = \frac{1}{2}$. In QIII, so $\sin 210^\circ < 0$; $\sin 210^\circ = -\frac{1}{2}$.
33. $\sin(-120^\circ)$ $\theta' = 60^\circ$; $\sin 60^\circ = \frac{\sqrt{3}}{2}$. In QIII, so $\sin(-120^\circ) < 0$:
 $\sin(-120^\circ) = -\frac{\sqrt{3}}{2}$
37. $\theta' = 60^\circ$; $\cot 60^\circ = \frac{1}{\tan 60^\circ} = \frac{\sqrt{3}}{3}$. 300° is in Quadrant IV, where cotangent is negative, so
 $\cot 300^\circ = -\frac{\sqrt{3}}{3}$.
41. $\csc 90^\circ = \frac{1}{\sin 90^\circ} = 1$
45. $\theta' = 30^\circ$. $\cos 30^\circ = \frac{\sqrt{3}}{2}$; 150° in Quadrant II, so $\cos 150^\circ = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$. $\sec 150^\circ = \frac{1}{\cos 150^\circ} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$.
49. 0.6898537919
53. 1.026304108
 13 $\boxed{+/-}$ $\boxed{\cos}$ $\boxed{1/x}$
 $\boxed{\text{TI-81}}$ $\boxed{(\text{COS})}$ $\boxed{(-)}$ 13 $\boxed{)}$ $\boxed{x^{-1}}$
 $\boxed{\text{ENTER}}$
57. -1.036744916
61. $\theta = \sin^{-1} 0.25 \approx 14.5^\circ$
65. $\theta = \tan^{-1}\left(-\frac{8}{5}\right) \approx -57.995^\circ \approx -58.0^\circ$
 8 $\boxed{+}$ 5 $\boxed{=}$ $\boxed{\pm}$ $\boxed{\text{SHIFT}}$ $\boxed{\tan}$
 $\boxed{\text{TI-81}}$ $\boxed{2\text{nd}}$ $\boxed{\text{TAN}}$ $\boxed{(-)}$ $\boxed{(-)}$ 8
 $\boxed{+}$ 5 $\boxed{)}$ $\boxed{\text{ENTER}}$
69. (a) $\theta = 0^\circ$ $E = 156 \sin(\theta + 45^\circ) = 156 \sin(0^\circ + 45^\circ) = 156 \sin 45^\circ \approx 110.31$
 (b) $\theta = 45^\circ$ $E = 156 \sin(45^\circ + 45^\circ) = 156 \sin 90^\circ = 156$
 (c) $\theta = 100^\circ$ $E = 156 \sin(100^\circ + 45^\circ) = 156 \sin 145^\circ \approx 89.48$
 (d) $\theta = -200^\circ$ $E = 156 \sin(-200^\circ + 45^\circ) = 156 \sin(-155^\circ) \approx -65.93$
 (e) $\theta = 13.3^\circ$ $E = 156 \sin(13.3^\circ + 45^\circ) = 156 \sin 58.3^\circ \approx 132.73$
 (f) $\theta = -45^\circ$ $E = 156 \sin(-45^\circ + 45^\circ) = 156 \sin 0^\circ = 0$
73. $\sin(2\theta) = 2 \sin \theta$ $\frac{\sqrt{3}}{2} = 2 \cdot \frac{1}{2}$
 $\sin(2 \cdot 30^\circ) = 2 \sin 30^\circ$ $\frac{\sqrt{3}}{2} = 1$ No
 $\sin 60^\circ = 2 \sin 30^\circ$

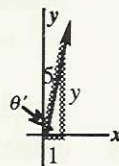
Exercise 2-4

1. $\sin \theta = 0.8251$, $\cos \theta > 0$
 $\sin \theta' = 0.8251$, so $\theta' \approx 55.6^\circ$
 Since $\sin \theta > 0$, $\cos \theta > 0$, θ is in QI.
 In QI $\theta = \theta'$, so $\theta \approx 55.6^\circ$.
5. $\sec \theta = -1.0642$, $\sin \theta < 0$
 $\cos \theta = -\frac{1}{1.0642}$ so $\cos \theta' = \frac{1}{1.0642}$; $\theta' \approx 20.0^\circ$.
 $\cos \theta < 0$, $\sin \theta < 0$, so θ is in QIII.
 $y = -\sqrt{22 - (-1)^2} = -\sqrt{21}$
 $\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{2} = -\frac{\sqrt{21}}{2}$ $\csc \theta = -\frac{2}{\sqrt{21}}$
 $\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{-1} = \frac{-\sqrt{21}}{-1} = \sqrt{21}$
 $\cot \theta = \frac{1}{\sqrt{21}}$
 $\cos \theta' = \frac{1}{2}$, so $\theta' = 60^\circ$.
 $\theta = 180^\circ + \theta' = 180^\circ + 60^\circ = 240^\circ$.
- 
21. $\tan \theta = 2$, $\cos \theta < 0$; $\cot \theta = \frac{1}{2}$
 $\tan \theta > 0$, $\cos \theta < 0$ so θ in QIII.
 $r = +\sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$
 $\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{-2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$
 $\csc \theta = -\frac{1}{2\sqrt{5}}$
- QIII.
 $\theta = 180^\circ + \theta' \approx 180^\circ + 20.0^\circ \approx 200.0^\circ$.
9. $\sin \theta = \frac{3}{8}$, $\cos \theta > 0$
 $\sin \theta' = \frac{3}{8}$, so $\theta' \approx 22.0^\circ$.
 $\sin \theta > 0$, $\cos \theta > 0$, so θ is in QI,
 and
 $\theta = \theta' \approx 22.0^\circ$.
- $\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{-1}{r} = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$
 $\sec \theta = -\sqrt{5}$
 $\tan \theta' = 2$, so $\theta' \approx 63.4^\circ$.
 $\theta = 180^\circ + \theta' \approx 180^\circ + 63.4^\circ \approx 243.4^\circ$.
- 
25. $\csc \theta = -1$; $\sin \theta = \frac{1}{\csc \theta} = \frac{1}{-1} = -1$;
 θ is 270° ; pick a point, say $(0, -1)$ on the terminal side of the angle.
 $r = \sqrt{x^2 + y^2} = \sqrt{0^2 + (-1)^2} = 1$
 $\cos \theta = \frac{x}{r} = \frac{0}{1} = 0$; $\sec \theta$ undefined
 $\tan \theta = \frac{y}{x} = \frac{-1}{0}$ (undefined); $\cot \theta = 0$
13. $\cos \theta = -\frac{5}{7}$, $\tan \theta > 0$
 $\cos \theta' = \frac{5}{7}$, so $\theta' \approx 44.4^\circ$.
 $\cos \theta < 0$, $\tan \theta > 0$ so θ is in QIII.
 $\theta = 180^\circ + \theta' \approx 180^\circ + 44.4^\circ \approx 224.4^\circ$.
29. $\sec \theta = 4$, $\csc \theta > 0$
 $\cos \theta = \frac{1}{4}$, $\sin \theta > 0$
 $\cos \theta > 0$, $\sin \theta > 0$ so θ in QI.
 $y = +\sqrt{4^2 - 1^2} = \sqrt{15}$
 $\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{4} = \frac{\sqrt{15}}{4}$; $\csc \theta = \frac{4}{\sqrt{15}}$
 $\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{1} = \sqrt{15}$; $\cot \theta = \frac{1}{\sqrt{15}}$
 $\cos \theta = \frac{1}{4}$, so $\theta \approx 75.5^\circ$
 $\theta = \theta' \approx 75.5^\circ$
- 

$$\begin{aligned}
 33. \quad \cos \theta &= -\frac{5}{13}, \sin \theta > 0 \\
 \sec \theta &= -\frac{13}{5} \\
 \cos \theta < 0, \sin \theta > 0, \text{ so } \theta \text{ in QII.} \\
 y &= +\sqrt{13^2 - (-5)^2} = 12 \\
 \sin \theta &= \frac{\text{opp}}{\text{hyp}} = \frac{y}{13} = \frac{12}{13}, \csc \theta = \frac{13}{12} \\
 \tan \theta &= \frac{\text{opp}}{\text{adj}} = \frac{y}{-5} = -\frac{12}{5}, \cot \theta = -\frac{5}{12} \\
 \cos \theta' &= \frac{5}{13} \\
 \text{so } \theta' &\approx 67.4^\circ \\
 \theta &= 180^\circ - \theta' \\
 &\approx 180^\circ - 67.4^\circ \\
 &\approx 112.6^\circ
 \end{aligned}$$



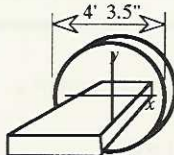
$$\begin{aligned}
 37. \quad \sec \theta &= 5, \tan \theta > 0 \\
 \cos \theta &= \frac{1}{5}, \tan \theta > 0 \\
 \cos \theta > 0, \tan \theta > 0 \text{ so } \theta \text{ in QI.} \\
 y &= +\sqrt{5^2 - 1^2} = 2\sqrt{6} \\
 \sin \theta &= \frac{\text{opp}}{\text{hyp}} = \frac{y}{5} = \frac{2\sqrt{6}}{5} \\
 \csc \theta &= \frac{5}{2\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{5\sqrt{6}}{12} = \frac{5}{12}\sqrt{6} \\
 \tan \theta &= \frac{\text{opp}}{\text{adj}} = \frac{y}{1} = y = 2\sqrt{6} \\
 \cot \theta &= \frac{1}{2\sqrt{6}} = \frac{1}{12}\sqrt{6} \\
 \cos \theta' &= \frac{1}{5} \text{ so } \theta' \\
 &\approx 78.5^\circ \\
 \theta &= \theta' \approx 78.5^\circ
 \end{aligned}$$



$$\begin{aligned}
 41. \quad (2, -5); r &= \sqrt{x^2 + y^2} = \sqrt{4 + 25} \\
 &= \sqrt{29} \\
 \sin \theta &= \frac{y}{r} = \frac{-5}{\sqrt{29}} = \frac{-5\sqrt{29}}{29} \\
 \csc \theta &= \frac{1}{\sin \theta} = -\frac{\sqrt{29}}{5} \\
 \cos \theta &= \frac{x}{r} = \frac{2}{\sqrt{29}} = \frac{2\sqrt{29}}{29} \\
 \sec \theta &= \frac{\sqrt{29}}{2} \\
 \tan \theta &= \frac{y}{x} = -\frac{5}{2}; \cot \theta = -\frac{2}{5} \\
 \tan \theta' &= 2.5, \theta' \approx 68.2^\circ \\
 \sin \theta < 0, \cos \theta > 0 \text{ so } \theta \text{ terminates in} \\
 \text{QIV. } \theta &= 360^\circ - \theta' \approx 360^\circ - 68.2^\circ = \\
 &291.8^\circ
 \end{aligned}$$

$$\begin{aligned}
 45. \quad \theta &= -134.4^\circ, r = 8.25. \\
 \sin \theta &= \frac{y}{r}, \text{ so } y = r \sin \theta, \text{ so } y = 8.25 \sin(-134.4^\circ) \approx -5.89 \text{ cm.} \\
 \cos \theta &= \frac{x}{r}, \text{ so } x = r \cos \theta, \text{ so } x = 8.25 \\
 \cos(-134.4^\circ) &\approx -5.77 \text{ cm}
 \end{aligned}$$

$$\begin{aligned}
 49. \quad 4' 3.5'' &= 4\frac{3.5}{12} \approx 4.292'; r \approx \frac{4.292}{2} \text{ ft} \approx 2.146 \text{ ft;} \\
 \theta &= 211.5^\circ, \sin \theta = \frac{y}{r}; y = r \sin \theta; \\
 y &\approx 2.146 \sin 211.5^\circ \approx -1.121 \text{ ft.} \\
 0.121 \text{ ft} \times 12''/\text{ft} &\approx 1.5'', \text{ so } y \approx -1' 1.5'' \\
 \cos \theta &= \frac{x}{r}; x = r \cos \theta \\
 x &\approx 2.146 \cos 211.5^\circ \approx -1.830 \text{ ft.} \\
 0.830 \text{ ft} \times 12''/\text{ft} &\approx 10.0'', \text{ so } \\
 x &\approx -1' 10.0''
 \end{aligned}$$



$$\begin{aligned}
 53. \quad \tan \theta &= u \text{ and } \theta \text{ terminates in quadrant III.} \\
 \text{Note that we use the values } \frac{-u}{-1} &\text{ to represent the value } u. \text{ This} \\
 &\text{is because the side adjacent to } \theta' \text{ must be negative in quadrant} \\
 &\text{III.}
 \end{aligned}$$

$$r = \sqrt{1^2 + u^2} = \sqrt{1 + u^2}$$

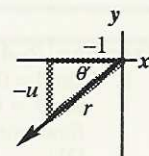
$$\cot \theta = \frac{1}{\tan \theta} = \frac{1}{u}$$

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{-u}{r} = -\frac{u}{\sqrt{1 + u^2}}$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{-1}{r} = -\frac{1}{\sqrt{1 + u^2}}$$

$$\sec \theta = \frac{1}{\cos \theta} = -\sqrt{1 + u^2}$$

$$\csc \theta = \frac{1}{\sin \theta} = -\frac{\sqrt{1 + u^2}}{u}$$



$$\begin{aligned}
 57. \quad \sin p &= \frac{AB \sin b}{AP} = \frac{512.4 \cdot \sin 28.3^\circ}{322.6} \approx 0.75302; \\
 p &\approx 48.852^\circ \\
 a &= 180^\circ - (b + p) \approx 102.848^\circ \\
 BP &= \frac{AP \sin a}{\sin b} \approx \frac{322.6 \cdot \sin 102.848^\circ}{\sin 28.3^\circ} \approx 663.4 \text{ ft.}
 \end{aligned}$$

Exercise 2-5

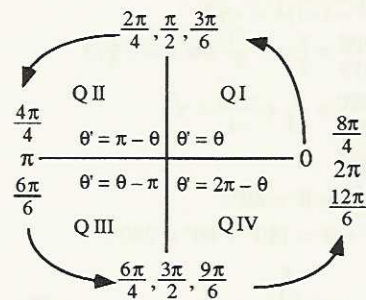
$$\begin{aligned}
 1. \quad x^2 + y^2 &= 1 \\
 5. \quad 100^\circ \quad \frac{s}{\pi} &= \frac{100^\circ}{180^\circ}; s = \frac{100^\circ \cdot \pi}{180^\circ} = \frac{5\pi}{9} \approx 1.75 \\
 9. \quad 270^\circ \quad \frac{s}{\pi} &= \frac{270^\circ}{180^\circ}; s = \frac{270^\circ \cdot \pi}{180^\circ} = \frac{3\pi}{2} \approx 4.71 \\
 13. \quad -305^\circ \quad \frac{s}{\pi} &= \frac{-305^\circ}{180^\circ}; s = \frac{-305^\circ \cdot \pi}{180^\circ} = -\frac{61\pi}{36} \\
 &\approx -5.32
 \end{aligned}$$

In problems 14 through 30 we solve the definition of radians for θ° :

$$\frac{s}{\pi} = \frac{\theta^\circ}{180^\circ}; \theta^\circ = \frac{180^\circ}{\pi} \cdot s$$

$$\begin{aligned}
 17. \quad \frac{3\pi}{5} \quad \theta^\circ &= \frac{180^\circ}{\pi} \cdot \frac{3\pi}{5} = 108^\circ \\
 21. \quad -\frac{17\pi}{6} \quad \theta^\circ &= \frac{180^\circ}{\pi} \cdot \left(-\frac{17\pi}{6}\right) = -510^\circ \\
 25. \quad -\frac{12}{17} \quad \theta^\circ &= \frac{180^\circ}{\pi} \cdot \left(-\frac{12}{17}\right) = -\frac{2160^\circ}{17\pi} \approx -40.44^\circ \\
 29. \quad -5 \quad \theta^\circ &= \frac{180^\circ}{\pi} \cdot (-5) = -\frac{900}{\pi} \approx -286.48^\circ
 \end{aligned}$$

In problems 31–44 add 2π to negative values until there is a coterminal angle which is positive. Then change each denominator of 3 to 6 to be able to use the figure. Then compare the expression to the figure to find in which quadrant the angle terminates. Then use the formula shown in the figure to find θ' .



$$\begin{aligned}
 33. \quad \frac{11\pi}{6}, \text{ in Q IV; } \theta' &= 2\pi - \theta = \frac{12\pi}{6} - \frac{11\pi}{6} = \frac{\pi}{6} \\
 37. \quad \frac{3\pi}{4}, \text{ in Q II; } \theta' &= \pi - \theta = \frac{4\pi}{4} - \frac{3\pi}{4} = \frac{\pi}{4} \\
 41. \quad -\frac{2\pi}{3} + 2\pi &= -\frac{2\pi}{3} + \frac{6\pi}{3} = \frac{4\pi}{3}, \frac{4\pi}{3} = \frac{8\pi}{6}, \text{ in Q III; } \theta' = \theta - \pi \\
 &= \frac{4\pi}{3} - \frac{3\pi}{3} = \frac{\pi}{3}
 \end{aligned}$$

$$45. \quad \text{See the left side of figure 2.14.}$$

$$49. \quad r = 4.5 \text{ mm, } L = 12 \text{ mm;}$$

$$L = rs; 12 = 4.5s; s \approx 2.7 \text{ radians}$$

53. Find L where $r = \frac{32.4}{2} = 16.2$ inches and

$$\theta^\circ = 85^\circ.$$

$$85^\circ = \frac{17}{36}\pi$$

$$L = rs$$

$$L = 16.2 \cdot \frac{17\pi}{36} \approx 24.0 \text{ in}$$

57. $A_p = \frac{sr^2}{2} = \frac{\frac{3\pi}{5} \cdot 6^2}{2} = 18 \cdot \frac{3\pi}{5} = \frac{54}{5}\pi \approx 33.93 \text{ cm}^2$

61. $A_p = \frac{\theta^\circ(\pi r^2)}{360^\circ} = \frac{15^\circ \cdot \pi \cdot 9^2}{360^\circ} = \frac{27}{8}\pi \approx 10.60 \text{ mm}^2$

63. $A_p = \frac{sr^2}{2}$

$$14.6 = \frac{s \cdot 4.85^2}{2}$$

$$29.2 = 4.85^2 s$$

$$s = \frac{29.2}{4.85^2} \approx 1.24 \text{ radians}$$

64. $A_p = \frac{\theta^\circ(\pi r^2)}{360^\circ}; r = 50/2 = 25 \text{ in}$

$$200 = \frac{\theta^\circ(\pi \cdot 25^2)}{360^\circ}$$

$$200 = \frac{625\pi}{360} \cdot \theta^\circ$$

$$\theta^\circ = \frac{200 \cdot 360}{625\pi} = \frac{576}{5\pi} \approx 36.7^\circ$$

65. The new area to be covered is the area of the larger, outer sector less the inner sector.

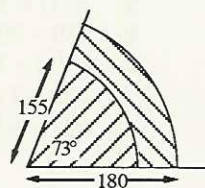
$$\frac{73^\circ \cdot \pi \cdot 180^2}{360^\circ} - \frac{73^\circ \cdot \pi \cdot 155^2}{360^\circ} \approx 5335.3 \text{ ft}^2$$

Dividing this by 150 ft² per gallon gives $\frac{5335.3}{150} \approx 35.57$ gallons, which is

$35\frac{1}{2}$ gallons to the nearest half gallon.

Note: The calculation can be simplified algebraically to

$$\frac{73\pi}{360}(180^2 - 155^2).$$



Exercise 2-6

1. $\frac{2\pi}{3}$ is in quadrant II, so $\sin \frac{2\pi}{3} > 0$ and

$$\theta' = \pi - \frac{2\pi}{3} = \frac{\pi}{3}. \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \text{ so } \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}.$$

5. $\frac{4\pi}{3}$ is in quadrant III, so $\cos \frac{4\pi}{3} < 0$ and $\theta' =$

$$= \frac{4\pi}{3} - \pi = \frac{\pi}{3}. \cos \frac{\pi}{3} = \frac{1}{2}, \text{ so } \cos \frac{4\pi}{3} = -\frac{1}{2}.$$

Make sure the calculator is in radian mode.

17. $\tan 0.5 \approx 0.5463$

21. $\sin 2.3 \approx 0.7457$

25. $\csc 2.5 = \frac{1}{\sin 2.5} \approx 1.6709$

29. $\theta' = \sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}$

$\sin \theta < 0, \tan \theta < 0$ so θ in qIV

$$\theta = 2\pi - \theta' = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

33. $\sin \theta = -\frac{1}{2}$ so $\theta' = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$

$\sin \theta < 0, \cos \theta > 0$ so θ in qIV.

$$\theta = 2\pi - \theta' = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

37. $\theta' = \cos^{-1} 0.885 \approx 0.484$

$\cos \theta < 0, \tan \theta > 0$ so θ in qIII

$$\theta = \pi + \theta' \approx \pi + 0.484 \approx 3.63$$

41. $3 \sin 2\theta = 1$

$$\sin 2\theta = \frac{1}{3}$$

$$2\theta = \sin^{-1} \frac{1}{3}$$

$$\theta = \frac{1}{2} \sin^{-1} \frac{1}{3} \approx 0.17$$

$$3 \left[\frac{1}{x} \right] \left[\sin^{-1} \right] \left[\div \right] \left[2 \right] \left[= \right]$$

$$\text{TI-81: } \left[\left(\right] \left[1 \right] \left[\div \right] \left[2 \right] \left[\right] \left[= \right]$$

$$\left[2\text{nd} \right] \left[\sin^{-1} \right] \left[\left(\right] \left[1 \right] \left[\div \right] \left[3 \right] \left[\right] \left[= \right]$$

Note: $3 \left[x^{-1} \right]$ is the same

$$\text{as } 1 \left[\div \right] 3$$

9. $\frac{5\pi}{3}$ is in quadrant IV, so $\tan \frac{5\pi}{3} < 0$ and $\theta' =$

$$2\pi - \frac{5\pi}{3} = \frac{\pi}{3}. \tan \frac{\pi}{3} = \sqrt{3}, \text{ so } \tan \frac{5\pi}{3} = -\sqrt{3}.$$

13. $-\frac{7\pi}{6}$ is coterminal with $\frac{5\pi}{6}$ so $\sin(-\frac{7\pi}{6}) = \sin \frac{5\pi}{6} = \frac{1}{2}$

45. $\sin \frac{\theta}{2} = 1$

$$\frac{\theta}{2} = \sin^{-1} 1$$

$$\frac{\theta}{2} = \frac{\pi}{2}$$

$$\theta = 2\left(\frac{\pi}{2}\right)$$

$$\theta = \pi$$

49. $\cos \theta(2 \cos \theta - 1) = 0$

$$\cos \theta = 0 \text{ or } 2 \cos \theta - 1 = 0$$

$$\theta = \cos^{-1} 0 \text{ or } 2 \cos \theta = 1$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = \cos^{-1} \frac{1}{2}$$

$$\theta = \frac{\pi}{2} \text{ or } \frac{\pi}{3}$$

53. $d = \frac{1}{3} \cos 8t - \frac{1}{4} \sin 8t$

(a) $t = \frac{1}{8} = 0.125:$

$$d = \frac{1}{3} \cos 8(0.125) - \frac{1}{4} \sin 8(0.125)$$

$$\approx -0.030$$

(b) $t = \frac{1}{4} = 0.25:$

$$d = \frac{1}{3} \cos 8(0.25) - \frac{1}{4} \sin 8(0.25)$$

$$\approx -0.366$$

57. $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

(a) $x = 0.1:$ $\sin 0.1 \approx 0.1 - \frac{0.1^3}{6} + \frac{0.1^5}{120} - \frac{0.1^7}{5040} \approx 0.0998334166$

(b) $x = 0.5:$ $\sin 0.5 \approx 0.5 - \frac{0.5^3}{6} + \frac{0.5^5}{120} - \frac{0.5^7}{5040} \approx 0.4794255332$

(c) $x = 1:$ $\sin 1 \approx 1 - \frac{1^3}{6} + \frac{1^5}{120} - \frac{1^7}{5040} \approx 0.8414682540$

(d) $x = \left(\frac{\pi}{6}\right):$ $\sin \frac{\pi}{6} \approx \frac{\pi}{6} - \frac{(\frac{\pi}{6})^3}{6} + \frac{(\frac{\pi}{6})^5}{120} - \frac{(\frac{\pi}{6})^7}{5040} \approx 0.4999999919$

Using Calculator

$$0.0998334166$$

$$0.4794255386$$

$$0.8414709848$$

$$0.5$$

Note: To calculate part (d) most easily, compute $\frac{\pi}{6}$ and store it in the calculator's memory. Assuming $\boxed{\text{Min}}$ is the key to enter a

value into memory, and $\boxed{\text{MRec}}$ is the key to recall the value in memory, then the following will calculate part d: $\boxed{\pi} \boxed{\div} \boxed{6} \boxed{=}$ $\boxed{\text{Min}}$

$$\boxed{\text{MRec}} \boxed{x^y} \boxed{3} \boxed{-} \boxed{6} \boxed{+} \boxed{\text{MRec}} \boxed{x^y} \boxed{5} \boxed{+} \boxed{120} \boxed{-} \boxed{\text{MRec}} \boxed{x^y} \boxed{7} \boxed{+} \boxed{5040} \boxed{=}$$

On the TI-81, store the equation as $Y_1:$ $\boxed{Y=}$ $\boxed{\text{CLEAR}}$ $\boxed{\text{XIT}}$ $\boxed{-}$ $\boxed{\text{XIT}}$ $\boxed{\text{MATH}}$ $\boxed{3}$ $\boxed{+}$ $\boxed{6}$ $\boxed{+}$ $\boxed{\text{XIT}}$ $\boxed{\wedge}$ $\boxed{5}$ $\boxed{+}$ $\boxed{120}$

$\boxed{-}$ $\boxed{\text{XIT}}$ $\boxed{\wedge}$ $\boxed{7}$ $\boxed{\div}$ $\boxed{5040}$ $\boxed{2\text{nd}}$ $\boxed{\text{CLEAR}}$. To compute the value for say $x = -1$, store this in X and calculate, as follows: 0.1

$$\boxed{\text{STO}} \boxed{\text{XIT}}$$
 $\boxed{2\text{nd}}$ $\boxed{\text{VARS}}$ $\boxed{1}$ $\boxed{\text{ENTER}}$

Chapter 2 Review

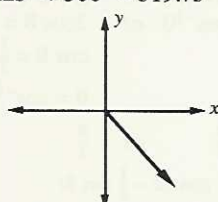
1. a. function because no first element repeats
b. domain: {1,4,6,7}
range: {5,7,10}
3. Not a function because a second element (7) repeats.

5. Not a function because a first element (2) repeats.

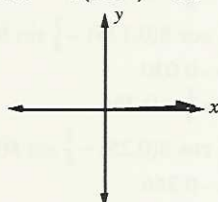
5. $f(x) = 3x^2 - 2x + 3$
 - a. $f(-1) = 3(-1)^2 - 2(-1) + 3 = 8$
 - b. $f(0) = 3(0)^2 - 2(0) + 3 = 3$
 - c. $f(\sqrt{5}) = 3(\sqrt{5})^2 - 2\sqrt{5} + 3 = 3(5) - 2\sqrt{5} + 3 = 18 - 2\sqrt{5}$
 - d. $f(\frac{1}{3}) = 3(\frac{1}{3})^2 - 2(\frac{1}{3}) + 3 = \frac{1}{3} - \frac{2}{3} + 3 = 2\frac{2}{3}$

7. $f(x) = \frac{3x}{x-1}$
 - a. $f(-1) = \frac{3(-1)}{-1-1} = \frac{-3}{-2} = \frac{3}{2}$
 - b. $f(0) = \frac{3(0)}{0-1} = 0$
 - c. $f(\sqrt{5}) = \frac{3\sqrt{5}}{\sqrt{5}-1}$
 - d. $f(\frac{1}{3}) = \frac{3(\frac{1}{3})}{\frac{1}{3}-1} = \frac{1}{-\frac{2}{3}} = -\frac{3}{2}$

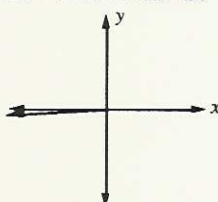
9. $-40.25^\circ + 360^\circ = 319.75^\circ$



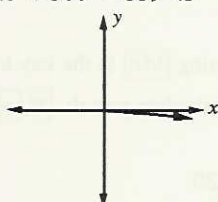
11. $1,800.6^\circ - 5(360^\circ) = 0.6^\circ$



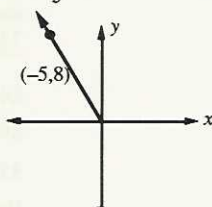
13. $547^\circ 26' - 360^\circ = 187^\circ 26'$



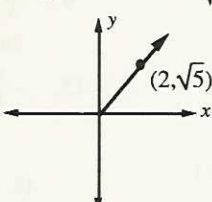
15. $-0^\circ 15' + 360^\circ = 359^\circ 45'$



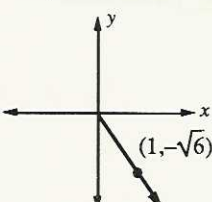
17. $(-5, 8) \quad r = \sqrt{25 + 64} = \sqrt{89}$
 $\sin \theta = \frac{8}{\sqrt{89}} = \frac{8}{\sqrt{89}} \quad \csc \theta = \frac{\sqrt{89}}{8}$
 $\cos \theta = \frac{-5}{\sqrt{89}} = -\frac{5}{\sqrt{89}} \quad \sec \theta = -\frac{\sqrt{89}}{5}$
 $\tan \theta = -\frac{8}{5} \quad \cot \theta = -\frac{5}{8}$



19. $(2, \sqrt{5}) \quad r = \sqrt{4 + 5} = 3$
 $\sin \theta = \frac{\sqrt{5}}{3} \quad \csc \theta = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$
 $\cos \theta = \frac{2}{3} \quad \sec \theta = \frac{3}{2}$
 $\tan \theta = \frac{\sqrt{5}}{2} \quad \cot \theta = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$



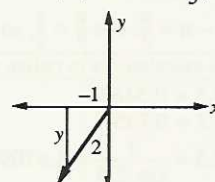
21. $(1, -\sqrt{6}) \quad r = \sqrt{1 + 6} = \sqrt{7}$
 $\sin \theta = \frac{-\sqrt{6}}{\sqrt{7}} = -\frac{\sqrt{42}}{7}$
 $\csc \theta = -\frac{\sqrt{7}}{\sqrt{6}} = -\frac{\sqrt{42}}{6}$
 $\cos \theta = \frac{1}{\sqrt{7}} = \frac{1}{7}\sqrt{7} \quad \sec \theta = \sqrt{7}$
 $\tan \theta = \frac{-\sqrt{6}}{1} = -\sqrt{6} \quad \cot \theta = -\frac{1}{\sqrt{6}} = -\frac{\sqrt{6}}{6}$



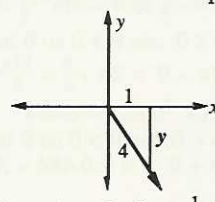
23. $\sin \theta < 0$ (III, IV), $\cos \theta < 0$ (II, III): III
25. $\cos \theta > 0$ (I, IV), $\tan \theta < 0$ (II, IV): IV
27. $\csc \theta > 0$ so $\sin \theta > 0$ (I, II), $\cot \theta < 0$ so $\tan \theta < 0$ (II, IV): II
29. $\tan \theta > 0$ (I, III), $\csc \theta < 0$ so $\sin \theta < 0$ (III, IV): III
31. 46.3°
33. $421^\circ 48' - 360^\circ = 61^\circ 48'$
35. $-248.7^\circ + 360^\circ = 111.3^\circ$; $180^\circ - 111.3^\circ = 68.7^\circ$
37. $242.57^\circ - 180^\circ = 62.57^\circ$
39. $\cos 240^\circ = -\frac{1}{2}$
41. $\csc 870^\circ = \frac{1}{\sin 870^\circ} = \frac{1}{\sin 150^\circ} = \frac{1}{\frac{1}{2}} = 2$
43. $\tan 213.9^\circ \approx 0.6720$

45. $\cos(-133^\circ 20') \approx -0.6862$
 $133 \boxed{+} 20 \boxed{+} 60 \boxed{=} \boxed{+/-} \boxed{\cos}$
 TI-81: $\boxed{\cos} \boxed{[]} \boxed{(-)} \boxed{133} \boxed{+} \boxed{20} \boxed{+}$
 $60 \boxed{)} \boxed{\text{ENTER}}$

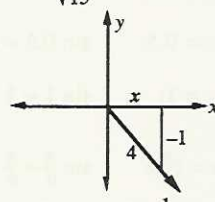
47. $\theta' = \sin^{-1} 0.3251 \approx 19.0^\circ$;
 $\sin \theta > 0$, $\cos \theta < 0$ so θ in qII
 $\theta = 180^\circ - \theta' \approx 180^\circ - 19.0^\circ = 161.0^\circ$
49. $\theta' = \tan^{-1} 0.6306 \approx 32.2^\circ$
 θ is in qIII, $\theta \approx 180^\circ + 32.2^\circ = 212.2^\circ$
51. $\cos \theta = -\frac{1}{2.0642}$
 $\theta' = \cos^{-1} \frac{1}{2.0642} \approx 61.0^\circ$
 $\cos \theta < 0$, $\sin \theta > 0$ so θ in qII
 $\theta = 180^\circ - \theta' \approx 180^\circ - 61.0^\circ = 119.0^\circ$
53. $\theta' = \sin^{-1} \frac{\sqrt{3}}{2} = 60^\circ$
 θ in qII so $\theta = 180^\circ - 60^\circ = 120^\circ$
55. $\theta' = \cos^{-1} \frac{5}{13} \approx 67.4^\circ$
 θ in qII so $\theta \approx 180^\circ - 67.4^\circ = 112.6^\circ$
57. $\cos \theta = -\frac{1}{2}$; $\sec \theta = -2 \quad y = -\sqrt{3}$
 $\sin \theta = -\frac{\sqrt{3}}{2}$; $\csc \theta = -\frac{2}{\sqrt{3}}$
 $\tan \theta = -\sqrt{3}$; $\cot \theta = -\frac{1}{\sqrt{3}}$



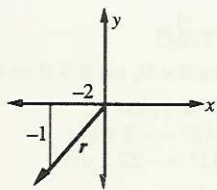
59. $\cos \theta = \frac{1}{4}$; $\sec \theta = 4 \quad y = -\sqrt{15}$
 $\sin \theta = -\frac{\sqrt{15}}{4}$; $\csc \theta = -\frac{4}{\sqrt{15}}$
 $\tan \theta = -\sqrt{15}$; $\cot \theta = -\frac{1}{\sqrt{15}}$



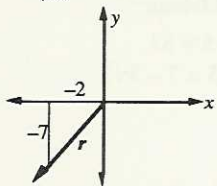
61. $\csc \theta = -4$ so $\sin \theta = -\frac{1}{4} \quad x = \sqrt{15}$
 $\cos \theta = \frac{\sqrt{15}}{4}$; $\sec \theta = \frac{4}{\sqrt{15}}$
 $\tan \theta = -\frac{1}{\sqrt{15}}$; $\cot \theta = -\sqrt{15}$



63. $\cot \theta = 2$ so $\tan \theta = \frac{1}{2} \quad r = \sqrt{5}$
 $\sin \theta = \frac{1}{\sqrt{5}} = \frac{1}{5}\sqrt{5}$; $\csc \theta = \sqrt{5}$
 $\cos \theta = \frac{2}{\sqrt{5}} = \frac{2}{5}\sqrt{5}$; $\sec \theta = \frac{\sqrt{5}}{2}$



65. $\tan \theta = \frac{7}{2}, \cot \theta = \frac{2}{7} \quad r = \sqrt{53}$
 $\sin \theta = \frac{-7}{\sqrt{53}} = -\frac{7}{\sqrt{53}}; \csc \theta = -\frac{\sqrt{53}}{7}$
 $\cos \theta = \frac{-2}{\sqrt{53}} = -\frac{2}{\sqrt{53}}; \sec \theta = -\frac{\sqrt{53}}{2}$



71. $\sin p = \frac{211.5 \sin 29.6^\circ}{185.7} \approx 0.5626$ so $p \approx 34.234^\circ$
 $a \approx 180^\circ - (29.6^\circ + 34.234^\circ) = 116.166^\circ$
 $BP = \frac{185.7 \sin 116.166^\circ}{\sin 29.6^\circ} \approx 337.4$ meters

73. $x = r \cos \theta = 8.075 \cos 10.2^\circ \approx 7.9$ inches
 $y = r \sin \theta = 8.075 \sin 10.2^\circ \approx 1.4$ inches

75. $\frac{120^\circ}{180^\circ} = \frac{s}{\pi}; \frac{2}{3} = \frac{s}{\pi}; s = \frac{2\pi}{3} \approx 2.09$

77. $\frac{430^\circ}{180^\circ} = \frac{s}{\pi}; s = \frac{43\pi}{18} \approx 7.50$

79. $\frac{\theta^\circ}{180^\circ} = \frac{11\pi}{3}; \theta^\circ = 180^\circ \cdot \frac{11\pi}{3} \cdot \frac{1}{\pi} = 660^\circ$

81. $\frac{\theta^\circ}{180^\circ} = \frac{2.5}{\pi}; \theta^\circ = 180^\circ \cdot \frac{2.5}{\pi} = \frac{450^\circ}{\pi} \approx 143.24^\circ$

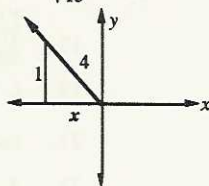
83. $L = rs; L = 3.9 \cdot \frac{6.3}{2} = 12.3$ inches

85. 150° is $\frac{5\pi}{6}$ radians
 $L = rs = \frac{15}{2} \cdot \frac{5\pi}{6} = (6\frac{1}{4})\pi$ mm

87. $A = \frac{1}{2}sr^2; 30^\circ$ is $\frac{\pi}{6}$ radians
 $A = \frac{1}{2} \cdot \frac{\pi}{6} \cdot 9^2 = \frac{27\pi}{4} \approx 21.21$ in²

89. $A = \frac{1}{2} \cdot \frac{11}{12} \cdot 6^2 = 16\frac{1}{2}$ mm²

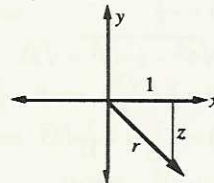
67. $\sin \theta = \frac{1}{4}; \csc \theta = 4 \quad x = -\sqrt{15}$
 $\cos \theta = \frac{-\sqrt{15}}{4}; \sec \theta = -\frac{4}{\sqrt{15}} = -\frac{4}{15}\sqrt{15}$
 $\tan \theta = -\frac{1}{\sqrt{15}} = -\frac{1}{15}\sqrt{15}; \cot \theta = -\sqrt{15}$



69. The triangle represents the reference triangle for an angle in qIV whose sine is z (z must represent a negative quantity).

$$r = \sqrt{1 + z^2}, \cos \theta = \frac{\text{adj}}{\text{hyp}}$$

$$= \frac{1}{r} = \frac{1}{\sqrt{1 + z^2}}$$



91. $\sin 1.9 \approx 0.9463$

93. $\tan 4.5 \approx 4.6373$

95. $\theta' = \frac{\pi}{4}; \theta$ in qII where $\tan \theta < 0$ so

$$\tan \frac{3\pi}{4} = -\tan \frac{\pi}{4} = -1$$

97. $\theta' = \frac{\pi}{6}; \theta$ in qIV where $\cos \theta > 0$;

$$\cos(-\frac{\pi}{6}) = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

99. $\theta' = \frac{\pi}{6}; \theta$ in qIII where $\sin \theta < 0$;

$$\sin(-\frac{5\pi}{6}) = -\sin \frac{\pi}{6} = -\frac{1}{2}$$

101. $\theta' = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}; \sin \theta < 0, \cos \theta > 0$ so θ in qIV;

$$\theta = 2\pi - \theta' = \frac{11\pi}{6}$$

103. $2 \sin \theta = 0.84$

$$\sin \theta = 0.42$$

$$\theta = \sin^{-1} 0.42 \approx 0.43$$

105. $(2 \cos \theta - 1)(\cos \theta - 1) = 0$

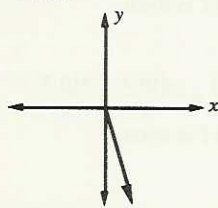
$$2 \cos \theta = 1 \quad \text{or} \quad \cos \theta = 1$$

$$\cos \theta = \frac{1}{2}$$

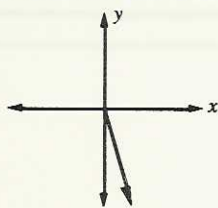
$$\theta = \frac{\pi}{3} \quad \text{or} \quad \theta = 0$$

Chapter 2 Test

1. a. $665^\circ - 360^\circ = 305^\circ$



b. $-417^\circ + 360^\circ = -57^\circ + 360^\circ = 303^\circ$

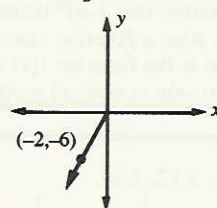


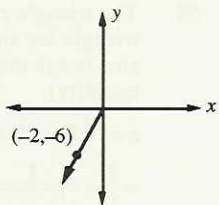
3. $(-2, -6) \quad r = \sqrt{4 + 36} = \sqrt{40} = 2\sqrt{10}$

$$\sin \theta = \frac{-6}{2\sqrt{10}} = -\frac{3}{\sqrt{10}}; \csc \theta = -\frac{\sqrt{10}}{3}$$

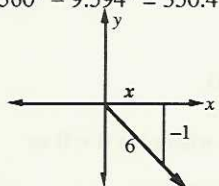
$$\cos \theta = \frac{-2}{2\sqrt{10}} = -\frac{1}{\sqrt{10}}; \sec \theta = -\sqrt{10}$$

$$\tan \theta = \frac{-6}{-2} = 3; \cot \theta = \frac{1}{3}$$





5. $\sin \theta = -\frac{1}{6}$ $\csc \theta = -6$
 $x = \sqrt{6^2 - (-1)^2} = \sqrt{35}$
 $\cos \theta = \frac{x}{6} = \frac{\sqrt{35}}{6}$ $\sec \theta = \frac{6}{\sqrt{35}}$
 $\tan \theta = \frac{-1}{\sqrt{35}} = -\frac{1}{\sqrt{35}}$ $\cot \theta = -\sqrt{35}$
 $\theta' = \sin^{-1} \frac{1}{6} \approx 9.594^\circ$
 $\theta = 360^\circ - \theta' \approx 360^\circ - 9.594^\circ = 350.4^\circ$



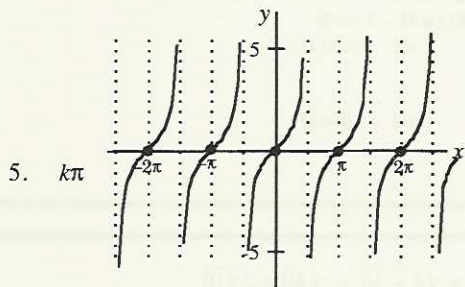
7. 246.2° in qIII so $\theta' = \theta - 180^\circ = 246.2^\circ - 180^\circ = 66.2^\circ$
9. $\sin 116.4^\circ \approx 0.8957$
11. $\csc 115^\circ 20' = \frac{1}{\sin 115^\circ 20'} \approx 1.1064$
115 $\boxed{+}$ 20 $\boxed{+}$ 60 $\boxed{=}$ $\boxed{\sin}$ $\boxed{1/x}$
TI-81: $\boxed{[}$ $\boxed{\sin}$ $\boxed{[}$ 115 $\boxed{+}$ 20 $\boxed{\div}$ 60 $\boxed{)]}$ $\boxed{)]}$ $\boxed{x^{-1}}$

13. $\sec \theta = -2.0642$ so $\cos \theta = \frac{-1}{2.0642}$
 $\theta' = \cos^{-1} \frac{1}{2.0642} \approx 61.0^\circ$; $\cos \theta < 0$, $\sin \theta > 0$ so θ in qII;
 $\theta = 180^\circ - \theta' \approx 180^\circ - 61.0^\circ = 119.0^\circ$
15. $x = r \cos \theta = 22.6 \cos 261.42^\circ \approx -3.4$ cm
 $y = r \sin \theta = 22.6 \sin 261.42^\circ \approx -22.3$ cm
17. $\frac{415^\circ}{180^\circ} = \frac{s}{\pi}$; $s = \frac{83}{36}\pi \approx 7.24$ radians
19. $L = rs$; $L = 5 \cdot \frac{8.2}{2} = 20.5$ inches
21. $\csc 2.5 = \frac{1}{\sin 2.5} \approx 1.6709$
23. $A = \frac{1}{2}sr^2 = \frac{1}{2}(\frac{\pi}{3})(16^2) \approx 134.04 \text{ mm}^2$
25. a. $f(-4) = (-4)^2 - 3(-4) + 5 = 33$
b. $f(\sqrt{2}) = (\sqrt{2})^2 - 3\sqrt{2} + 5 = 7 - 3\sqrt{2}$
27. $\theta = \tan^{-1} \sqrt{3} = \frac{\pi}{3}$; θ in qI
 $\theta = \pi + \theta' \approx \pi + \frac{\pi}{3} = \frac{4\pi}{3}$
29. $2 \tan 3\theta = 4.2$
 $\tan 3\theta = 2.1$
 $3\theta = \tan^{-1} 2.1$
 $\theta = \frac{1}{3} \tan^{-1} 2.1 \approx 0.38$
31. $d = \frac{1}{3} \cos 4(\frac{\pi}{16}) - \frac{1}{4} \sin 4(\frac{\pi}{16})$
 $= \frac{1}{3} \cos \frac{\pi}{4} - \frac{1}{4} \sin \frac{\pi}{4}$
 $= \frac{1}{3} \cdot \frac{\sqrt{2}}{2} - \frac{1}{4} \cdot \frac{\sqrt{2}}{2}$
 $= \frac{\sqrt{2}}{6} - \frac{\sqrt{2}}{8} = \frac{4\sqrt{2}}{24} - \frac{3\sqrt{2}}{24}$
 $= \frac{\sqrt{2}}{24} \approx 0.0589$

Chapter 3

Exercise 3-1

1. a. See figure 3-3; b. See figure 3-6;
c. See figure 3-9



9. $-\frac{5\pi}{3}$ $\sin(-\frac{5\pi}{3}) = -\sin \frac{5\pi}{3} = -(-\frac{\sqrt{3}}{2}) = \frac{\sqrt{3}}{2}$
 $\cos(-\frac{5\pi}{3}) = \cos \frac{5\pi}{3} = \frac{1}{2}$
 $\tan(-\frac{5\pi}{3}) = -\tan \frac{5\pi}{3} = -(-\sqrt{3}) = \sqrt{3}$

In problems 10–25 remember that $(-x)^n$ is the same as x^n if n is even and $-x^n$ if n is odd. Also a function cannot be both even and odd. (The only exception is the function $f(x) = 0$.)
Remember that $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$, $\tan(-x) = -\tan x$.

Exercise 3-2

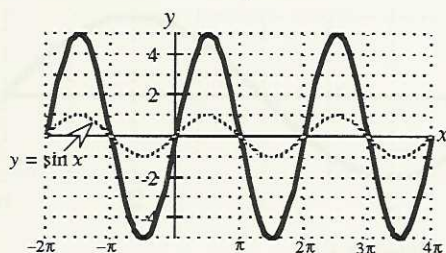
1. See figures 3.11, 3.12, 3.14.
3. $\csc(-x) = \frac{1}{\sin(-x)} = \frac{1}{-\sin(x)} = -\frac{1}{\sin x} = -\csc x$
5. $\cot(-x) = \frac{1}{\tan(-x)} = \frac{1}{-\tan(x)} = -\frac{1}{\tan x} = -\cot x$

13. $f(x) = -x^2$
 $f(-x) = -3(-x)^2 = -3x^2$
 $f(-x) = f(x)$ so f is even.
17. $f(x) = 3x - 2x^3$
 $f(-x) = 3(-x) - 2(-x)^3 = -3x - (-2x^3) = -3x + 2x^3$
 $-f(x) = -(3x - 2x^3) = -3x + 2x^3$
 $f(-x) = -f(x)$ so f is odd
21. $f(x) = \frac{x^5 - x^3}{x}$
 $f(-x) = \frac{(-x)^5 - (-x)^3}{-x} = \frac{-x^5 - (-x^3)}{-x} = \frac{-x^5 + x^3}{-x} = \frac{-(x^5 - x^3)}{-x} = \frac{x^5 - x^3}{x}$
 $f(-x) = f(x)$ so f is even.
25. $f(x) = \frac{\sin x}{x}$
 $f(-x) = \frac{\sin(-x)}{-x} = \frac{-\sin x}{-x} = \frac{\sin x}{x}$
 $f(-x) = f(x)$ so f is even.

Exercise 3-3

1. $y = 5 \sin x$

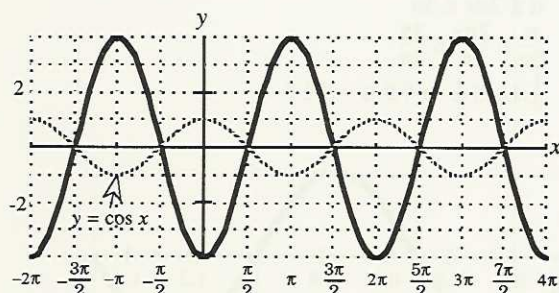
Amplitude is 5.



5. $y = -4 \cos x$

Amplitude is 4.

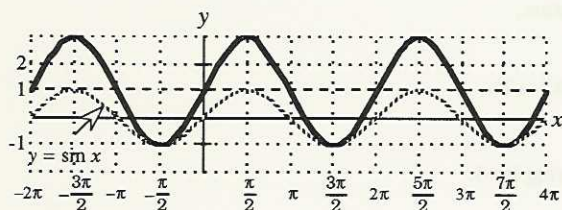
Graph is flipped about the x -axis relative to the graph of $y = \cos x$.



9. $y = 2 \sin x + 1$

Amplitude is 2.

Graph is raised vertically 1 unit.



13. $y = 2 \sin 4x$

Amplitude is 2.

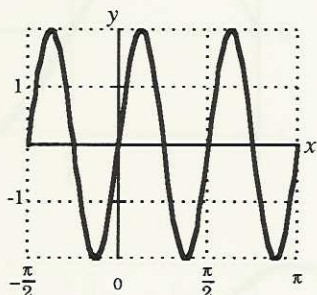
$0 \leq 4x \leq 2\pi$

$0 \leq \frac{4x}{4} \leq \frac{2\pi}{4}$

Divide each member by 4.

$0 \leq x \leq \frac{\pi}{2}$; one basic sine cycle between 0 and $\frac{\pi}{2}$.

Phase shift is 0; period is $\frac{\pi}{2}$.



17. $y = \frac{2}{3} \sin(3x + \pi)$

Amplitude is $\frac{2}{3}$.

$0 \leq 3x + \pi \leq 2\pi$

$-\pi \leq 3x \leq \pi$

Subtract π from each member.

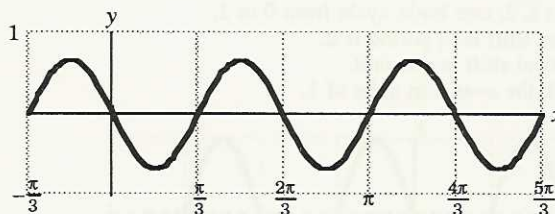
$-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$

Divide each member by 3.

One basic cycle between $-\frac{\pi}{3}$ and $\frac{\pi}{3}$.

Phase shift is $-\frac{\pi}{3}$; period is $\frac{\pi}{3} - (-\frac{\pi}{3}) = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$.

For convenience mark the x -axis in terms of $\frac{\pi}{3}$.



21. $y = -\cos(2x + \frac{\pi}{2})$

Amplitude is 1.

Graph is reflected about the x -axis relative to the graph of $y = \cos x$.

$0 \leq 2x + \frac{\pi}{2} \leq 2\pi$

$-\frac{\pi}{2} \leq 2x \leq 2\pi - \frac{\pi}{2}$

Subtract $\frac{\pi}{2}$ from each member.

$-\frac{\pi}{2} \leq 2x \leq \frac{3\pi}{2}$

$\frac{1}{2}(-\frac{\pi}{2}) \leq \frac{1}{2}(2x) \leq \frac{1}{2} \cdot \frac{3\pi}{2}$

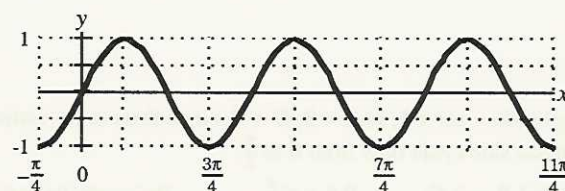
Multiply each member by $\frac{1}{2}$.

$-\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$

One basic cycle between $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

Phase shift is $-\frac{\pi}{4}$; period is $\frac{3\pi}{4} - (-\frac{\pi}{4}) = \pi$.

Mark the x -axis in terms of $\frac{\pi}{4}$.



25. $y = \cos 2\pi x$

Amplitude is 1.

$0 \leq 2\pi x \leq 2\pi$

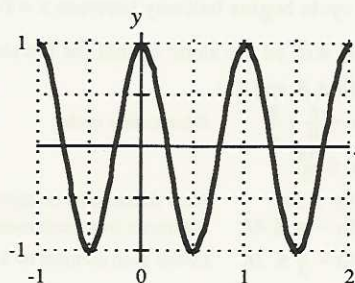
$0 \leq x \leq 1$

Divide each member by 2π .

Basic cycle from 0 to 1.

Phase shift is 0; period is $1 - 0 = 1$.

Mark the x -axis in multiples of $\frac{1}{2}$.



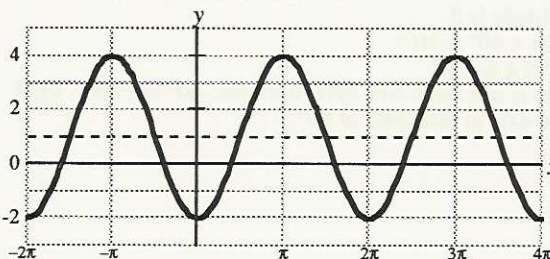
29. $y = -3 \cos x + 1$

Amplitude is 3.

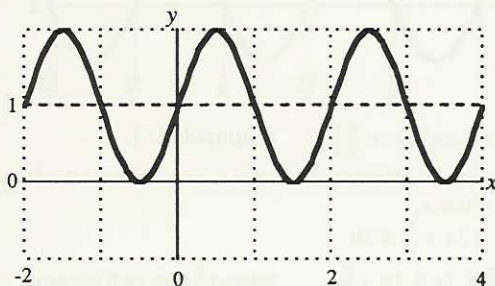
The graph is flipped about the horizontal line $y = 1$.

One basic cycle between 0 and 2π . Phase shift is 0; period is 2π .

Vertical shift 1 unit upwards.



33. $y = \sin \pi x + 1$ Amplitude is 1.
 $0 \leq \pi x \leq 2\pi$
 $0 \leq x \leq 2$; one basic cycle from 0 to 2.
Phase shift is 0; period is 2.
Vertical shift is one unit.
Mark the x -axis in units of 1.



37. $y = -\cos(-3x)$
 $y = -[\cos 3x]$
 $y = -\cos 3x$
41. $y = \sin(-x) - 3$
 $y = -\sin x - 3$

53. amplitude = $|A| = 3$, so $A = 3$; $D = 0$, since there is no vertical translation.
A basic sine cycle runs from 0 to $\frac{\pi}{2}$.

To find B and C : $0 \leq x \leq \frac{\pi}{2}$ Basic cycle; the goal is to convert $\frac{\pi}{2}$ to 2π .
 $0 \leq 2x \leq \pi$ Multiply each term by 2.
 $0 \leq 4x \leq 2\pi$ Multiply each term by 2.

Of course we could have combined the previous two steps by multiplying by 4.
Thus $Bx + C = 4x$, so $B = 4$, $C = 0$.
The equation is $y = 3 \sin 4x$.

57. $A = 3$, $D = 0$

A cosine cycle begins halfway between $x = 0$ and $x = \frac{\pi}{4}$, at $\frac{\pi}{8}$.

The period will be the same as that for the sine function, $\frac{\pi}{2}$.

Thus, to find B and C :

$$\frac{\pi}{8} \leq x \leq \frac{\pi}{8} + \frac{\pi}{2} \quad \text{Basic cosine cycle.}$$

$$\frac{\pi}{8} \leq x \leq \frac{5\pi}{8}$$

$$\pi \leq 8x \leq 5\pi \quad \text{Clear fractions by multiplying each member by 8.}$$

$$0 \leq 8x - \pi \leq 4\pi \quad \text{Subtract } \pi \text{ from each member so the left member is 0.}$$

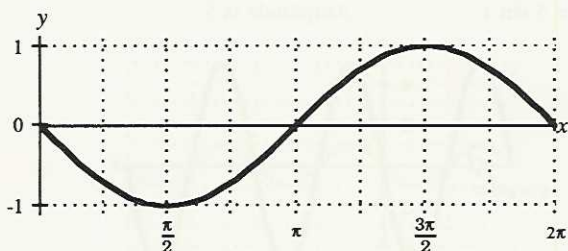
$$0 \leq 4x - \frac{\pi}{2} \leq 2\pi \quad \text{Divide each member by 2, so the right member is } 2\pi.$$

$$\text{Thus } Bx + C = 4x - \frac{\pi}{2}, \text{ so } B = 4, C = -\frac{\pi}{2}.$$

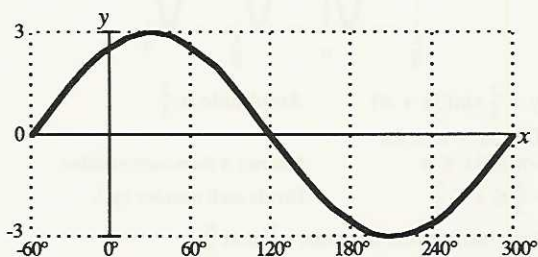
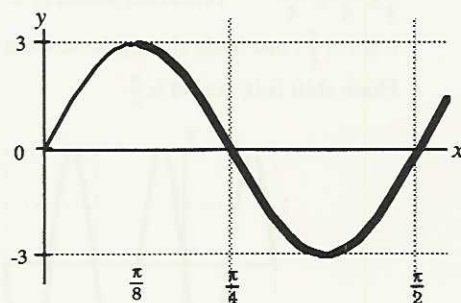
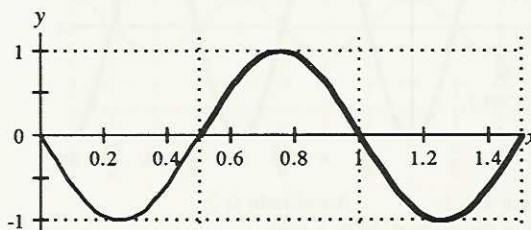
$$\text{The equation is } y = 3 \cos\left(4x - \frac{\pi}{2}\right).$$

61. amplitude is 3.
 $0^\circ \leq x + 60^\circ \leq 360^\circ$
 $-60^\circ \leq x \leq 300^\circ$
There is one basic cycle between -60° and 300° . Mark the x -axis in multiples of 60° .

45. $y = \sin(-x)$
 $y = -\sin x$



49. $y = -\sin(-2\pi x + \pi)$
 $y = -\sin[-(2\pi x - \pi)]$
 $y = \sin(2\pi x - \pi)$
 $0 \leq 2\pi x - \pi \leq 2\pi$
 $\pi \leq 2\pi x \leq 3\pi$
 $\frac{\pi}{2\pi} \leq \frac{2\pi x}{2\pi} \leq \frac{3\pi}{2\pi}$
 $\frac{1}{2} \leq x \leq \frac{3}{2}$, or $0.5 \leq x \leq 1.5$



65. $90^\circ \leq x \leq 90^\circ + 54^\circ$

One cycle is between the phase shift and phase shift plus period.

$0^\circ \leq x - 90^\circ \leq 54^\circ$

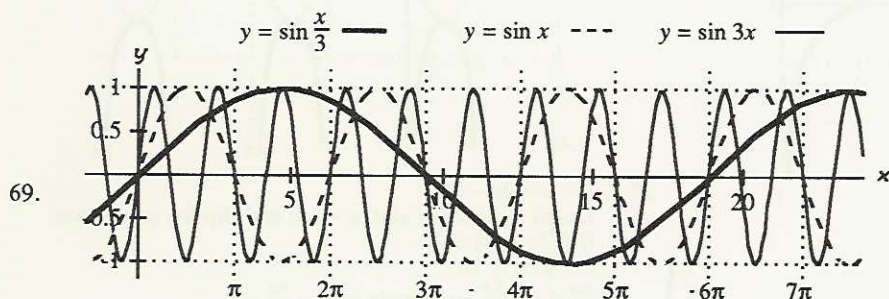
Subtract 90° so the left member is 0.

$\frac{360^\circ}{54^\circ} \cdot 0^\circ \leq \frac{360^\circ}{54^\circ} \cdot (x - 90^\circ) \leq \frac{360^\circ}{54^\circ} \cdot 54^\circ$

Multiply each term by $\frac{360^\circ}{54^\circ}$, so the right member becomes 360° .

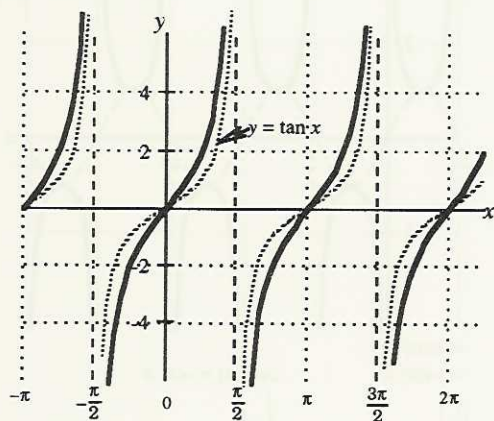
$0^\circ \leq \frac{20}{3}x - 600^\circ \leq 360^\circ$

$y = 60 \sin\left(\frac{20}{3}x - 600^\circ\right)$



Exercise 3-4

1. $y = 5 \tan x$

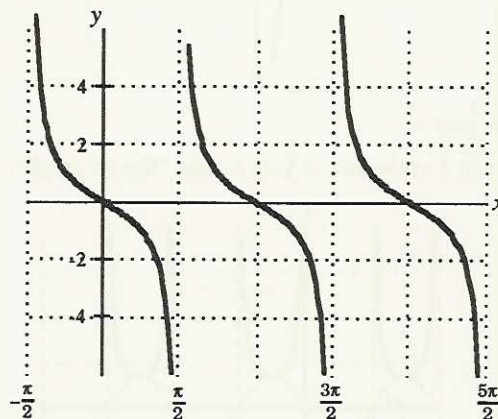


5. $y = \cot\left(x - \frac{\pi}{2}\right)$

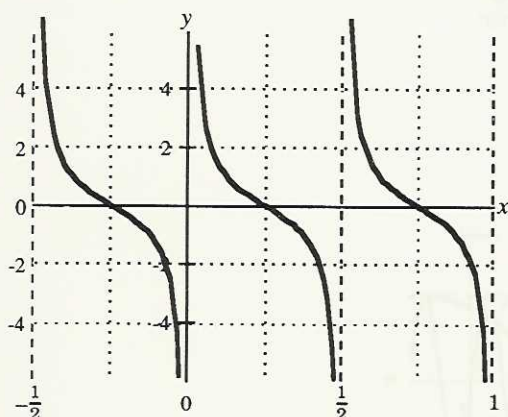
$0 \leq x - \frac{\pi}{2} \leq \pi$

$\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$

Basic cycle.

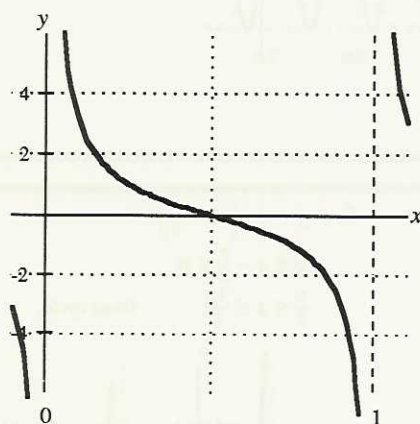


9. $y = \cot 2\pi x$
 $0 \leq 2\pi x \leq \pi$
 $0 \leq x \leq \frac{\pi}{2\pi}$
 $0 \leq x \leq \frac{1}{2}$



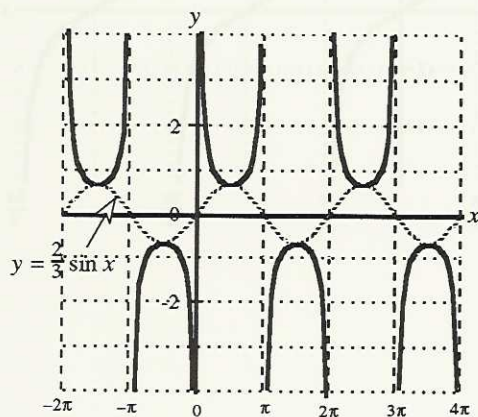
13. $y = -\cot(-\pi x)$
 $= -[-\cot \pi x]$
 $= \cot \pi x$
 $0 \leq \pi x \leq \pi$
 $0 \leq x \leq 1$

Divide each term by π .



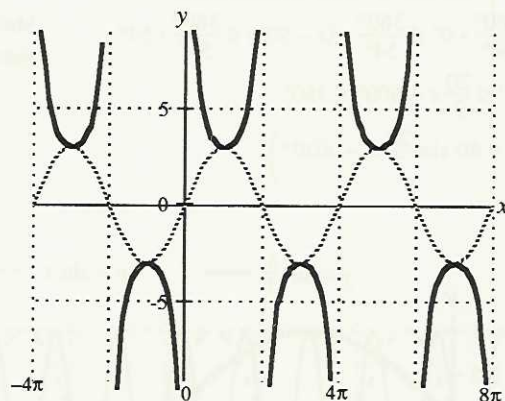
17. $y = \frac{2}{3} \csc x$

Graph 3 cycles of $y = \frac{2}{3} \sin x$, then "flip the graph".



21. $y = 3 \csc \frac{x}{2}$

Graph 3 cycles of $3 \sin \frac{x}{2}$ and "flip".



25. $y = \csc(2x - 3\pi)$

Graph 3 cycles of $\sin(2x - 3\pi)$ and flip the graph over.

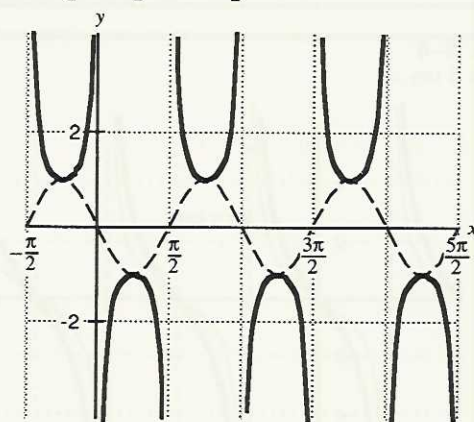
$0 \leq 2x - 3\pi \leq 2\pi$

$3\pi \leq 2x \leq 5\pi$

$\frac{3\pi}{2} \leq x \leq \frac{5\pi}{2}$; one cycle is $\frac{5\pi}{2} - \frac{3\pi}{2} = \pi$

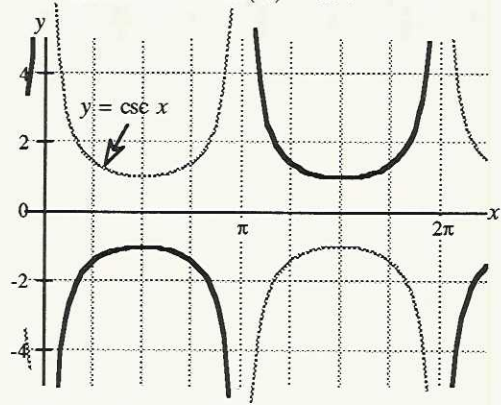
Subtracting π tells where other cycles begin.

$\frac{3\pi}{2} - \pi = \frac{\pi}{2}$, and $\frac{\pi}{2} - \pi = -\frac{\pi}{2}$

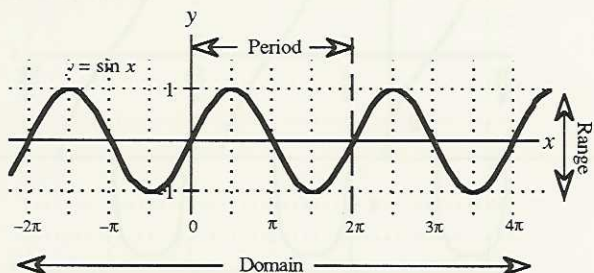


29. $y = \csc(-x)$
 $= -\csc x$

$\csc(-\theta) = -\csc \theta$



Chapter 3 Review



1. Domain: R ; Range: $-1 \leq y \leq 1$; Period: 2π .

3. $\cos(-\frac{\pi}{6}) = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ cosine is even so $\cos(-x) = \cos x$

5. $\sin(-\frac{5\pi}{6}) = -\sin \frac{5\pi}{6} = -\frac{1}{2}$

7. $f(x) = \frac{x^2 - 1}{x}$
 $f(-x) = \frac{(-x)^2 - 1}{-x} = \frac{x^2 - 1}{-x} = -\frac{x^2 - 1}{x} = -f(x)$
 $-f(x) = -\frac{x^2 - 1}{x} = \frac{-x^2 + 1}{x}$

$f(-x) = -f(x)$, so f is odd.

9. $f(x) = \tan x \cdot \cos x$

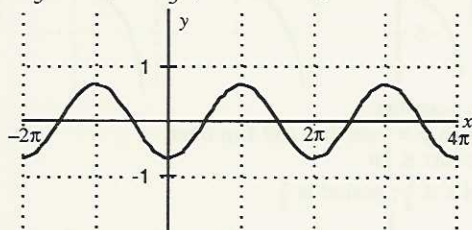
$f(-x) = \tan(-x) \cdot \cos(-x)$
 $= -\tan x \cdot \cos x$ $\tan(-x) = -\tan x$; $\cos(-x) = \cos x$
 $= -(\tan x \cdot \cos x)$
 $= -f(x)$.

Since $f(-x) = -f(x)$, f is an odd function.

11. domain: $x \neq k\pi$, k an integer; range: R

13. $f(x) = \sec x \cdot \sin^2 x + x^4$
 $f(-x) = \sec(-x) \cdot [\sin(-x)]^2 + (-x)^4$
 $= \sec x \cdot [-\sin x]^2 + x^4$
 $= \sec x \cdot \sin^2 x + x^4$
 $f(-x) = f(x)$ so f is even.

15. $y = -\frac{2}{3} \cos x$; $A = \frac{2}{3}$, period = 2π , phase shift = 0

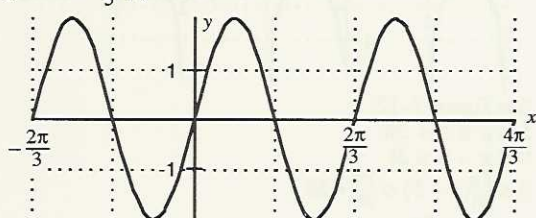


17. $y = 2 \sin 3x$; $A = 2$

$0 \leq 3x \leq 2\pi$

$0 \leq x \leq \frac{2\pi}{3}$ divide each member by 3

period = $\frac{2\pi}{3}$; phase shift = 0



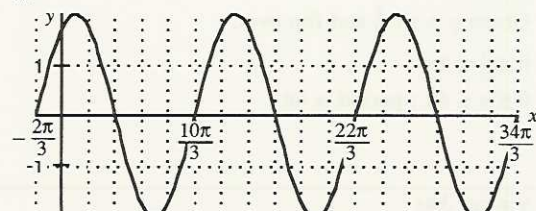
19. $y = 2 \sin(\frac{x}{2} + \frac{\pi}{3})$; $A = 2$

$0 \leq \frac{x}{2} + \frac{\pi}{3} \leq 2\pi$

$0 \leq 3x + 2\pi \leq 12\pi$ Multiply each member by 3

$-2\pi \leq 3x \leq 10\pi$

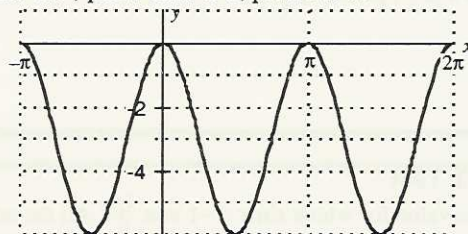
$-\frac{2\pi}{3} \leq x \leq \frac{10\pi}{3}$; phase shift is $-\frac{2\pi}{3}$; period is $\frac{10\pi}{3} - (-\frac{2\pi}{3}) = 4\pi$



21. $y = 3 \cos 2x - 3$; $A = 3$; vertical translation is -3 .

$0 \leq 2x \leq 2\pi$

$0 \leq x \leq \pi$; phase shift is 0, period is π .



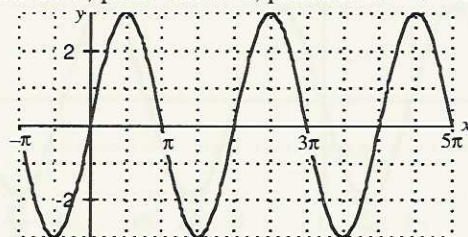
23. $y = 3 \sin(-x + \pi)$; $A = 1$

$y = 3 \sin[-(x - \pi)]$

$y = -3 \sin(x - \pi)$

$0 \leq x - \pi \leq 2\pi$

$\pi \leq x \leq 3\pi$; phase shift is π , period is $3\pi - \pi = 2\pi$



25. $y = A \cos(Bx + C) + D$

$A = 3$, A cycle starts at $\frac{1}{2} \cdot \frac{\pi}{3} = \frac{\pi}{6}$; cycle length is $\frac{2\pi}{3} - (-\frac{\pi}{3}) = \pi$, so one basic cycle is between $\frac{\pi}{6}$ and $\frac{\pi}{6} + \pi = \frac{7\pi}{6}$. $D = -1$.

$\frac{\pi}{6} \leq x \leq \frac{7\pi}{6}$

$0 \leq x - \frac{\pi}{6} \leq \pi$

Subtract $\frac{\pi}{6}$ from each member.

$0 \leq 2x - \frac{\pi}{3} \leq 2\pi$

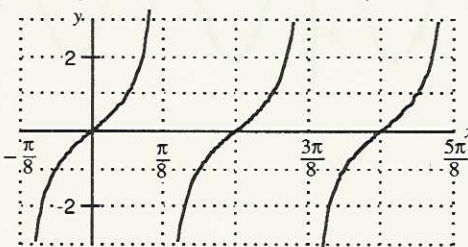
Multiply each member by 2.

$y = 3 \cos(2x - \frac{\pi}{3}) - 1$

27. $y = \tan 4x$

$-\frac{\pi}{2} \leq 4x \leq \frac{\pi}{2}$

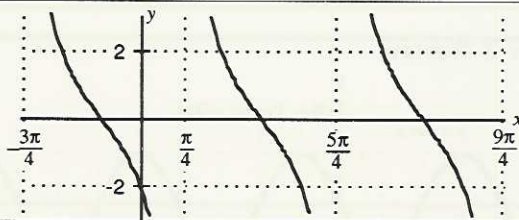
$-\frac{\pi}{8} \leq x \leq \frac{\pi}{8}$ multiply each member by $\frac{1}{4}$



29. $y = 2 \cot(x - \frac{\pi}{4})$

$0 \leq x - \frac{\pi}{4} \leq \pi$

$\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$

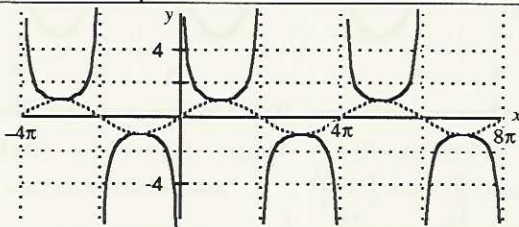


31. $y = \csc \frac{x}{2}$

Graph $y = \sin \frac{x}{2}$ and flip over.

$0 \leq \frac{x}{2} \leq 2\pi$

$0 \leq x \leq 4\pi$; period is 4π

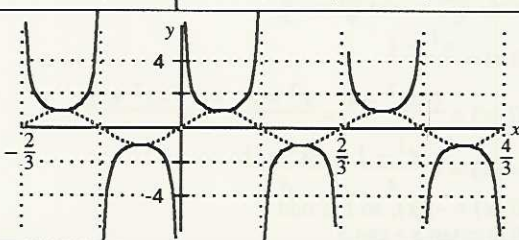


33. $y = \csc 3\pi x$

Graph $y = \sin 3\pi x$ and flip over.

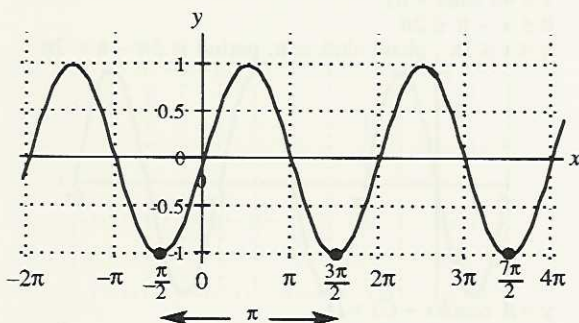
$0 \leq 3\pi x \leq 2\pi$

$0 \leq x \leq \frac{2}{3}$; period is $\frac{2}{3}$



Chapter 3 Test

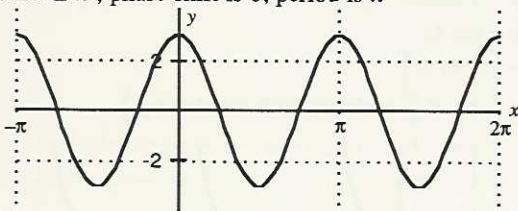
1. One value for which $\sin x = -1$ is at $\frac{3\pi}{2}$. All the other values are integer multiples of 2π units from this value. Thus the values are $x = \frac{3\pi}{2} + 2k\pi$, k an integer.



3. $f(x) = x + \sin x$
 $f(-x) = (-x) + \sin(-x)$
 $= (-x) + (-\sin x)$
 $= -(x + \sin x)$
 $= -f(x)$.

Since $f(-x) = -f(x)$, f is an odd function.

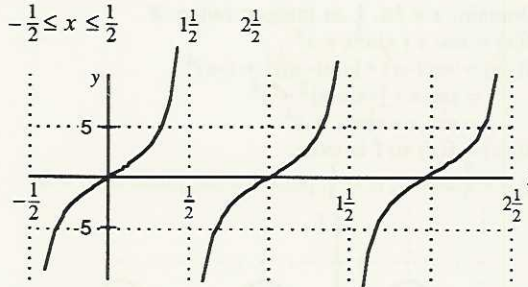
5. $y = 3 \cos 2x$; $A = 3$
 $0 \leq 2x \leq 2\pi$
 $0 \leq x \leq \pi$; phase shift is 0, period is π



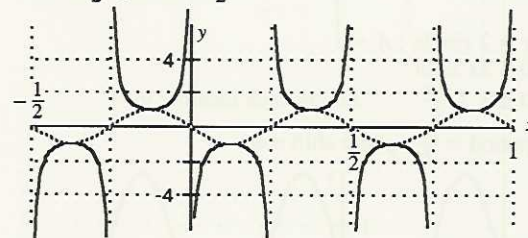
7. $3 \tan \pi x$

$-\frac{\pi}{2} \leq \pi x \leq \frac{\pi}{2}$

$-\frac{1}{2} \leq x \leq \frac{1}{2}$



9. $y = -\csc 4\pi x$
 Graph $y = -\sin 4\pi x$ and flip over.
 $0 \leq 4\pi x \leq 2\pi$
 $0 \leq x \leq \frac{1}{2}$; period is $\frac{1}{2}$



11. See figure 3-12.
 13. $5 \leq x \leq 5 + 28$
 $0 \leq x - 5 \leq 28$
 $0 \leq \frac{2\pi}{28}(x - 5) \leq \frac{2\pi}{28} \cdot 28$
 $0 \leq \frac{\pi}{14}x - \frac{5\pi}{14} \leq 2\pi$
 $y = \sin(\frac{\pi}{14}x - \frac{5\pi}{14})$, x in days

Exercise 4-1

1. $f(x) = 2x - 7$; $g(x) = \frac{1}{2}x + 3\frac{1}{2}$

(1) Let $y = f(x) = 2x - 7$. Show $g(y) = x$:

$$g(y) = \frac{1}{2}y + 3\frac{1}{2}$$

$$= \frac{1}{2}(2x - 7) + \frac{7}{2}$$

$$= x$$

(2) Let $y = g(x) = \frac{1}{2}x + \frac{7}{2}$. Show $f(y) = x$:

$$f(y) = 2y - 7 = 2(\frac{1}{2}x + \frac{7}{2}) - 7 = x$$

5. $f(x) = 2x - 5$; $g(x) = \frac{1}{2}(x + 5)$

(1) Let $y = f(x) = 2x - 5$. Show $g(y) = x$:

$$g(y) = \frac{1}{2}(y + 5) = \frac{1}{2}[(2x - 5) + 5] = x$$

(2) Let $y = g(x) = \frac{1}{2}(x + 5)$. Show $f(y) = x$:

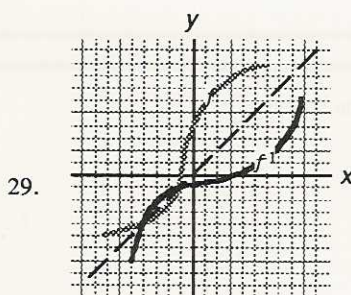
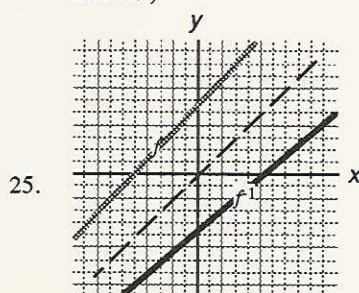
$$f(y) = 2y - 5 = 2[\frac{1}{2}(x + 5)] - 5 = x$$

9. $f(x) = 7 - \frac{3}{x}$; $g(x) = \frac{3}{7 - x}$

(1) Let $y = f(x) = 7 - \frac{3}{x}$. Show $g(y) = x$:

$$g(y) = \frac{3}{7 - y} = \frac{3}{7 - (7 - \frac{3}{x})} = \frac{3}{\frac{3}{x}} = x$$

21. Function, not one to one (passes the vertical line test, fails the horizontal line test)



(2) Let $y = g(x) = \frac{3}{7 - x}$. Show $f(y) = x$:

$$f(y) = 7 - \frac{3}{y} = 7 - \frac{3}{\frac{3}{7 - x}} = 7 - (7 - x) = x$$

13. $f(x) = x^2 - 9$, $x \geq 0$; $g(x) = \sqrt{x + 9}$

(1) Let $y = f(x) = x^2 - 9$. Show $g(y) = x$:

$$g(y) = \sqrt{y + 9} = \sqrt{(x^2 - 9) + 9} = \sqrt{x^2} = x \text{ if } x \geq 0.$$

(2) Let $y = g(x) = \sqrt{x + 9}$. Show $f(y) = x$:

$$f(y) = y^2 - 9 = (\sqrt{x + 9})^2 - 9 = (x + 9) - 9 = x$$

17. $f(x) = x^2 - 2x + 3$, $x \geq 1$; $g(x) = \sqrt{x - 2} + 1$

(1) Let $y = f(x) = x^2 - 2x + 3$. Show $g(y) = x$:

$$g(y) = \sqrt{y - 2} + 1 = \sqrt{(x^2 - 2x + 3) - 2} + 1$$

$$= \sqrt{x^2 - 2x + 1} + 1 = \sqrt{(x - 1)^2} + 1 = x - 1 + 1 = x.$$

(2) Let $y = g(x) = \sqrt{x - 2} + 1$. Show $f(y) = x$:

$$f(y) = y^2 - 2y + 3$$

$$= (\sqrt{x - 2} + 1)^2 - 2(\sqrt{x - 2} + 1) + 3$$

$$= [(x - 2) + 2\sqrt{x - 2} + 1] - 2\sqrt{x - 2} - 2 + 3 = x$$

33. $A(x) = 4(x + 4)$

$y = 4(x + 4)$

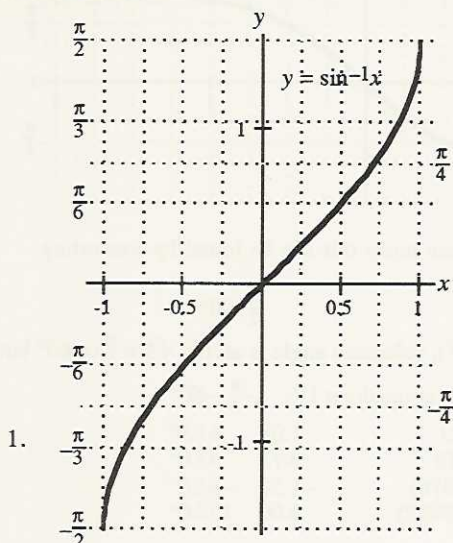
$x = 4(y + 4)$

$\frac{x}{4} = y + 4$

$y = \frac{x}{4} - 4$

$A^{-1}(x) = \frac{x}{4} - 4$

Exercise 4-2



5. $\sin^{-1}0 = 0$ since $\sin 0 = 0$ and $-\frac{\pi}{2} \leq 0 \leq \frac{\pi}{2}$; 0 or 0°

Use degree mode to obtain answers in degrees and radian mode to obtain answers in radians.

9. $\sin^{-1}0.8823$ 1.08 61.9°

13. $\sin^{-1}(-0.9976)$ -1.50 -86.0°

.9976 $\boxed{+/-}$ $\boxed{2nd}$ $\boxed{\sin^{-1}}$

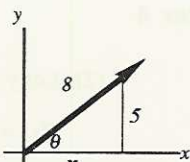
TI-81: $\boxed{2nd}$ $\boxed{\sin^{-1}}$ $\boxed{[]}$ $\boxed{(-)}$.9976 $\boxed{] }$ $\boxed{ENTER}$

$$17. \tan(\arcsin \frac{5}{8})$$

$$x = \sqrt{8^2 - 5^2} = \sqrt{39}; \theta = \arcsin \frac{5}{8};$$

$$\tan \theta = \frac{5}{x} = \frac{5}{\sqrt{39}} \cdot \frac{\sqrt{39}}{\sqrt{39}}$$

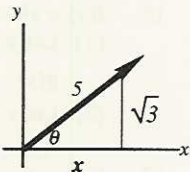
$$= \frac{5\sqrt{39}}{39} \text{ or } \frac{5}{39}\sqrt{39}$$



$$21. \cot(\sin^{-1} \frac{\sqrt{3}}{5})$$

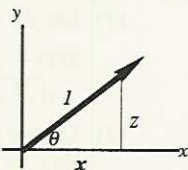
$$x = \sqrt{5^2 - (\sqrt{3})^2} = \sqrt{22}; \tan \theta = \frac{\sqrt{3}}{x} = \frac{\sqrt{3}}{\sqrt{22}};$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{\sqrt{22}}{\sqrt{3}} = \frac{\sqrt{66}}{3}.$$



$$25. \cos(\sin^{-1} z), z > 0$$

$$x = \sqrt{1 - z^2}; \cos \theta = \frac{x}{1} = \sqrt{1 - z^2}.$$



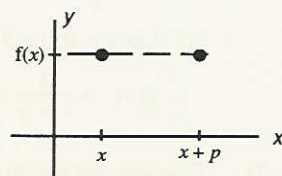
$$33. \arcsin(\cos \frac{2\pi}{3})$$

$$\arcsin(-\frac{1}{2})$$

$$-\arcsin \frac{1}{2}$$

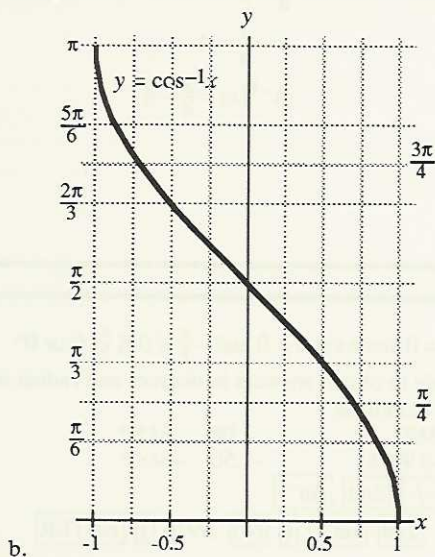
$$-\frac{\pi}{6}$$

37. A periodic function cannot be one to one. There are two ways to see this. First of all, let x be in the domain of f . Then there is a number $p > 0$ such that $f(x + kp) = f(x)$. This means there are two ordered pairs in f where the second element is repeated: $(x, f(x))$ and $(x + kp, f(x))$. The second way to see this is pictorially. The graph of f must fail the horizontal lines test.

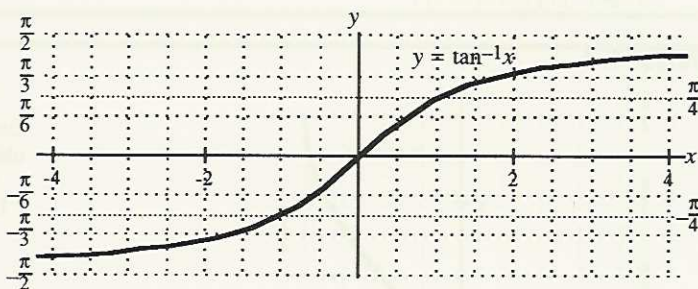


Exercise 4-3

1. See problem 1, exercise 4-2 for the inverse sine function.



c.



The following table is taken from the text. It will be used in the following problems.

θ	$0, 0^\circ$	$\frac{\pi}{6}, 30^\circ$	$\frac{\pi}{4}, 45^\circ$	$\frac{\pi}{3}, 60^\circ$	$\frac{\pi}{2}, 90^\circ$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Not defined

Note: A calculator will give exact answers in degree mode when the reference angle is 30° , 45° or 60° . If you know the

$$29. \sin(\arccos \frac{5}{8})$$

$$y = \sqrt{39}; \sin \theta = \frac{y}{8} = \frac{\sqrt{39}}{8}$$

$$29. \sec(\arcsin \sqrt{2z})$$

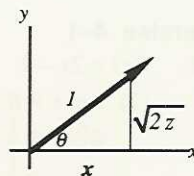
Note that $\sqrt{2z} \geq 0$, so $\theta = \arcsin \sqrt{2z}$ terminates in quadrant I (or is quadrantal).

$$x = \sqrt{1^2 - (\sqrt{2z})^2} = \sqrt{1 - 2z};$$

$$\cos \theta = \frac{x}{1} = x = \sqrt{1 - 2z};$$

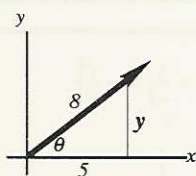
$$\sec \theta = \frac{1}{\cos \theta} = \frac{1}{\sqrt{1 - 2z}} \cdot \frac{\sqrt{1 - 2z}}{\sqrt{1 - 2z}} = \frac{\sqrt{1 - 2z}}{1 - 2z}$$

Note: If $2z = 0$ the angle is quadrantal, but the result is still valid.

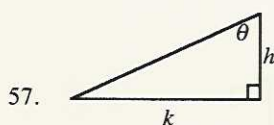
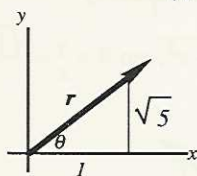


corresponding radian mode this can be found by converting degrees to radians.

5. $\cos^{-1} 0$ $\frac{\pi}{2}, 90^\circ$
9. $\arctan(-\sqrt{3})$; reference angle is $\arctan \sqrt{3} = \frac{\pi}{3}$ or 60° but the angle is in quadrant IV: $-\frac{\pi}{3}, -60^\circ$
13. $\sin^{-1} 0.8823$ 1.08 61.9°
17. $\tan^{-1} 0.9316$ 0.75 43.0°
21. $\sin^{-1}(-0.9976)$ -1.50 -86.0°
25. $\arccos(-0.9902)$ 3.00 172.0°



33. $\sin(\tan^{-1} \sqrt{5})$
 $r = \sqrt{6}$; $\sin \theta = \frac{\sqrt{5}}{r} = \frac{\sqrt{5}}{\sqrt{6}} = \frac{\sqrt{30}}{6}$.



57. $\tan \theta = \frac{k}{h}$, so $\theta = \tan^{-1} \frac{k}{h}$.

65. $\tan \theta = 4.1$
 $\theta = \tan^{-1} 4.1$

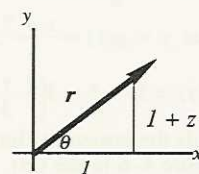
69. $\frac{\tan \theta}{5} = 10$
 $\tan \theta = 50$
 $\theta = \tan^{-1} 50$

73. $\sin \frac{3\theta}{2} = -0.56$
 $\frac{3\theta}{2} = \sin^{-1}(-0.56)$
 $\theta = \frac{2}{3} \sin^{-1}(-0.56)$

61. $\sin \theta = \frac{3500}{z}$, so
 $\theta = \sin^{-1} \frac{3500}{z}$.

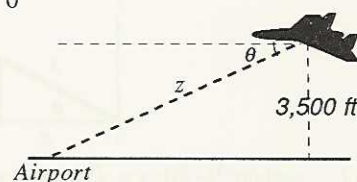
77. $\frac{6 \sin 5\theta}{5} = \frac{10}{13}$
 $6 \sin 5\theta = \frac{50}{13}$
 $\sin 5\theta = \frac{1}{6} \cdot \frac{50}{13}$
 $\sin 5\theta = \frac{25}{39}$

45. $\sec[\tan^{-1}(1+z)], z > 0$
 $r = \sqrt{1^2 + (1+z)^2} = \sqrt{z^2 + 2z + 2}$
 $\cos \theta = \frac{1}{r}$; $\sec \theta = \frac{1}{\cos \theta}$
 $= r = \sqrt{z^2 + 2z + 2}$.



49. $\tan^{-1}(\sin \frac{\pi}{2})$
 $\tan^{-1} 1$
 $\frac{\pi}{4}$

53. $\arccos(\tan \frac{5\pi}{4})$
 $\arccos(1)$
 0



50. $\sin^{-1} \frac{25}{39}$
 $\theta = \frac{1}{5} \sin^{-1} \frac{25}{39}$
 81. $\sin(2x+3) = 0.6$
 $2x+3 = \sin^{-1} 0.6$
 $2x = \sin^{-1} 0.6 - 3$
 $x = \frac{1}{2} (\sin^{-1} 0.6 - 3)$

Exercise 4-4

1. $\csc^{-1} 2 = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$ or 30°

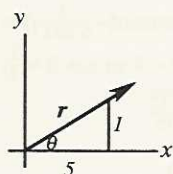
5. $\sec^{-1}(-2) = \cos^{-1}(-\frac{1}{2}) = \pi - \cos^{-1} \frac{1}{2} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ or 120°

9. $\operatorname{arccot} 0 = \frac{\pi}{2} - \arctan 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ or 90°

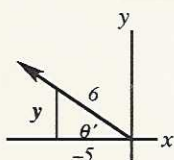
In the following problems use degree mode to obtain degree results and radian mode to obtain radian results.

25. $\csc(\operatorname{arccot} 5)$ If $\theta = \operatorname{arccot} 5$, then $\cot \theta = 5$, and $\tan \theta = \frac{1}{5}$.

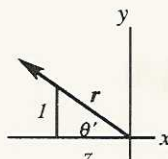
$r = \sqrt{26}$;
 $\csc \theta = \frac{1}{\sin \theta}$
 $= \frac{1}{\frac{1}{r}} = r$
 $r = \sqrt{26}$



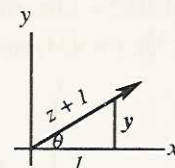
29. $\tan(\sec^{-1}(-\frac{6}{5})) = \tan(\cos^{-1}(-\frac{5}{6}))$
 $y = \sqrt{11}$; $\tan \theta = \frac{y}{-5} = -\frac{\sqrt{11}}{5}$.



33. $\sec(\cot^{-1} z), z < 0$
 If $\theta = \cot^{-1} z, z < 0$, then θ is an angle terminating in quadrant II, and $\tan \theta = \frac{1}{z}, r = \sqrt{z^2 + 1}$;
 $\sec \theta = \frac{1}{\cos \theta} = \frac{1}{\frac{z}{r}} = \frac{r}{z} = \frac{\sqrt{z^2 + 1}}{z}$.



37. $\tan[\sec^{-1}(z+1)], z+1 > 0$
 $= \tan(\cos^{-1} \frac{1}{z+1})$
 $y = \sqrt{(z+1)^2 - 1^2} = \sqrt{z^2 + 2z}$
 $\tan \theta = \frac{y}{1} = y = \sqrt{z^2 + 2z}$.



Chapter 4 Review

1. $f(x) = 3x - 5$; $g(x) = \frac{x+5}{3}$

(1) Let $y = f(x) = 3x - 5$; show that $g(y) = x$.

$$g(y) = \frac{y+5}{3} = \frac{(3x-5)+5}{3} = x$$

(2) Let $y = g(x) = \frac{x+5}{3}$; show that $f(y) = x$.

$$f(y) = 3y - 5 = 3\left(\frac{x+5}{3}\right) - 5 = x$$

3. No; fails the horizontal line test.

5. See figure 4-6 in the text.

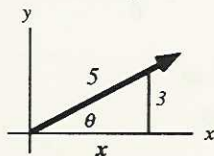
7. $\sin^{-1}(-\frac{1}{2}) = -\sin^{-1}\frac{1}{2} = -\frac{\pi}{6}, -30^\circ$

9. $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}, 30^\circ$

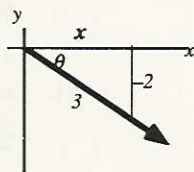
11. $\arcsin 0 = 0, 0^\circ$

13. $\sin^{-1}0.9737 \approx 1.34, 76.8^\circ$

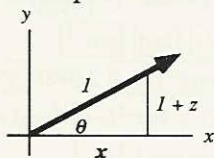
15. $\cos(\sin^{-1}\frac{3}{5})$; $x = 4$; $\cos \theta = \frac{x}{5} = \frac{4}{5}$



17. $\sec[\sin^{-1}(-\frac{2}{3})]$; $x = \sqrt{5}$; $\cos \theta = \frac{\sqrt{5}}{3}$, $\sec \theta = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$



19. $\cos(\arcsin(1+z))$, $1+z > 0$; $x = \sqrt{1^2 - (1+z)^2}$
 $= \sqrt{-2z - z^2}$; $\cos \theta = \frac{x}{1} = x = \sqrt{-2z - z^2}$



21. domain: R ; range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

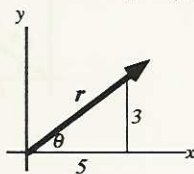
23. $\arccos\frac{\sqrt{3}}{2} = 30^\circ, \frac{\pi}{6}$ (Remember that a calculator will give the exact answer when in degree mode.)

25. $\arcsin(-\frac{\sqrt{2}}{2}) = -\arcsin\frac{\sqrt{2}}{2} = -45^\circ, -\frac{\pi}{4}$

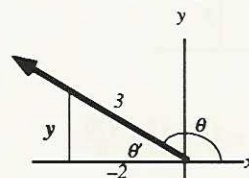
27. $\tan^{-1}(-1) = -45^\circ, -\frac{\pi}{2}$

29. $\arccos 0.4882 \approx 1.06, 60.8^\circ$

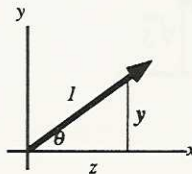
31. $\cos(\tan^{-1}\frac{3}{5})$; $r = \sqrt{34}$; $\cos \theta = \frac{5}{r} = \frac{5}{\sqrt{34}} = \frac{5}{34}\sqrt{34}$



33. $\tan[\cos^{-1}(-\frac{2}{3})]$; $y = \sqrt{5}$; $\tan \theta = \frac{y}{-2} = -\frac{\sqrt{5}}{2}$

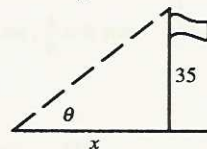


35. $\tan(\cos^{-1}z)$, $z > 0$; $y = \sqrt{1-z^2}$; $\tan \theta = \frac{y}{z} = \frac{\sqrt{1-z^2}}{z}$



37. $\sin \theta = \frac{j}{k}$ so $\theta = \sin^{-1}\frac{j}{k}$

39. $\tan \theta = \frac{35}{x}$ so $\theta = \tan^{-1}\frac{35}{x}$



41. $\sin \theta = -0.88$

$\theta = \sin^{-1}(-0.88)$

43. $2 \tan \theta = 10$

$\tan \theta = 5$

$\theta = \tan^{-1}5$

45. $2 \cos 3\theta = 1.4$

$\cos 3\theta = 0.7$

$3\theta = \cos^{-1}0.7$

$\theta = \frac{1}{3} \cos^{-1}0.7$

47. $\sin 2\theta + 3 = 3.6$

$\sin 2\theta = 0.6$

$2\theta = \sin^{-1}0.6$

$\theta = \frac{1}{2} \sin^{-1}0.6$

49. $\operatorname{arccsc}(-\frac{2\sqrt{3}}{3}) = \arcsin(-\frac{3}{2\sqrt{3}}) = \arcsin(-\frac{\sqrt{3}}{2}) = -60^\circ, -\frac{\pi}{3}$

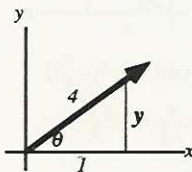
51. $\cot^{-1}(-\sqrt{3}) = \frac{\pi}{2} - \tan^{-1}(-\sqrt{3}) = \frac{\pi}{2} - (-\frac{\pi}{3}) = \frac{5\pi}{6}, 150^\circ$

53. $\cot^{-1}1.5601 = \frac{\pi}{2} - \tan^{-1}1.5601 \approx 0.57, 32.7^\circ$

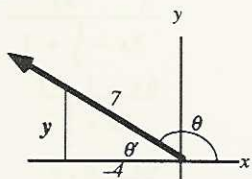
55. $\operatorname{arccsc}(-6.6917) = \arcsin(-\frac{1}{6.6917}) \approx -0.15, -8.6^\circ$

57. $\cot(\operatorname{arcsec} 4)$; $\sec \theta = 4$ so $\cos \theta = \frac{1}{4}$, $y = \sqrt{15}$

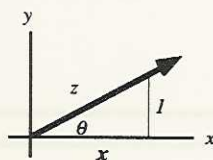
$\cot \theta = \frac{1}{y} = \frac{1}{\sqrt{15}} = \frac{\sqrt{15}}{15}$



59. $\csc[\operatorname{arcsec}(-\frac{7}{4})]$; $\sec \theta = -\frac{7}{4}$ so $\cos \theta = -\frac{4}{7}$; $y = \sqrt{33}$
 $\csc \theta = \frac{7}{y} = \frac{7}{\sqrt{33}} = \frac{7\sqrt{33}}{33}$

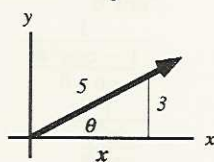


61. $\cot(\csc^{-1}z)$, $z > 0$; $\csc \theta = z$ so $\sin \theta = \frac{1}{z}$; $x = \sqrt{z^2 - 1}$
 $\cot \theta = \frac{x}{1} = x = \sqrt{z^2 - 1}$

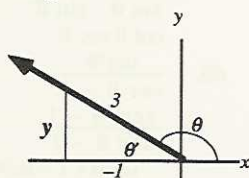


Chapter 4 test

- See figure 4-10 in the text.
- $\sec^{-1}(-2) = \cos^{-1}(-\frac{1}{2}) = 120^\circ, \frac{2\pi}{3}$
- $\csc^{-1}2 = \sin^{-1}\frac{1}{2} = 30^\circ, \frac{\pi}{6}$
- $\tan^{-1}1.4617 \approx 0.97, 55.6^\circ$
- $\operatorname{arcsec}(-1.4830) = \arccos(\frac{-1}{1.4830}) \approx 2.31, 132.4^\circ$
- $\cot(\sin^{-1}\frac{3}{5})$; $x = 4$; $\cot \theta = \frac{x}{3} = \frac{4}{3}$

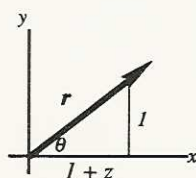
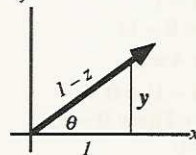


13. $\tan[\sec^{-1}(-3)]$; $\sec \theta = -3$ so $\cos \theta = -\frac{1}{3}$
 $y = 2\sqrt{2}$; $\tan \theta = \frac{y}{-1} = -y = -2\sqrt{2}$

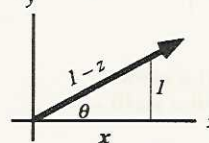


15. $\sin[\operatorname{arccot}(1+z)]$, $1+z > 0$; $\cot \theta = 1+z$ so $\tan \theta = \frac{1}{1+z}$
 $r = \sqrt{1^2 + (1+z)^2} = \sqrt{z^2 + 2z + 2}$; $\sin \theta = \frac{1}{r}$
 $= \frac{1}{\sqrt{z^2 + 2z + 2}}$

63. $\csc[\sec^{-1}(1-z)]$, $1-z > 0$; $\sec \theta = 1-z$ so $\cos \theta = \frac{1}{1-z}$;
 $y = \sqrt{(1-z)^2 - 1} = \sqrt{z^2 - 2z}$; $\sin \theta = \frac{y}{1-z}$ so
 $\csc \theta = \frac{1-z}{y} = \frac{1-z}{\sqrt{z^2 - 2z}}$



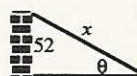
17. $\sec[\csc^{-1}(1-z)]$, $1-z > 0$; $\csc \theta = 1-z$, so $\sin \theta = \frac{1}{1-z}$;
 $x = \sqrt{(1-z)^2 - 1} = \sqrt{z^2 - 2z}$; $\cos \theta = \frac{x}{1-z} = \frac{\sqrt{z^2 - 2z}}{1-z}$;
 $\sec \theta = \frac{1}{\cos \theta} = \frac{1-z}{\sqrt{z^2 - 2z}}$



19. $\sin \theta = \frac{52}{x}$, $\theta = \sin^{-1}\frac{52}{x}$

21. $2 \sin \theta = -1.88$
 $\sin \theta = \frac{-1.88}{2} = -0.94$
 $\theta = \sin^{-1}(-0.94)$

23. $4 \cos(2\theta - 3) = 2.4$
 $\cos(2\theta - 3) = 0.6$
 $2\theta - 3 = \cos^{-1}0.6$
 $2\theta = \cos^{-1}0.6 + 3$
 $\theta = \frac{1}{2}(\cos^{-1}0.6 + 3)$



Exercise 5-0

- $$\begin{aligned} 17. \quad & 2 \sec^2 \theta + 3 \sec \theta - 7 = 0 \\ & \sec \theta = \frac{-3 \pm \sqrt{3^2 - 4(2)(-7)}}{2(2)} \\ & \sec \theta = \frac{-3 \pm \sqrt{65}}{4} \\ 21. \quad & \frac{2\left(3x - \frac{1}{2}\right)^2 - 3\left(3x - \frac{1}{2}\right) + 1}{\left(3x - \frac{1}{2}\right)^2 - 1} \\ & u = 3x - \frac{1}{2} \\ & \frac{2u^2 - 3u + 1}{u^2 - 1} \\ & \frac{(u-1)(2u-1)}{(u-1)(u+1)} \\ & \frac{2u-1}{u+1} \end{aligned}$$

$$\frac{2\left(3x - \frac{1}{2}\right) - 1}{3x - \frac{1}{2} + 1} = \frac{6x - 1 - 1}{3x + \frac{1}{2}} = \frac{6x - 2}{3x + \frac{1}{2}} \cdot \frac{2}{2} = \frac{12x - 4}{6x + 1}$$

1. $\frac{\sin \theta}{\tan \theta}$
 $\frac{\sin \theta}{\frac{\sin \theta}{\cos \theta}}$
 $\sin \theta \cdot \frac{\cos \theta}{\sin \theta}$
 $\cos \theta$
5. $\cot^2 \theta \sin^2 \theta$
 $\frac{\cos^2 \theta}{\sin^2 \theta} \cdot \sin^2 \theta$
 $\cos^2 \theta$
9. $\frac{(\sec \theta - 1)(\sec \theta + 1)}{\sin^2 \theta}$
 $\frac{\sec^2 \theta - 1}{\sin^2 \theta}$
 $\frac{\tan^2 \theta}{\sin^2 \theta}$ $\tan^2 \theta + 1 = \sec^2 \theta,$
 so $\tan^2 \theta = \sec^2 \theta - 1$
 $\tan^2 \theta \cdot \frac{1}{\sin^2 \theta}$
 $\frac{\sin^2 \theta}{\cos^2 \theta} \cdot \frac{1}{\sin^2 \theta}$
 $\frac{1}{\cos^2 \theta}$
 $\sec^2 \theta$
13. $\cos \theta (\sec \theta - \cos \theta)$
 $\cos \theta \sec \theta - \cos^2 \theta$
 $\cos \theta \cdot \frac{1}{\cos \theta} - \cos^2 \theta$
 $1 - \cos^2 \theta$
 $\sin^2 \theta$
17. $\frac{\cos^2 \theta}{\sin^2 \theta} - \frac{1}{\sin^2 \theta}$
 $\frac{\cos^2 \theta - 1}{\sin^2 \theta}$
 $\frac{-\sin^2 \theta}{\sin^2 \theta}$
 -1
21. $\sec^2 \theta (\csc^2 \theta - 1)$
 $\sec^2 \theta [(\cot^2 \theta + 1) - 1]$
 $\sec^2 \theta \cot^2 \theta$
 $\frac{1}{\cos^2 \theta} \cdot \frac{\cos^2 \theta}{\sin^2 \theta}$
 $\frac{1}{\sin^2 \theta}$
 $\csc^2 \theta$

25. $\frac{\sin x + \cos x \cot x}{\sin x + \cos x \frac{\cos x}{\sin x}}$
 $\frac{\frac{\sin^2 x}{\sin x} + \frac{\cos^2 x}{\sin x}}{\frac{\sin^2 x + \cos^2 x}{\sin x}}$
 $\frac{1}{\sin x}$
 $\csc x$
29. $\sec x - \tan x \sin x$
 $\sec x - \frac{\sin x}{\cos x} \cdot \sin x$
 $\frac{1}{\cos x} - \frac{\sin^2 x}{\cos x}$
 $\frac{1 - \sin^2 x}{\cos x}$
 $\sin^2 \theta + \cos^2 \theta = 1$, so $\cos^2 \theta = 1 - \sin^2 \theta$
 $\frac{\cos^2 \theta}{\cos \theta}$
 $\cos \theta$
33. $\csc \theta + \cot \theta$
 $\frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta}$
 $\frac{1 + \cos \theta}{\sin \theta}$
37. $\frac{1 + \csc \theta}{1 + \sec \theta}$
 $\frac{1 + \frac{1}{\sin \theta}}{1 + \frac{1}{\cos \theta}}$
 $\frac{\frac{\sin \theta + 1}{\sin \theta}}{\frac{\cos \theta + 1}{\cos \theta}}$
 $\frac{\sin \theta + 1}{\sin \theta} \cdot \frac{\cos \theta}{\cos \theta + 1}$
 $\frac{\cos \theta}{\sin \theta} \cdot \frac{\sin \theta + 1}{\cos \theta + 1}$
 $\cot \theta \left(\frac{1 + \sin \theta}{1 + \cos \theta} \right)$
41. $\sec y - \cos y$
 $\frac{1}{\cos y} - \cos y$
 $\frac{1 - \cos^2 y}{\cos y}$

$$\begin{array}{l}
\frac{\sin 2y}{\cos y} \\
\frac{\sin y}{\cos y} \cdot \sin y \\
\tan y \sin y \\
45. \quad \frac{1}{\sec \theta - \cos \theta} \\
\frac{1}{\cos \theta} - \cos \theta \\
\frac{1}{1 - \cos^2 \theta} \\
\frac{1}{\cos \theta} \\
\frac{\sin^2 \theta}{\cos \theta} \\
\cos \theta \\
\frac{\sin^2 \theta}{\cos \theta} \\
\cos \theta \cdot \frac{1}{\sin \theta} \\
\cot \theta \csc \theta \\
49. \quad \frac{\tan^2 \theta}{\sec \theta - 1} \\
\frac{\sec^2 \theta - 1}{\sec \theta - 1} \\
\sec \theta + 1 \\
\tan^2 \theta + 1 = \sec^2 \theta, \text{ so } \tan^2 \theta = \sec^2 \theta - 1. \\
\frac{(\sec \theta - 1)(\sec \theta + 1)}{\sec \theta - 1} \\
\sec \theta + 1 \\
53. \quad \frac{\sin x}{1 + \cos x} \\
\frac{\sin x}{1 + \cos x} \cdot \frac{1 - \cos x}{1 - \cos x} \\
\frac{\sin x(1 - \cos x)}{1 - \cos^2 x} \\
\frac{\sin x(1 - \cos x)}{\sin^2 x} \\
\frac{1 - \cos x}{\sin x} \quad \text{Alternate solution in text} \\
57. \quad \frac{1}{1 - \sin x} + \frac{1}{1 + \sin x} \\
\frac{(1 + \sin x) + (1 - \sin x)}{(1 - \sin x)(1 + \sin x)} \\
\frac{\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}}{\frac{2}{1 - \sin^2 x} = \frac{2}{\cos^2 x} = 2 \sec^2 x}
\end{array}$$

$$61. \frac{\cot^2 x - \cos^2 x}{\frac{\cos^2 x}{\sin^2 x} - \cos^2 x} = \frac{\cos^2 x - \cos^2 x \sin^2 x}{\sin^2 x} = \frac{\cos^2 x(1 - \sin^2 x)}{\sin^2 x}$$

$$\frac{\cos^2 x}{\sin^2 x} (1 - \sin^2 x) = \cot^2 x - \cos^2 x$$

$$65. \text{Left side: } \frac{1 - \sin x}{1 + \sin x} \cdot \frac{1 - \sin x}{1 + \sin x} = \frac{1 - \sin x}{1 + \sin x} \cdot \frac{1 - \sin x}{1 - \sin x} = \frac{1 - \sin^2 x}{1 - 2 \sin x + \sin^2 x} = \frac{\cos^2 x}{1 - 2 \sin x + \sin^2 x} = \frac{1}{\cos^2 x} - \frac{2 \sin x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \sec^2 x - 2 \frac{\sin x}{\cos x} + \tan^2 x$$

$$\sec^2 x - 2 \tan x \sec x + \tan^2 x$$

Right side:

$$(\tan x - \sec x)^2 = \tan^2 x - 2 \tan x \sec x + \sec^2 x$$

$$69. \frac{2 \cos^2 y - 1}{2 \cos^2 y - (\sin^2 y + \cos^2 y)} = \frac{2 \cos^2 y - 1}{\cos^2 y - \sin^2 y}$$

In problems 73 and 77 let $\theta = \frac{\pi}{3}$; $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$,

$$\cos \frac{\pi}{3} = \frac{1}{2}; \tan \frac{\pi}{3} = \sqrt{3}.$$

Different values are used in the text for θ .

$$73. \sec \theta = \frac{1}{\csc \theta}$$

$$2 = \frac{1}{\frac{\sqrt{3}}{2}}$$

$$2 \neq \frac{\sqrt{3}}{2}$$

$$77. \csc \theta + \sec \theta \cot \theta = 2$$

$$\frac{2}{\sqrt{3}} + 2 \cdot \frac{1}{\sqrt{3}} = 2$$

$$\frac{4}{\sqrt{3}} \neq 2$$

$$81. (a) (1 - \csc^2 \theta)(1 - \sec^2 \theta) = 1$$

$$\theta = \frac{\pi}{6}: (1 - \csc^2 \frac{\pi}{6})(1 - \sec^2 \frac{\pi}{6}) = (1 - 2^2)(1 - (\frac{2}{\sqrt{3}})^2) = (-3)(1 - \frac{4}{3}) = (-3)(-\frac{1}{3}) = 1$$

$$(b) \theta = \frac{\pi}{4}: (1 - \csc^2 \frac{\pi}{4})(1 - \sec^2 \frac{\pi}{4}) = (1 - (\sqrt{2})^2)(1 - (\sqrt{2})^2) = (1 - 2)(1 - 2) = 1$$

$$(c) \text{ Yes: } (1 - \csc^2 \theta)(1 - \sec^2 \theta) = 1$$

$$(-\cot^2 \theta)(-\tan^2 \theta) = \frac{1}{\tan^2 \theta} \tan^2 \theta = 1$$

Exercise 5-2

$$1. \sin 18^\circ = \cos(90^\circ - 18^\circ) = \cos 72^\circ$$

$$5. \sec \frac{\pi}{3} = \csc(\frac{\pi}{2} - \frac{\pi}{3}) = \csc \frac{\pi}{6}$$

$$9. \sec(-\frac{3\pi}{4}) = \csc(\frac{\pi}{2} - (-\frac{3\pi}{4})) = \csc \frac{5\pi}{4}$$

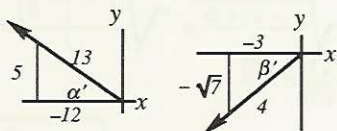
$$13. \cos 20^\circ \csc 70^\circ = \cos 20^\circ \sec 20^\circ = \cos 20^\circ \cdot \frac{1}{\cos 20^\circ} = 1$$

$$29. \sin \frac{5\pi}{12} = \sin(\frac{\pi}{4} + \frac{\pi}{6}) = \sin \frac{\pi}{4} \cos \frac{\pi}{6} + \cos \frac{\pi}{4} \sin \frac{\pi}{6} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

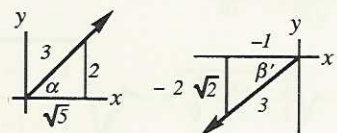
$$33. \sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$37. \tan 75^\circ = \tan(45^\circ + 30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} = \frac{1 + \frac{\sqrt{3}}{3}}{1 - 1(\frac{\sqrt{3}}{3})} = \frac{\frac{3 + \sqrt{3}}{3}}{\frac{3 - \sqrt{3}}{3}} = \frac{3 + \sqrt{3}}{3 - \sqrt{3}} = \frac{12 + 6\sqrt{3}}{6} = 2 + \sqrt{3}$$

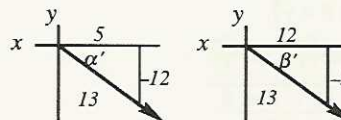
$$41. \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} = \frac{-\frac{5}{12} - \frac{\sqrt{7}}{3}}{1 + (-\frac{5}{12})(\frac{\sqrt{7}}{3})} = \frac{-\frac{15 - 12\sqrt{7}}{36}}{1 - \frac{5\sqrt{7}}{36}} = \frac{-15 - 12\sqrt{7}}{36 - 5\sqrt{7}}$$



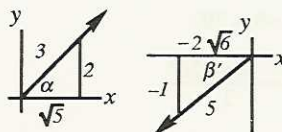
$$45. \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta = \frac{\sqrt{5}}{3}(-\frac{1}{3}) + \frac{2}{3}(-\frac{2\sqrt{2}}{3}) = \frac{-\sqrt{5} - 4\sqrt{2}}{9}$$



$$49. \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} = \frac{-\frac{12}{5} - (-\frac{5}{12})}{1 + (-\frac{12}{5})(-\frac{5}{12})} = \frac{-\frac{12}{5} + \frac{5}{12}}{1 + 1} = \frac{-\frac{144 + 25}{60}}{2} = \frac{1}{2}(-\frac{119}{60}) = -\frac{119}{120}$$



$$53. \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta = \frac{2}{3}(-\frac{2\sqrt{6}}{5}) - \frac{\sqrt{5}}{3}(-\frac{1}{5}) = \frac{-4\sqrt{6} + \sqrt{5}}{15}$$



$$57. \sin(\pi - \theta) = \sin \pi \cos \theta - \cos \pi \sin \theta \\ = 0 \cos \theta - (-1) \sin \theta \\ = \sin \theta$$

$$61. \tan(\pi - \theta) = \frac{\tan \pi - \tan \theta}{1 + \tan \pi \tan \theta} \\ = \frac{0 - \tan \theta}{1 + 0 \tan \theta} \\ = -\tan \theta$$

$$73. \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha + (-\beta)) = \cos \alpha \cos(-\beta) - \sin \alpha \sin(-\beta)$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta - \sin \alpha [-\sin \beta]$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$$

$$77. \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha + (-\beta)) = \sin \alpha \cos(-\beta) + \cos \alpha \sin(-\beta)$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta + \cos \alpha [-\sin \beta]$$

$$= \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$81. \csc\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\sin\left(\frac{\pi}{2} - \theta\right)} = \frac{1}{\cos \theta} = \sec \theta$$

$$65. \tan(\theta + \pi) = \frac{\tan \theta + \tan \pi}{1 - \tan \theta \tan \pi} \\ = \frac{\tan \theta + 0}{1 - \tan \theta (0)} \\ = \tan \theta$$

$$69. \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)] \\ = \frac{1}{2} [\cos \alpha \cos \beta + \sin \alpha \sin \beta - (\cos \alpha \cos \beta - \sin \alpha \sin \beta)] \\ = \frac{1}{2} [2 \sin \alpha \sin \beta] = \sin \alpha \sin \beta$$

This was shown true in the text.

Replace β by $-\beta$. This is valid since the identity is true for all angles α and β .

$\alpha + (-\beta) = \alpha - \beta$; $\cos(-\theta) = \cos \theta$; $\sin(-\theta) = -\sin \theta$.

This statement is true since the preceding statements are true.

Identity [3].

Cosine is an even function, sine is odd.

This is identity [4].

Exercise 5-3

$$1. 2 \sin \frac{\pi}{4} \cos \frac{\pi}{4}$$

$$2 \sin \alpha \cos \alpha = \sin 2\alpha$$

$$\text{Let } \alpha = \frac{\pi}{4}, \text{ so } 2\alpha = \frac{\pi}{2}, \text{ and } \sin 2\alpha \text{ is } \sin \frac{\pi}{2}.$$

$$5. 1 - 2 \sin^2 \frac{\pi}{10}$$

$$1 - 2 \sin^2 \alpha = \cos 2\alpha$$

$$\text{Let } \alpha = \frac{\pi}{10}, \text{ so } 2\alpha = \frac{\pi}{5}, \text{ so } \cos 2\alpha = \cos \frac{\pi}{5}.$$

$$9. 2 \sin 6\theta \cos 6\theta$$

$$2 \sin \alpha \cos \alpha = \sin 2\alpha$$

$$\text{Let } \alpha = 6\theta.$$

$$\sin 2\alpha \text{ becomes } \sin 12\theta.$$

$$13. \frac{10 \tan 3\theta}{1 - \tan^2 3\theta}$$

$$\frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \tan 2\alpha$$

Identity.

$$\frac{10 \tan \alpha}{1 - \tan^2 \alpha} = 5 \tan 2\alpha$$

Multiply each member by 5.

$$\text{Let } \alpha = 3\theta, \text{ so } 2\alpha = 6\theta. \text{ Thus } 5 \tan 2\alpha \text{ is } 5 \tan 6\theta.$$

$$17. 3 \cos^2 3\theta - 3 \sin^2 3\theta$$

$$\cos^2 \alpha - \sin^2 \alpha = \cos 2\alpha$$

Identity.

$$3 \cos^2 \alpha - 3 \sin^2 \alpha = 3 \cos 2\alpha$$

Multiply each member by 3. α represents 3θ , so 2α is 6θ .

Then $3 \cos 2\alpha$ means $3 \cos 6\theta$.

$$21. \cos \frac{5\pi}{6} = \cos^2 \theta - \sin^2 \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$2\theta = \frac{5\pi}{6}, \text{ so } \theta = \frac{5\pi}{12}.$$

$$\cos \frac{5\pi}{6} = \cos^2 \frac{5\pi}{12} - \sin^2 \frac{5\pi}{12}.$$

$$25. \sin 10^\circ = \sqrt{\frac{1 - \cos \theta}{2}}$$

$$\sin \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{2}}$$

$$\frac{\alpha}{2} = 10^\circ, \text{ so } \alpha = \theta = 20^\circ.$$

$$\sin 10^\circ = \sqrt{\frac{1 - \cos 20^\circ}{2}}$$

$$29. \tan \frac{2\pi}{5} = \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}$$

$$\tan \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$$

$$\frac{\alpha}{2} = \frac{2\pi}{5}, \text{ so } \alpha = \theta = \frac{4\pi}{5}, \text{ and}$$

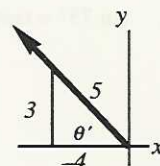
$$\tan \frac{2\pi}{5} = \sqrt{\frac{1 - \cos \frac{4\pi}{5}}{1 + \cos \frac{4\pi}{5}}}$$

$$33. \cos \theta = -\frac{4}{5}, \frac{\pi}{2} < \theta < \pi$$

$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \cdot \frac{3}{5} \left(-\frac{4}{5}\right) = -\frac{24}{25}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta \\ = \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = -\frac{24}{7}$$



$$37. \sec \theta = -\frac{5}{2}, \pi < \theta < \frac{3\pi}{2}: \cos \theta = -\frac{2}{5}$$

$$\frac{\pi}{2} \leq \frac{\theta}{2} \leq \frac{3\pi}{4}, \left(\frac{\theta}{2} \text{ in qII}\right) \text{ so } \sin \frac{\theta}{2} > 0, \cos \frac{\theta}{2} < 0, \tan \frac{\theta}{2} < 0.$$

$$\sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{2}} = \sqrt{\frac{1 - \left(-\frac{2}{5}\right)}{2}} = \sqrt{\frac{1 + \frac{2}{5}}{2}} \\ = \sqrt{\frac{\frac{7}{5}}{2}} = \frac{\sqrt{7}}{\sqrt{10}} = \frac{\sqrt{70}}{10}$$

$$\cos \frac{\theta}{2} = -\sqrt{\frac{1 + \cos \theta}{2}} = -\sqrt{\frac{1 + \left(-\frac{2}{5}\right)}{2}} = -\sqrt{\frac{1 - \frac{2}{5}}{2}} \\ = -\sqrt{\frac{\frac{3}{5}}{2}} = -\frac{\sqrt{30}}{10}$$

$$\tan \frac{\theta}{2} = -\sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} = -\sqrt{\frac{1 - \left(-\frac{2}{5}\right)}{1 + \left(-\frac{2}{5}\right)}} \\ = -\sqrt{\frac{\frac{7}{5}}{\frac{3}{5}}} = -\sqrt{\frac{7}{3}} = -\frac{\sqrt{21}}{3}$$

$$41. \quad 15^\circ, \text{ or } \frac{\pi}{12} \quad \sin 15^\circ = \sqrt{\frac{1 - \cos 30^\circ}{2}} = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2 - \sqrt{3}}{4}} = \frac{\sqrt{2 - \sqrt{3}}}{2}$$

$$\cos 15^\circ = \sqrt{\frac{1 + \cos 30^\circ}{2}} = \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2 + \sqrt{3}}{4}} = \frac{\sqrt{2 + \sqrt{3}}}{2}$$

$$\tan 15^\circ = \frac{1 - \cos 30^\circ}{\sin 30^\circ} = \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{2 - \sqrt{3}}{1} = 2 - \sqrt{3}$$

$$45. \quad \cos 37.5^\circ = \cos(15^\circ + 22.5^\circ) = \cos 15^\circ \cos 22.5^\circ - \sin 15^\circ \sin 22.5^\circ$$

$$= \frac{\sqrt{2 + \sqrt{3}}}{2} \cdot \frac{\sqrt{2 + \sqrt{2}}}{2} - \frac{\sqrt{2 - \sqrt{3}}}{2} \cdot \frac{\sqrt{2 - \sqrt{2}}}{2} = \frac{\sqrt{4 + \sqrt{6} + 2\sqrt{3} + 2\sqrt{2}} - \sqrt{4 + \sqrt{6} - 2\sqrt{3} - 2\sqrt{2}}}{4}$$

Note: A simpler solution is as follows: Use $\cos 75^\circ = \cos(30^\circ + 45^\circ)$ and expand. This gives $\cos 75^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$. Use this value in

the identity $\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$, with $\alpha = 75^\circ$. This produces the solution $\frac{1}{4} \sqrt{2\sqrt{6} - 2\sqrt{2} + 8}$.

$$49. \quad \frac{\sin 2\theta + 1}{2 \sin \theta \cos \theta + 1}$$

$$\frac{2 \sin \theta \cos \theta + \sin^2 \theta + \cos^2 \theta}{\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta}$$

$$\frac{(\sin \theta + \cos \theta)^2}{(\sin \theta + \cos \theta)^2}$$

(Alternate solution in text)

$$53. \quad \frac{1 + \cos 2\theta}{1 - \cos 2\theta}$$

$$\frac{1 + (2 \cos^2 \theta - 1)}{1 - (1 - 2 \sin^2 \theta)}$$

$$\frac{2 \cos^2 \theta}{2 \sin^2 \theta}$$

$$\frac{\cos^2 \theta}{\sin^2 \theta}$$

$$\cot^2 \theta$$

$$57. \quad \frac{\sin 2\theta - 4 \sin^3 \theta \cos \theta}{2 \sin \theta \cos \theta - 4 \sin^3 \theta \cos \theta}$$

$$\frac{2 \sin \theta \cos \theta (1 - 2 \sin^2 \theta)}{\sin 2\theta \cos 2\theta}$$

(Alternate solution in text)

$$61. \quad \tan 2\theta = \frac{2(\tan \theta + \tan^3 \theta)}{1 - \tan^4 \theta}$$

$$\frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2 \tan \theta (1 + \tan^2 \theta)}{(1 - \tan^2 \theta)(1 + \tan^2 \theta)}$$

$$= \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$65. \quad \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$$

$$\frac{1 - \frac{\sin^2 \theta}{\cos^2 \theta}}{1 + \frac{\sin^2 \theta}{\cos^2 \theta}}$$

$$\frac{\frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta}}{\frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta}}$$

$$\frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin^2 \theta}$$

$$\frac{\cos 2\theta}{\cos^2 \theta}$$

$$69. \quad \frac{\cos 2\theta \sin^2 \theta}{\cos^2 \theta}$$

$$\frac{1}{\cos 2\theta} \cdot \frac{\cos^2 \theta \sin^2 \theta}{\cos^2 \theta}$$

$$\left(\pm \sqrt{\frac{1 + \cos \theta}{2}} \right)^2 \left(\pm \sqrt{\frac{1 - \cos \theta}{2}} \right)^2$$

$$\frac{1 + \cos \theta}{2} \cdot \frac{1 - \cos \theta}{2}$$

$$\frac{1 - \cos^2 \theta}{4}$$

$$\frac{\sin^2 \theta}{4}$$

$$73. \quad \frac{\sin^2 \theta}{2} = \frac{\csc \theta - \cot \theta}{2 \csc \theta}$$

$$\sin^2 \theta = \frac{1 - \cos \theta}{2} \cdot \frac{\sin \theta}{\sin \theta}$$

$$\left(\sqrt{\frac{1 - \cos \theta}{2}} \right)^2 = \frac{1 - \cos \theta}{2}$$

$$\frac{1 - \cos \theta}{2}$$

$$77. \quad 2 \cos^2 \frac{\theta}{2} - \cos \theta = 1$$

$$2 \left(\sqrt{\frac{1 + \cos \theta}{2}} \right)^2 - \cos \theta$$

$$2 \frac{1 + \cos \theta}{2} - \cos \theta$$

$$1 + \cos \theta - \cos \theta$$

$$1$$

81. We know that $\sin 4\theta = 4 \cos^3 \theta \sin \theta - 4 \sin^3 \theta \cos \theta$ and $\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1$ from problem 80.

a. $\sin 5\theta$

$$= \sin(\theta + 4\theta)$$

$$= \sin \theta \cos 4\theta + \cos \theta \sin 4\theta$$

$$= \sin \theta (8 \cos^4 \theta - 8 \cos^2 \theta + 1) + \cos \theta (4 \sin \theta \cos \theta - 8 \sin^3 \theta \cos \theta)$$

$$= 4 \sin \theta \cos^2 \theta - 8 \sin^3 \theta \cos^2 \theta + 8 \sin \theta \cos^4 \theta - 8 \sin \theta \cos^2 \theta + \sin \theta$$

We know $\cos^2 \theta = 1 - \sin^2 \theta$, so that $\cos^4 \theta = (1 - \sin^2 \theta)^2 = 1 - 2 \sin^2 \theta + \sin^4 \theta$.

Replace $\cos^2 \theta$ and $\cos^4 \theta$ in the equation above:

$$= 4 \sin \theta (1 - \sin^2 \theta) - 8 \sin^3 \theta (1 - \sin^2 \theta) + 8 \sin \theta (1 - 2 \sin^2 \theta + \sin^4 \theta)$$

$$= 4 \sin \theta - 4 \sin^3 \theta - 8 \sin^3 \theta + 8 \sin^5 \theta + 8 \sin \theta - 16 \sin^3 \theta + 8 \sin^5 \theta$$

$$= 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$$

b. $\cos 5\theta$

$$= \cos(\theta + 4\theta)$$

$$= \cos \theta \cos 4\theta - \sin \theta \sin 4\theta$$

$$= \cos \theta (8 \cos^4 \theta - 8 \cos^2 \theta + 1) - \sin \theta (4 \cos^3 \theta \sin \theta - 4 \sin^3 \theta \cos \theta)$$

$$= 8 \cos^5 \theta - 8 \cos^3 \theta + \cos \theta - 4 \cos^3 \theta \sin^2 \theta + 4 \sin^4 \theta \cos \theta$$

$$= 8 \cos^5 \theta - 8 \cos^3 \theta + \cos \theta - 4 \cos^3 \theta (1 - \cos^2 \theta) + 4 (1 - \cos^2 \theta)^2 \cos \theta$$

$$= 8 \cos^5 \theta - 8 \cos^3 \theta + \cos \theta - 4 \cos^3 \theta + 4 \cos^5 \theta + 4 \cos \theta - 8 \cos^3 \theta + 4 \cos^5 \theta$$

$$= 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$$

85. (a) Problem 41 shows that $\sin 15^\circ = \frac{\sqrt{2} - \sqrt{3}}{2}$.

(b) $\sin 15^\circ = \sin(45^\circ - 30^\circ)$
 $= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$
 $= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$
 $= \frac{\sqrt{6} - \sqrt{2}}{4}$

(c) $\frac{\sqrt{6} - \sqrt{2}}{4} = \frac{\sqrt{2} - \sqrt{3}}{2}$ Square both values.
 $\left(\frac{\sqrt{6} - \sqrt{2}}{4}\right)^2 = \left(\frac{\sqrt{2} - \sqrt{3}}{2}\right)^2$
 $\frac{(\sqrt{6} - \sqrt{2})^2}{16} = \frac{2 - \sqrt{3}}{4}$
 $\frac{6 - 2\sqrt{12} + 2}{16} = \frac{2 - \sqrt{3}}{4}$
 $\frac{8 - 4\sqrt{3}}{16} = \frac{2 - \sqrt{3}}{4}$
 $\frac{4(2 - \sqrt{3})}{16} = \frac{2 - \sqrt{3}}{4}$
 $\frac{2 - \sqrt{3}}{4} = \frac{2 - \sqrt{3}}{4}$

93. $\cos 2\alpha + \cos 2\beta = 2 \cos(\alpha + \beta) \cdot \cos(\alpha - \beta)$
 $(2 \cos^2 \alpha - 1) + (2 \cos^2 \beta - 1) = 2(\cos \alpha \cos \beta - \sin \alpha \sin \beta)(\cos \alpha \cos \beta + \sin \alpha \sin \beta)$
 $2 \cos^2 \alpha + 2 \cos^2 \beta - 2 = 2(\cos^2 \alpha \cos^2 \beta - \sin^2 \alpha \sin^2 \beta)$
 $2[\cos^2 \alpha \cos^2 \beta - (1 - \cos^2 \alpha)(1 - \cos^2 \beta)]$
 $2[\cos^2 \alpha \cos^2 \beta - (1 - \cos^2 \beta - \cos^2 \alpha + \cos^2 \alpha \cos^2 \beta)]$
 $2[-1 + \cos^2 \beta + \cos^2 \alpha]$
 $-2 + 2\cos^2 \beta + 2\cos^2 \alpha$

89. $\cos \theta_2 = \cos \frac{\theta}{2} = \frac{8}{9}$; $\cos \theta = \frac{8}{x}$

$\cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}}$

The angles are acute, so we need the positive value.

$\frac{8}{9} = \sqrt{\frac{1 + \frac{8}{x}}{2}}$

Substitute values.

$\frac{64}{81} = \frac{1 + \frac{8}{x}}{2}$

Square both members.

$128 = 81\left(1 + \frac{8}{x}\right)$

If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

$128 = 81 + \frac{648}{x}$

$47 = \frac{648}{x}$

$x = \frac{648}{47} = 13\frac{37}{47}$

Exercise 5-4

1. $\tan \theta + 1 = 0$
 $\tan \theta = -1$

$\theta' = \tan^{-1} 1 = \frac{\pi}{4} (45^\circ)$

tangent is negative in qII and qIV, so $\theta = \pi - \theta' = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$(180^\circ - 45^\circ = 135^\circ)$

or $2\pi - \theta' = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4} (360^\circ - 45^\circ = 315^\circ)$.

5. $\sqrt{3} \tan \theta - 1 = 0$

$\tan \theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$;

$\theta' = \tan^{-1} \frac{\sqrt{3}}{3} = \frac{\pi}{6} (30^\circ)$.

$\tan \theta > 0$ in qI and qIII, so $\theta = \theta' = \frac{\pi}{6} (30^\circ)$ and

$\pi + \theta' = \pi + \frac{\pi}{6} = \frac{7\pi}{6} (180^\circ + 30^\circ = 210^\circ)$.

9. $3 \sin^2 \theta - 3 = 0$

$\sin^2 \theta - 1 = 0$

Divide each member by 3.

$\sin^2 \theta = 1$

$\sin \theta = \pm 1$

$\sin \theta = 1$ at $\frac{\pi}{2} (90^\circ)$ and $\sin \theta = -1$ at $\frac{3\pi}{2} (270^\circ)$.

13. $(\cos \theta - 1)(\sin \theta + 1) = 0$

$\cos \theta - 1 = 0$ or $\sin \theta + 1 = 0$

Zero product property.

$\cos \theta = 1$ or $\sin \theta = -1$

$0 (0^\circ)$

$\frac{3\pi}{2} (270^\circ)$

17. $\sin^2 \theta - \sin \theta = 0$

$\sin \theta(\sin \theta - 1) = 0$

$\sin \theta = 0$ or $a^2 - a = a(a - 1)$

$0 (0^\circ)$ and $\pi (180^\circ)$

or $\sin \theta - 1 = 0$

$\sin \theta = 1$

$\frac{\pi}{2} (90^\circ)$

21. $2 \sin^2 \theta + \sin \theta - 1 = 0$

$(2 \sin \theta - 1)(\sin \theta + 1) = 0$

Factor like $2u^2 + u - 1 = 0$.

$2 \sin \theta - 1 = 0$ or $\sin \theta + 1 = 0$

$2 \sin \theta = 1$ or $\sin \theta = -1$

$\sin \theta = -1$

$\sin \theta = \frac{1}{2}$

$\frac{3\pi}{2} (270^\circ)$

$\theta' = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$; $\sin \theta > 0$ in qI and qII.

$\theta = \theta' = \frac{\pi}{6} (30^\circ)$ and $\pi - \frac{\pi}{6} = \frac{5\pi}{6} (150^\circ)$

25. $2 \sin \theta \cos \theta - \sin \theta = 0$

$\sin \theta(2 \cos \theta - 1) = 0$

$\sin \theta = 0$ or $2 \cos \theta - 1 = 0$

$0 (0^\circ), \pi (180^\circ)$

$\cos \theta = \frac{1}{2}$

$\frac{\pi}{3} (60^\circ)$ and $\frac{5\pi}{3} (300^\circ)$

(See problem 3.)

29. $\tan x \cot x = 0$

$\tan x = 0$ or $\cot x = 0$

$\frac{\sin x}{\cos x} = 0$ or $\frac{\cos x}{\sin x} = 0$

$\sin x = 0$ or $\cos x = 0$

$0 (0^\circ), \pi (180^\circ)$

$\frac{\pi}{2} (90^\circ), \frac{3\pi}{2} (270^\circ)$

33. $\tan x + \cot x = -2$

$\tan x + \frac{1}{\tan x} = -2$

$\tan^2 x + 1 = -2 \tan x$

$\tan^2 x + 2 \tan x + 1 = 0$

$(\tan x + 1)^2 = 0$

$\tan x + 1 = 0$

$\tan x = -1$

$\frac{3\pi}{4} (135^\circ)$ and $\frac{7\pi}{4} (315^\circ)$ (Problem 1.)

37. $4 \tan^2 x = 3 \sec^2 x$

$4 \tan^2 x = 3(\tan^2 x + 1)$

$4 \tan^2 x = 3 \tan^2 x + 3$

$\tan^2 x = 3$

$\tan x = \pm \sqrt{3}$

$\theta' = \tan^{-1} \sqrt{3} = \frac{\pi}{3} (60^\circ)$

Since $\tan \theta$ is both positive and negative there are solutions in each quadrant.

$$qI: \quad \theta = \theta' = \frac{\pi}{3} (60^\circ)$$

$$qII: \quad \theta = \pi - \theta' = \pi - \frac{\pi}{3} = \frac{2\pi}{3} (180^\circ - 60^\circ = 120^\circ)$$

$$qIII: \quad \theta = \pi + \theta' = \pi + \frac{\pi}{3} = \frac{4\pi}{3} (180^\circ + 60^\circ = 240^\circ)$$

$$qIV: \quad \theta = 2\pi - \theta' = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} (360^\circ - 60^\circ = 300^\circ)$$

$$41. \quad \begin{aligned} 2 \tan^2 x \sin x &= \tan^2 x \\ 2 \tan^2 x \sin x - \tan^2 x &= 0 \\ \tan^2 x (2 \sin x - 1) &= 0 \\ \tan^2 x &= 0 & \text{or} & \quad 2 \sin x - 1 = 0 \\ \tan x &= 0 & \text{or} & \quad \sin x = \frac{1}{2} \\ 0 (0^\circ), \pi (180^\circ) & \text{(Prob. 19)} & \quad \frac{\pi}{6} (30^\circ) \text{ and } \frac{5\pi}{6} (150^\circ) & \text{(Prob. 21.)} \end{aligned}$$

$$45. \quad \begin{aligned} \cot^2 x - 3 \cot x - 2 &= 0 \\ a = 1, b = -3, c = -2: \\ \cot x &= \frac{3 \pm \sqrt{9 - 4(-2)}}{2} = \frac{3 \pm \sqrt{17}}{2} \\ \cot x &= \frac{3 + \sqrt{17}}{2} & \cot x &= \frac{3 - \sqrt{17}}{2} \\ \tan x &= \frac{2}{3 + \sqrt{17}} & \tan x &= \frac{2}{3 - \sqrt{17}} \\ x' &= \tan^{-1} \frac{2}{3 + \sqrt{17}} & x' &= \tan^{-1} \frac{2}{3 - \sqrt{17}} \\ x' &\approx 0.274 (15.68^\circ) & x' &\approx 1.059 (60.68^\circ) \\ \tan x > 0 \text{ in } qI \text{ and } qIII. & \tan x < 0 \text{ in } qII \text{ and } qIV. \\ x = x' &\approx 0.27 & x &= \pi - x' \approx \pi - 1.059 \\ & & &\approx 2.08 \\ & & &= 2\pi - x' \approx 2\pi - 1.059 \\ & & &\approx 5.22 \\ x^\circ &= x' \approx 15.7^\circ & x^\circ &= 180^\circ - x' \\ & & &\approx 180^\circ - 60.7^\circ \approx 119.3^\circ \\ & & &= 360^\circ - x' \\ & & &\approx 360^\circ - 60.7^\circ \\ & & &\approx 299.3^\circ \\ & & &= 180^\circ + x' \approx 180^\circ + 15.7^\circ \\ & & &= 195.7^\circ \end{aligned}$$

$$49. \quad \begin{aligned} \tan x + 2 \sec x &= 3 \\ \tan x &= 3 - 2 \sec x \\ (\tan x)^2 &= (3 - 2 \sec x)^2 & \text{Since we are squaring both sides} \\ \tan^2 x &= 9 - 12 \sec x + 4 \sec^2 x & \text{we must check all answers.} \\ \sec^2 x - 1 &= 9 - 12 \sec x + 4 \sec^2 x \\ 3 \sec^2 x - 12 \sec x + 10 &= 0 \\ a = 3, b = -12, c = 10: \sec x &= \frac{12 \pm \sqrt{144 - 120}}{6} = \frac{12 \pm 2\sqrt{6}}{6} \\ &= \frac{6 \pm \sqrt{6}}{3} \end{aligned}$$

$$\begin{aligned} \sec x &= \frac{6 + \sqrt{6}}{3} & \sec x &= \frac{6 - \sqrt{6}}{3} \\ \cos x &= \frac{3}{6 + \sqrt{6}} \approx 0.3551 & \cos x &= \frac{3}{6 - \sqrt{6}} \approx 0.8449 \\ x &= \cos^{-1} \frac{3}{6 + \sqrt{6}} \approx 1.208 (69.2^\circ) & x &= \cos^{-1} \frac{3}{6 - \sqrt{6}} \approx 0.564 (32.3^\circ) \\ \cos x > 0 \text{ in } qI \text{ and } qIV. & \cos x > 0 \text{ in } qI \text{ and } qIV. \\ x = x' &\approx 1.21 & x &= x' \approx 0.56 \\ &= 2\pi - x' \approx 2\pi - 1.208 \approx 5.08 & &= 2\pi - x' \approx 2\pi - 0.56 \\ & & &\approx 5.72 \\ x^\circ &= x' \approx 69.2^\circ & x^\circ &= x' \approx 32.3^\circ \\ &= 360^\circ - x' \approx 360^\circ - 69.2^\circ & &= 360^\circ - x' \approx 360^\circ - 32.3^\circ \\ &\approx 290.8^\circ & &\approx 327.7^\circ \end{aligned}$$

As stated above we must check the solutions. 1.21 and 5.72 do not check:

$$\tan(1.21) + 2 \sec(1.21) \approx 8.3, \text{ not } 3, \text{ and}$$

$$\tan(5.72) + 2 \sec(5.72) \approx 1.7, \text{ not } 3.$$

Thus the solutions are 5.08 (290.8°) and 0.56 (32.3°).

$$53. \quad \begin{aligned} \cot x &= -\sqrt{3}; \quad \tan x = -\frac{\sqrt{3}}{3} \\ x' &= \tan^{-1} \frac{\sqrt{3}}{3} = \frac{\pi}{6} (30^\circ). \end{aligned}$$

Primary solutions are in qII and qIV: $\frac{5\pi}{6} (150^\circ)$ and $\frac{11\pi}{6} (330^\circ)$.

For all solutions we add $k\pi$ to these. Since the primary solutions differ by $k\pi$ ($k = 1$), we only need mention one primary solution to describe all solutions.

All solutions: $\frac{5\pi}{6} + k\pi$ ($150^\circ + k \cdot 180^\circ$).

$$57. \quad \begin{aligned} \tan x &= 1 \\ x &= \tan^{-1} 1 = \frac{\pi}{4} (45^\circ) \end{aligned}$$

Primary solutions are in qI and qIII: $\frac{\pi}{4} (45^\circ)$ and $\frac{5\pi}{4} (225^\circ)$.

These differ by π (180°), so we can write all solutions with one of them: $\frac{\pi}{4} + k\pi$ ($45^\circ + k \cdot 180^\circ$).

$$61. \quad \begin{aligned} \sin \frac{x}{2} &= \frac{\sqrt{3}}{2} \\ \left(\frac{x}{2}\right)' &= \sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3} (60^\circ) \end{aligned}$$

Primary solutions to $\frac{x}{2}$ are in qI and qII: $\frac{\pi}{3} (60^\circ)$ and $\frac{2\pi}{3} (120^\circ)$.

All solutions:

$$\frac{x}{2} = \frac{\pi}{3} + 2k\pi (60^\circ + k \cdot 360^\circ) \text{ and } \frac{2\pi}{3} + 2k\pi (120^\circ + k \cdot 360^\circ).$$

$$x = \frac{2\pi}{3} + 4k\pi (120^\circ + k \cdot 720^\circ) \text{ and } \frac{4\pi}{3} + 4k\pi (240^\circ + k \cdot 720^\circ).$$

$$65. \quad \begin{aligned} 3 \cot 2x &= \sqrt{3} \\ \cot 2x &= \frac{\sqrt{3}}{3} \\ \tan 2x &= \sqrt{3} \\ (2x)' &= \tan^{-1} \sqrt{3} = \frac{\pi}{3} (60^\circ) \end{aligned}$$

Primary solutions: $2x = \frac{\pi}{3} (60^\circ)$ and $\frac{4\pi}{3} (240^\circ)$

All solutions:

$$2x = \frac{\pi}{3} + k\pi (60^\circ + k \cdot 180^\circ) \text{ and } \frac{4\pi}{3} + k\pi (240^\circ + k \cdot 180^\circ)$$

$$2x = \frac{\pi}{3} + k\pi (60^\circ + k \cdot 180^\circ) \quad \text{The second expression is redundant.}$$

$$x = \frac{\pi}{6} + k \frac{\pi}{2} (30^\circ + k \cdot 90^\circ)$$

$$69. \quad \begin{aligned} 2 \cos 2x + 1 &= 0 \\ \cos 2x &= -\frac{1}{2} \\ (2x)' &= \cos^{-1} \frac{1}{2} = \frac{\pi}{3} (60^\circ) \end{aligned}$$

Primary solutions: $2x = \frac{2\pi}{3} (120^\circ)$ and $\frac{4\pi}{3} (240^\circ)$

All solutions:

$$2x = \frac{2\pi}{3} + 2k\pi (120^\circ + k \cdot 360^\circ) \text{ and } \frac{4\pi}{3} + 2k\pi (240^\circ + k \cdot 360^\circ)$$

$$x = \frac{\pi}{3} + k\pi (60^\circ + k \cdot 180^\circ) \text{ and } \frac{2\pi}{3} + k\pi (120^\circ + k \cdot 180^\circ)$$

$$73. \quad \begin{aligned} 2 \sin 2\theta &= 1 \\ \sin 2\theta &= \frac{1}{2} \\ 2\theta &= \sin^{-1} \frac{1}{2} = \frac{\pi}{6} (30^\circ) \end{aligned}$$

Primary solutions:

$$2\theta = \frac{\pi}{6} + 2k\pi (30^\circ + k \cdot 360^\circ) \text{ and } \frac{5\pi}{6} + 2k\pi (150^\circ + k \cdot 360^\circ)$$

All solutions:

$$\theta = \frac{\pi}{12} + k\pi (15^\circ + k \cdot 180^\circ) \text{ and } \frac{5\pi}{12} + k\pi (75^\circ + k \cdot 180^\circ)$$

$$77. \quad \begin{aligned} \sqrt{3} \tan \frac{\theta}{4} &= 1 \\ \tan \frac{\theta}{4} &= \frac{\sqrt{3}}{3} \end{aligned}$$

$$\left(\frac{\theta}{4}\right)' = \tan^{-1} \frac{\sqrt{3}}{3} = \frac{\pi}{6} (30^\circ)$$

Primary solutions: $\frac{\theta}{4} = \frac{\pi}{6} (30^\circ)$ and $\frac{7\pi}{6} (210^\circ)$

All solutions:

$$\frac{\theta}{4} = \frac{\pi}{6} + k\pi (30^\circ + k \cdot 180^\circ) \text{ and } \frac{7\pi}{6} + k\pi (210^\circ + k \cdot 180^\circ)$$

$$\frac{\theta}{4} = \frac{\pi}{6} + k\pi \quad (30^\circ + k \cdot 180^\circ) \quad \text{The second expression is redundant.}$$

$$\theta = \frac{2\pi}{3} + 4k\pi \quad (120^\circ + k \cdot 720^\circ)$$

$$81. \quad \sin 2\theta + \sin \theta = 0$$

$$2 \sin \theta \cos \theta + \sin \theta = 0$$

$$\sin \theta (2 \cos \theta + 1) = 0$$

$$\sin \theta = 0$$

$$0 (0^\circ), \pi (180^\circ)$$

$$2 \cos \theta + 1 = 0$$

$$\cos \theta = -\frac{1}{2}$$

$$\frac{2\pi}{3} (120^\circ), \frac{4\pi}{3} (240^\circ)$$

$$85. \quad \sin \frac{\theta}{2} = \tan \frac{\theta}{2}$$

$$\sin \frac{\theta}{2} = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}$$

$$\sin \frac{\theta}{2} \cos \frac{\theta}{2} = \sin \frac{\theta}{2}$$

$$\sin \frac{\theta}{2} \cos \frac{\theta}{2} - \sin \frac{\theta}{2} = 0$$

$$\sin \frac{\theta}{2} (\cos \frac{\theta}{2} - 1) = 0$$

$$\sin \frac{\theta}{2} = 0$$

$$\frac{\theta}{2} = 0 (0^\circ), \pi (180^\circ)$$

$$\theta = 0 (0^\circ), 2\pi (360^\circ)$$

$$\text{Primary solutions: } 0 (0^\circ)$$

$$\cos \frac{\theta}{2} = 1$$

$$\frac{\theta}{2} = 0 (0^\circ)$$

$$\theta = 0 (0^\circ)$$

$$89. \quad \tan \frac{\theta}{2} = \cos \theta - 1$$

$$\pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} = \cos \theta - 1$$

$$\left(\pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} \right)^2 = (\cos \theta - 1)^2 \quad \text{All solutions must be checked, since we square both members.}$$

$$\frac{1 - \cos \theta}{1 + \cos \theta} = \cos^2 \theta - 2 \cos \theta + 1$$

$$1 - \cos \theta = (\cos \theta + 1)(\cos^2 \theta - 2 \cos \theta + 1)$$

$$1 - \cos \theta = \cos^3 \theta - \cos^2 \theta - \cos \theta + 1$$

$$\cos^3 \theta - \cos^2 \theta = 0$$

$$\cos^2 \theta (\cos \theta - 1) = 0$$

$$\cos^2 \theta = 0$$

$$\cos \theta = 0$$

$$\frac{\pi}{2} (90^\circ), \frac{3\pi}{2} (270^\circ)$$

$$\cos \theta - 1 = 0$$

$$\cos \theta = 1$$

$$0 (0^\circ)$$

These results must be checked because we squared both members in the first step.

$$\frac{\pi}{2} \text{ does not check: } \tan \frac{\pi}{2} = \cos \frac{\pi}{2} - 1$$

$$\tan \frac{\pi}{4} = \cos \frac{\pi}{2} - 1$$

$$1 = 0 - 1$$

$$1 \neq -1$$

The rest of the solutions check.

Thus the solutions are $0 (0^\circ)$ and $\frac{3\pi}{2} (270^\circ)$.

$$93. \quad \text{If } B = 0.7, x = 2, y = -8, \text{ find } A \text{ to the nearest } 0.01.$$

$$-8 = 2 \cos A \cos 0.7 - 4 \cos A \sin 0.7 - 8 \sin A$$

$$-8 = (2 \cos 0.7) \cos A - (4 \sin 0.7) \cos A - 8 \sin A$$

$$-8 = 1.5297 \cos A - 2.5769 \cos A - 8 \sin A$$

$$-8 = -1.0472 \cos A - 8 \sin A$$

$$8 \sin A - 8 = -1.0472 \cos A$$

$$\sin A - 1 = -0.1309 \cos A \quad \text{Divide each member by 8.}$$

$$\sin A = 1 - 0.1309 \cos A$$

$$(\sin A)^2 = (1 - 0.1309 \cos A)^2$$

$$\sin^2 A = 1 - 0.2618 \cos A + 0.017134 \cos^2 A$$

$$1 - \cos^2 A = 1 - 0.2618 \cos A + 0.017134 \cos^2 A$$

$$0 = 1.0171 \cos^2 A - 0.2618 \cos A$$

$$0 = \cos A (1.0171 \cos A - 0.2618)$$

$$\cos A = 0 \text{ or } 1.0171 \cos A - 0.2618 = 0$$

$$A = \cos^{-1} 0 \text{ or } 1.0171 \cos A = 0.2618$$

$$A = \frac{\pi}{2} \text{ or } \cos A = 0.25739$$

$$A \approx 1.310476103$$

Thus A is $\frac{\pi}{2}$ or 1.31

Chapter 5 Review

$$1. \quad \frac{\cot \theta}{\cos \theta}$$

$$\frac{\cos \theta}{\sin \theta} \cdot \frac{1}{\cos \theta}$$

$$\frac{1}{\sin \theta}$$

$$\csc \theta$$

$$3. \quad \text{Left side:}$$

$$\frac{\tan^2 \theta}{\sec^2 \theta - 1}$$

$$\frac{\tan^2 \theta}{\tan^2 \theta}$$

$$1$$

$$\text{Right side:}$$

$$\sin^4 \theta \csc^4 \theta$$

$$\sin^4 \theta \cdot \frac{1}{\sin^4 \theta}$$

$$1$$

$$5. \quad \frac{\csc \theta \tan \theta}{\sin \theta}$$

$$\csc \theta \tan \theta \cdot \frac{1}{\sin \theta}$$

$$\frac{1}{\sin \theta} \cdot \frac{\sin \theta}{\cos \theta} \cdot \frac{1}{\sin \theta}$$

$$\frac{1}{\cos \theta} \cdot \frac{1}{\sin \theta}$$

$$\csc \theta \sec \theta$$

$$7. \quad \frac{\csc \theta - \sec \theta}{\frac{1}{\sin \theta} - \frac{1}{\cos \theta}}$$

$$\frac{\frac{1}{\sin \theta} - \frac{1}{\cos \theta}}{\frac{\cos \theta - \sin \theta}{\sin \theta \cos \theta}}$$

$$9. \quad \frac{\sec \theta}{\tan \theta - \cot \theta}$$

$$\frac{1}{\cos \theta}$$

$$\frac{\sin \theta}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta}$$

$$\frac{1}{\cos \theta}$$

$$\frac{\sin^2 \theta - \cos^2 \theta}{\cos \theta \sin \theta}$$

$$\frac{1}{\cos \theta} \cdot \frac{\cos \theta \sin \theta}{\sin^2 \theta - \cos^2 \theta}$$

$$\frac{\sin \theta}{\sin^2 \theta - \cos^2 \theta}$$

$$11. \quad \frac{\sec^2 \theta - 1}{\sec^2 \theta}$$

$$\frac{\sec^2 \theta - 1}{\sec^2 \theta} = \frac{1}{\sec^2 \theta}$$

$$\frac{1 - \cos^2 \theta}{\sin^2 \theta}$$

$$13. \quad \frac{\csc x - \tan x \cot x}{\frac{1}{\sin x} - \tan x \cdot \frac{1}{\tan x}}$$

$$\csc x - 1$$

$$15. \quad \frac{\csc^2 x \sec^2 x (\cos^2 x - \sin^2 x)}{\csc^2 x \sec^2 x \cos^2 x - \csc^2 x \sec^2 x \sin^2 x}$$

$$\csc^2 x \cdot \frac{1}{\cos^2 x} \cos^2 x - \frac{1}{\sin^2 x} \sec^2 x \sin^2 x$$

$$\csc^2 x - \sec^2 x$$

$$17. \quad \frac{\sec x (\sin x - \cos x)}{\sec x \sin x - \sec x \cos x}$$

$$\frac{1}{\cos x} \sin x - \frac{1}{\cos x} \cos x$$

$$\tan x - 1$$

$$19. \quad \frac{1}{1 + \csc x} + \frac{1}{1 - \csc x}$$

$$\frac{(1 - \csc x) + (1 + \csc x)}{(1 + \csc x)(1 - \csc x)}$$

$$\frac{2}{1 - \csc^2 x}$$

$$\frac{2}{-(\csc^2 x - 1)}$$

$$\frac{2}{-\cot^2 x}$$

$$-2 \tan^2 x$$

$$21. \quad \frac{\tan^4 x + \tan^2 x}{\tan^2 x (\tan^2 x + 1)}$$

$$\tan^2 x \cdot \sec^2 x$$

$$\frac{1}{\cot^2 x} \cdot \sec^2 x$$

Alternate solution in text

$$23. \sin \theta + \cos \theta = 1$$

Let $\theta = \frac{\pi}{4}$: $\sin \frac{\pi}{4} + \cos \frac{\pi}{4} =$

$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}; \sqrt{2} \neq 1$$

$$25. \frac{1}{\cot \theta - \csc \theta} = \sec \theta$$

Let $\theta = \frac{\pi}{4}$: $\frac{1}{1 - \sqrt{2}} \neq \sqrt{2}$

$$27. \tan\left(-\frac{\pi}{12}\right) = -\tan \frac{\pi}{12}$$

$$= -\tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= -\frac{\tan \frac{\pi}{4} - \tan \frac{\pi}{6}}{1 + \tan \frac{\pi}{4} \tan \frac{\pi}{6}}$$

$$= -\frac{1 - \frac{\sqrt{3}}{3}}{1 + 1\left(\frac{\sqrt{3}}{3}\right)} \cdot \frac{3}{3}$$

$$33. \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta + \sin \alpha \sin \beta} =$$

$$\frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\cos \alpha \sin \beta}{\sin \alpha \sin \beta} =$$

$$\frac{\cos \alpha \cos \beta}{\sin \alpha \sin \beta} + \frac{\sin \alpha \sin \beta}{\sin \alpha \sin \beta} =$$

$$\frac{\cot \beta + \cot \alpha}{\cot \alpha \cot \beta + 1}$$

$$35. \sin\left(\frac{\pi}{4} - \theta\right) = \sin \frac{\pi}{4} \cos \theta - \cos \frac{\pi}{4} \sin \theta$$

$$= \frac{\sqrt{2}}{2} \cos \theta - \frac{\sqrt{2}}{2} \sin \theta = \frac{\sqrt{2}}{2} (\cos \theta - \sin \theta)$$

$$37. \tan(\theta + \pi) = \frac{\tan \theta + \tan \pi}{1 - \tan \theta \tan \pi}$$

$$= \frac{\tan \theta + 0}{1 - 0 \cdot \tan \theta} = \tan \theta$$

$$\cot(\theta + \pi) = \frac{1}{\tan(\theta + \pi)} = \frac{1}{\tan \theta} = \cot \theta$$

$$39. \sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\sin \theta = 2 \sin 5\pi \cos 5\pi; \alpha = 5\pi \text{ so } 2\alpha = \theta = 10\pi$$

$$41. \cos 2\beta = 1 - 2 \sin^2 \beta$$

$$\cos 24^\circ = 1 - 2 \sin^2 \theta; 2\beta = 24^\circ \text{ so } \beta = \theta = 12^\circ$$

$$43. \tan 2a = \frac{2 \tan a}{1 - \tan^2 a}$$

$$2 \tan 2a = \frac{4 \tan a}{1 - \tan^2 a}; a = 4\theta \text{ so } 2a = 8\theta;$$

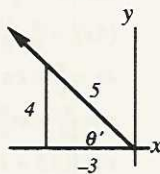
$$2 \tan 8\theta = \frac{4 \tan 4\theta}{1 - \tan^2 4\theta}$$

The result is $2 \tan 8\theta$.

$$45. (a) \cos 2\theta = 2 \cos^2 \theta - 1 = 2\left(-\frac{3}{5}\right)^2 - 1 = \frac{18}{25} - 1 = -\frac{7}{25}$$

$$(b) \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2\left(-\frac{4}{3}\right)}{1 - \left(-\frac{4}{3}\right)^2}$$

$$= \frac{-\frac{8}{3}}{1 - \frac{16}{9}} \cdot \frac{9}{9} = \frac{-24}{9 - 16} = \frac{24}{7}$$



$$47. \sin 2x - \cos x = \cos x(2 \sin x - 1)$$

$$2 \sin x \cos x - \cos x$$

$$\cos x(2 \sin x - 1)$$

$$= \frac{3 - \sqrt{3}}{3 + \sqrt{3}} \cdot \frac{3 - \sqrt{3}}{3 - \sqrt{3}}$$

$$= \frac{12 - 6\sqrt{3}}{6} = -2 + \sqrt{3}$$

$$29. \tan(-15^\circ) = -\tan 15^\circ = -\tan(45^\circ - 30^\circ)$$

$$= -\frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$$

$$= -\frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \cdot \frac{1}{\sqrt{3}}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

$$= -\frac{\sqrt{3} - 1}{\sqrt{3} + 1} \cdot \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= -\frac{3 - 2\sqrt{3} + 1}{3 - 1} =$$

$$= -\frac{4 - 2\sqrt{3}}{2} = -2 + \sqrt{3}$$

$$31. a. \cos(\alpha + \beta) =$$

$$\cos \alpha \cos \beta - \sin \alpha \sin \beta =$$

$$= \frac{\sqrt{3}}{2} \left(-\frac{\sqrt{15}}{4}\right) - \frac{1}{2} \left(-\frac{1}{4}\right) =$$

$$\frac{3\sqrt{3}}{8} + \frac{1}{8} = \frac{3\sqrt{3} + 1}{8}$$

$$b. \tan(\alpha - \beta) =$$

$$\frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} =$$

$$49. \frac{\cos 2x}{2 - 4 \sin^2 x}$$

$$\frac{\cos 2x}{\cos 2x}$$

$$\frac{2(1 - 2 \sin^2 x)}{\cos 2x} = \frac{1}{2}$$

$$51. \sin 2x - \cos 2x$$

$$2 \sin x \cos x - (2 \cos^2 x - 1)$$

$$2 \sin x \cos x - 2 \cos^2 x + 1$$

$$2 \cos x(\sin x - \cos x) + 1$$

$$53. \cos(-15^\circ) = \cos 15^\circ \text{ (cosine is an even function)}$$

$$= \cos \frac{30^\circ}{2} = \sqrt{\frac{1 + \cos 30^\circ}{2}} = \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} =$$

$$\sqrt{\frac{2 + \sqrt{3}}{4}} = \frac{\sqrt{2 + \sqrt{3}}}{2}$$

$$55. \cos \frac{3\pi}{8} = \cos \frac{3\pi}{4} = \sqrt{\frac{1 + \cos \frac{3\pi}{4}}{2}} = \sqrt{\frac{1 + (-\frac{\sqrt{2}}{2})}{2}}$$

$$= \sqrt{\frac{2 - \sqrt{2}}{4}} = \frac{\sqrt{2 - \sqrt{2}}}{2}$$

$$57. \text{ Given } \sin x = -\frac{2}{3}, \pi < x < \frac{3\pi}{2}. \text{ Find (a) } \cos \frac{x}{2}; \text{ (b) } \tan \frac{x}{2}.$$

$\pi < x < \frac{3\pi}{2}$, so x terminates in quadrant III; also, $\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$, so $\frac{x}{2}$ terminates in quadrant II, where cosine is negative and tangent is negative.

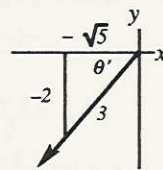
$$\cos \frac{x}{2} = -\sqrt{\frac{1 + \cos x}{2}} = -\sqrt{\frac{1 + (-\frac{2}{3})}{2}} = -\sqrt{\frac{3 - 2}{6}}$$

$$\sec \frac{x}{2} = -\sqrt{\frac{6}{3 - 2}}$$

$$\tan \frac{x}{2} = \frac{1 - \cos \theta}{\sin \theta} = \frac{1 - (-\frac{2}{3})}{-\frac{2}{3}} \cdot \frac{3}{3}$$

$$= \frac{3 + \sqrt{5}}{-2} = -\frac{3 + \sqrt{5}}{2}$$

$$\cot \frac{x}{2} = -\frac{2}{3 + \sqrt{5}} \cdot \frac{3 - \sqrt{5}}{3 - \sqrt{5}} = \frac{-3 + \sqrt{5}}{2}$$



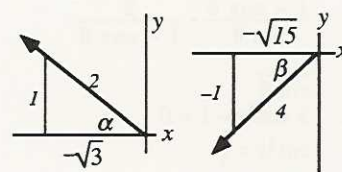
$$= \frac{-\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{15}}}{1 + (-\frac{1}{\sqrt{3}})(\frac{1}{\sqrt{15}})} \cdot \frac{\sqrt{45}}{\sqrt{45}} =$$

$$\frac{-\sqrt{15} - \sqrt{3}}{\sqrt{45} - 1} = \frac{-\sqrt{15} - \sqrt{3}}{3\sqrt{5} - 1} \cdot \frac{3\sqrt{5} + 1}{3\sqrt{5} + 1} =$$

$$\frac{-3\sqrt{75} - \sqrt{15} - 3\sqrt{15} - \sqrt{3}}{45 - 1} =$$

$$\frac{-15\sqrt{3} - 4\sqrt{15} - \sqrt{3}}{44} = \frac{-16\sqrt{3} - 4\sqrt{15}}{44}$$

$$= \frac{-4\sqrt{3} - \sqrt{15}}{11}$$



$$59. \cot \frac{\theta}{2} = \frac{1}{\tan \frac{\theta}{2}} = \frac{1}{\frac{\sin \theta}{1 + \cos \theta}} = \frac{1 + \cos \theta}{\sin \theta}$$

$$61. \tan \frac{\theta}{2} \csc^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta} \cdot \frac{1}{\sin^2 \frac{\theta}{2}} = \frac{1 - \cos \theta}{\sin \theta} \cdot \frac{1}{\frac{1 - \cos \theta}{2}} = \frac{1 - \cos \theta}{\sin \theta} \cdot \frac{2}{1 - \cos \theta} = \frac{2}{\sin \theta}$$

$$63. 3 \cot^2 x - 1 = 0$$

$$\cot^2 x = \frac{1}{3}$$

$$\cot x = \pm \frac{1}{\sqrt{3}}$$

$$\tan x = \pm \sqrt{3}$$

$$x' = \tan^{-1} \sqrt{3} = \frac{\pi}{3}$$

$$x = \frac{\pi}{3}$$

$$65. (4 \sin^2 x - 1)(\sec x - 2) = 0$$

$$4 \sin^2 x - 1 = 0$$

$$4 \sin^2 x = 1$$

$$\sin^2 x = \frac{1}{4}$$

$$\sin x = \pm \frac{1}{2}$$

$$\frac{\pi}{6}$$

$$67. \sec^2 \theta - 4 = 0$$

$$\sec^2 \theta = 4$$

$$\sec \theta = \pm 2$$

$$\cos \theta = \pm \frac{1}{2}$$

$$60^\circ, 120^\circ, 240^\circ, 300^\circ$$

$$69. 2 \sin \theta - \csc \theta + 1 = 0$$

$$2 \sin \theta - \frac{1}{\sin \theta} + 1 = 0$$

$$2 \sin^2 \theta - 1 + \sin \theta = 0$$

$$2 \sin^2 \theta + \sin \theta - 1 = 0$$

$$(2 \sin \theta - 1)(\sin \theta + 1) = 0$$

$$\sin \theta = \frac{1}{2}$$

$$30^\circ, 150^\circ$$

$$71. 2 \sin^2 \theta - 3 \cos \theta = 3$$

$$2(1 - \cos^2 \theta) - 3 \cos \theta = 3$$

$$2 - 2 \cos^2 \theta - 3 \cos \theta = 3$$

$$2 \cos^2 \theta + 3 \cos \theta + 1 = 0$$

$$(2 \cos \theta + 1)(\cos \theta + 1) = 0$$

$$2 \cos \theta + 1 = 0$$

$$\cos \theta = -\frac{1}{2}$$

$$120^\circ, 240^\circ$$

$$\sec x - 2 = 0$$

$$\sec x = 2$$

$$\cos x = \frac{1}{2}$$

$$\frac{\pi}{3}$$

$$\sin \theta = -1$$

$$270^\circ$$

$$\cos \theta + 1 = 0$$

$$\cos \theta = -1$$

$$180^\circ$$

$$\text{If } ax^2 + bx + c = 0 \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$73. \tan^2 x - 3 \tan x - 3 = 0; a = 1, b = -3, c = -3$$

$$\tan x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-3)}}{2(1)} = \frac{3 \pm \sqrt{9 - (-12)}}{2}$$

$$= \frac{3 \pm \sqrt{21}}{2}$$

$$x = \tan^{-1} \frac{3 + \sqrt{21}}{2} \approx -0.669$$

$$x \approx -0.67 + k\pi \text{ or } 1.31 + k\pi$$

$$x = \tan^{-1} \frac{3 - \sqrt{21}}{2} \approx 1.31$$

$$75. \sec^2 x - \sec x = 2$$

$$\sec^2 x - \sec x - 2 = 0$$

$$(\sec x - 2)(\sec x + 1) = 0$$

$$\sec x - 2 = 0$$

$$\sec x = 2$$

$$\cos x = \frac{1}{2}$$

$$x' = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$x = \frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi, \pi + 2k\pi$$

$$\sec x + 1 = 0$$

$$\sec x = -1$$

$$\cos x = -1$$

$$x' = \pi$$

$$77. 2 \cos \frac{x}{2} - \sqrt{3} = 0$$

$$\cos \frac{x}{2} = \frac{\sqrt{3}}{2}$$

$$\frac{x}{2} = \cos^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{6} (30^\circ)$$

Since we will be doubling final solutions, there is no need to add multiples of 2π to primary solutions. (Otherwise, doubling will produce values greater than 4π , which are not primary solutions.)

$$\frac{x}{2} \text{ in quadrant I: } \frac{x}{2} = \frac{\pi}{6} (30^\circ), \text{ so } x = \frac{\pi}{3} (60^\circ);$$

$$\frac{x}{2} \text{ in quadrant IV: } \frac{x}{2} = \frac{11\pi}{6} (330^\circ), \text{ so } x = \frac{11\pi}{3} (660^\circ).$$

Discarding the solution greater than 2π , we obtain $x = \frac{\pi}{3} (60^\circ)$.

$$79. \sin^2 \frac{x}{4} = \frac{1}{2}$$

$$\sin \frac{x}{4} = \pm \frac{\sqrt{2}}{2}$$

$$\left(\frac{x}{4}\right)' = \sin^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4} (45^\circ)$$

$\frac{x}{4} = \frac{\pi}{4} (45^\circ), \frac{3\pi}{4} (135^\circ), \frac{5\pi}{4} (225^\circ), \frac{7\pi}{4} (315^\circ)$, so, multiplying by 4 and discarding non-primary solutions, $x = \pi (180^\circ)$.

$$81. 3 \tan^2 \frac{\theta}{4} = 9$$

$$\tan^2 \frac{\theta}{4} = 3$$

$$\tan \frac{\theta}{4} = \pm \sqrt{3}$$

$$\left(\frac{\theta}{4}\right)' = \tan^{-1} \sqrt{3} = \frac{\pi}{3} (60^\circ)$$

$\frac{\theta}{4} = \frac{\pi}{3} (60^\circ), \frac{2\pi}{3} (120^\circ), \frac{4\pi}{3} (240^\circ), \frac{5\pi}{3} (300^\circ)$. Multiply by 4, discard values greater than 2π (360°): $\theta = \frac{4\pi}{3} (240^\circ)$

$$83. \cos \theta + \sin 2\theta = 0$$

$$\cos \theta + 2 \sin \theta \cos \theta = 0$$

$$\cos \theta (1 + 2 \sin \theta) = 0$$

$$\cos \theta = 0$$

$$\frac{\pi}{2} (90^\circ), \frac{3\pi}{2} (270^\circ)$$

$$\sin \theta = -\frac{1}{2}$$

$$\frac{7\pi}{6} (210^\circ), \frac{11\pi}{6} (330^\circ)$$

$$85. (\cot 4x - \sqrt{3})(\csc 3x + 2) = 0$$

$$\cot 4x - \sqrt{3} = 0$$

$$\cot 4x = \sqrt{3}$$

$$\tan 4x = \frac{\sqrt{3}}{3}$$

$$(4x)' = \frac{\pi}{6}, \frac{7\pi}{6}$$

$$4x = \frac{\pi}{6} + k\pi$$

$$x = \frac{\pi}{24} + k\frac{\pi}{4} \text{ or } \frac{7\pi}{18} + k\frac{2\pi}{3} \text{ or } \frac{11\pi}{18} + k\frac{2\pi}{3}$$

$$x \approx 0.13 + k\frac{\pi}{4} \text{ or } 1.22 + k\frac{\pi}{4} \text{ or } 1.92 + k\frac{\pi}{4}$$

$$\csc 3x + 2 = 0$$

$$\csc 3x = -2$$

$$\sin 3x = -\frac{1}{2}$$

$$(3x)' = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$3x = \frac{7\pi}{6} + 2k\pi, \frac{11\pi}{6} + 2k\pi$$

Chapter 5 Test

1. $\csc^2 x \sin x \cos x$

$$\frac{1}{\sin^2 x} \sin x \cos x$$

$$\frac{1}{\sin x} \cos x$$

$$\frac{\cos x}{\sin x}$$

$$\cot x$$

3. $\cot x - 2 \tan x$

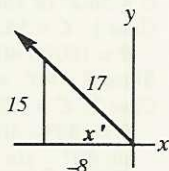
$$\cot \frac{\pi}{4} - 2 \tan \frac{\pi}{4}$$

$$1 - 2(1)$$

$$-1 \neq 1$$

5. $\sin 2x = 2 \sin x \cos x$

$$= 2 \frac{15}{17} \left(-\frac{8}{17} \right) = -\frac{240}{289}$$



7. $\cos 22.5^\circ = \cos \frac{45^\circ}{2} = \sqrt{\frac{1 + \cos 45^\circ}{2}} = \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}}$

$$= \sqrt{\frac{2 + \sqrt{2}}{4}} = \frac{\sqrt{2 + \sqrt{2}}}{2} \cdot \sec 22.5^\circ = \frac{1}{\cos 22.5^\circ}$$

$$= \frac{2}{\sqrt{2 + \sqrt{2}}} \cdot \frac{\sqrt{2 - \sqrt{2}}}{\sqrt{2 - \sqrt{2}}} = \frac{2\sqrt{2 - \sqrt{2}}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2(2 - \sqrt{2})}}{2}$$

$$= \sqrt{4 - 2\sqrt{2}}. \text{ (Note: } \frac{2}{\sqrt{2 + \sqrt{2}}} \text{ given in the text)}$$

9. $4 \cos^2 x = 1$

$$\cos^2 x = \frac{1}{4}$$

$$\cos x = \pm \frac{1}{2}$$

$$x' = \cos^{-1} \frac{1}{2} = \frac{\pi}{3} (60^\circ)$$

$$x = x', \pi - x', \pi + x', 2\pi - x'$$

$$= \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

11. $(6 \sin x - 1)(\sin x + 1) = 0$

$$6 \sin x - 1 = 0$$

$$\sin x = \frac{1}{6}$$

$$x' = \sin^{-1} \frac{1}{6} \approx 0.167$$

$$x = x' \text{ or } \pi - x'$$

$$= 0.17 \text{ or } 2.97$$

$$\sin x + 1 = 0$$

$$\sin x = -1$$

$$x = \frac{3\pi}{2}$$

The solutions are $x \approx 0.17 + 2k\pi, 2.97 + 2k\pi$, or $\frac{3\pi}{2} + 2k\pi$.

13. $\sec \frac{2x}{4} = 2$

$$\cos \frac{2x}{4} = \frac{1}{2}$$

$$\cos \frac{x}{4} = \pm \frac{\sqrt{2}}{2}$$

$$\left(\frac{x}{4}\right)' = \frac{\pi}{4}$$

Primary solutions for $\frac{x}{4}$

are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

$\frac{x}{4}$	x (all solutions)
$\frac{\pi}{4} + 2k\pi$	$\pi + 8k\pi$
$\frac{3\pi}{4} + 2k\pi$	$3\pi + 8k\pi$
$\frac{5\pi}{4} + 2k\pi$	$5\pi + 8k\pi$
$\frac{7\pi}{4} + 2k\pi$	$7\pi + 8k\pi$

Exercise 6-1

- 1.
- $a = 12.5$
- ,
- $A = 35^\circ$
- ,
- $B = 49^\circ$

$$C = 180^\circ - A - B = 96^\circ$$

$$\frac{\sin 35^\circ}{12.5} = \frac{\sin 49^\circ}{b} = \frac{\sin 96^\circ}{c}$$

$$\frac{\sin 35^\circ}{12.5} = \frac{\sin 49^\circ}{b}$$

$$b = \frac{12.5 \sin 49^\circ}{\sin 35^\circ}$$

$$b \approx 16.4$$

Keystrokes for b:

$$12.5 \text{ [X]} 49 \text{ [sin]} \text{ [+]} 35 \text{ [sin]} \text{ [=]}$$

$$\text{TI-81: } 12.5 \text{ [sin]} 49 \text{ [+]} \text{ [sin]} 35 \text{ [ENTER]}$$

- 5.
- $b = 92.5$
- ,
- $A = 47^\circ$
- ,
- $B = 100^\circ$

$$C = 180^\circ - 47^\circ - 100^\circ = 33^\circ$$

$$\frac{\sin 47^\circ}{a} = \frac{\sin 100^\circ}{92.5} = \frac{\sin 33^\circ}{c}$$

$$\frac{\sin 100^\circ}{92.5} = \frac{\sin 33^\circ}{c}$$

$$c = \frac{92.5 \sin 33^\circ}{\sin 100^\circ}$$

$$c \approx 51.2^\circ$$

- 9.
- $c = 5.00$
- ,
- $A = 100^\circ$
- ,
- $B = 45^\circ$

$$C = 180^\circ - 100^\circ - 45^\circ = 35^\circ$$

$$\frac{\sin 100^\circ}{a} = \frac{\sin 45^\circ}{b} = \frac{\sin 35^\circ}{5}$$

$$\frac{\sin 100^\circ}{a} = \frac{\sin 35^\circ}{5}$$

$$a = \frac{5 \sin 100^\circ}{\sin 35^\circ}$$

$$a \approx 8.58$$

- 13.
- $a = 4.25$
- ,
- $c = 2.86$
- ,
- $A = 132^\circ$

$$\frac{\sin 132^\circ}{4.25} = \frac{\sin B}{b} = \frac{\sin C}{2.86}$$

$$\frac{\sin 132^\circ}{4.25} = \frac{\sin C}{2.86} \text{ so } \sin C = \frac{2.86 \sin 132^\circ}{4.25}$$

$$C' = \sin^{-1} \frac{2.86 \sin 132^\circ}{4.25} \approx 30.01^\circ$$

$$C \approx 30.01^\circ \text{ or } 180^\circ - 30.01^\circ \approx 149.99^\circ$$

Case 1: $C \approx 30.01^\circ$

$$B \approx 180^\circ - 132^\circ - 30.01^\circ \approx 17.99^\circ$$

$$\frac{\sin 132^\circ}{4.25} = \frac{\sin 17.99^\circ}{b}$$

$$b \approx 1.77$$

Thus, the solution is $B \approx 18.0^\circ$, $C \approx 30.0^\circ$, $b \approx 1.77$.Case 2: $C \approx 149.99^\circ$

$$B = 180^\circ - 132^\circ - 149.99^\circ \approx -101.99^\circ \text{ (No solution.)}$$

- 17.
- $a = 4$
- ,
- $b = 22$
- ,
- $A = 30^\circ$

$$\frac{\sin 30^\circ}{4} = \frac{\sin B}{22} = \frac{\sin C}{c}$$

$$\frac{\sin 30^\circ}{4} = \frac{\sin B}{22} \text{ so } \sin B = \frac{22 \sin 30^\circ}{4}$$

$$B' = \sin^{-1} \frac{22 \sin 30^\circ}{4} = \sin^{-1} 2.75$$

 B' is undefined, since $0 < \sin B < 1$, and we have $\sin B = 2.75$ from this data.

Thus, there is no solution possible.

- 21.
- $c = 5.00$
- ,
- $b = 8.00$
- ,
- $B = 45.0^\circ$

$$\frac{\sin A}{a} = \frac{\sin 45^\circ}{8} = \frac{\sin C}{5}$$

$$\frac{\sin 45^\circ}{8} = \frac{\sin C}{5} \text{ so } \sin C = \frac{5 \sin 45^\circ}{8}$$

$$C' = \sin^{-1} \frac{5 \sin 45^\circ}{8} \approx 26.23^\circ$$

$$C \approx 26.23^\circ \text{ or } 180^\circ - 26.23^\circ \approx 153.77^\circ$$

Case 1: $C \approx 26.23^\circ$

$$A \approx 180^\circ - 45^\circ - 26.23^\circ \approx 108.77^\circ$$

$$\frac{\sin 108.77^\circ}{a} = \frac{\sin 45^\circ}{8}$$

$$a \approx 10.71$$

Solution: $C \approx 26.2^\circ$, $A \approx 108.77^\circ$, $a \approx 10.71$ Case 2: $C \approx 153.77^\circ$

$$A \approx 180^\circ - 45^\circ - 153.77^\circ \approx -18.77^\circ$$

No solution.

25. The acute angle at the asteroid is
- $180^\circ - 81.5^\circ - 88^\circ = 10.5^\circ$
- .

$$\frac{\sin 10.5^\circ}{153,800} = \frac{\sin 88^\circ}{d}; d \approx 843,449 \approx 843,400 \text{ miles.}$$

- 29.
- $\frac{\sin 40^\circ}{58} = \frac{\sin C}{75}$
- ;
- $\sin C = \frac{75 \sin 40^\circ}{58}$
- ;
- $C' \approx 56.2^\circ$

$$C \approx 56.2^\circ \text{ or } 180^\circ - 56.2^\circ \approx 123.8^\circ$$

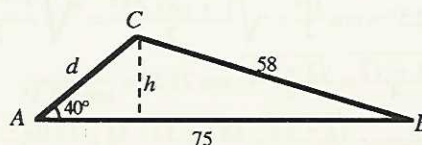
Case 1: $C \approx 56.2^\circ$

$$B = 180^\circ - 40^\circ - 56.2^\circ \approx 83.8^\circ$$

Since $B < 40^\circ$, we solve the second case.Case 2: $C \approx 123.8^\circ$

$$B = 180^\circ - 40^\circ - 123.8^\circ \approx 16.2^\circ$$

$$\frac{\sin 40^\circ}{58} = \frac{\sin 16.2^\circ}{d}; d \approx 25.17$$

Thus $d \approx 25$ miles.

33. Let
- (x, y)
- be the point at
- B
- . It is on the terminal side of angle

 A . Then $\cos A = \frac{x}{r}$, where r is the length of AB . But $r = c$,so $\cos A = \frac{x}{c}$ and so $c \cos A = x$.Using right triangles we see that in each figure $\cos C =$ $\frac{b-x}{a}$ so that $a \cos C = b - x$. Note that when A is obtuse(the right hand figure) x is negative, so $b - x$ is the length of $|b| + |x|$.

$$c \cos A = x$$

$$a \cos C = b - x$$

$$a \cos C + c \cos A = x + (b - x) \quad \text{Add}$$

$$a \cos C + c \cos A = b$$

Thus (2) is true.

(1) and (3) can be shown true by putting angles B and C in standard position and proceeding in the same manner. In fact this is not really necessary, since the labelling in a triangle is arbitrary, and thus, for example, we could obtain (1) by changing the label B to A , C to B , and A to C , and labelling the sides appropriately. (Also see the discussion in the text).

37. (a) It can be seen that the sum of the area of the four triangles shown in the figure is

$$\frac{1}{2} ab \sin A + \frac{1}{2} cd \sin C + \frac{1}{2} ad \sin D + \frac{1}{2} bc \sin B.$$

This total is twice as large as the total area of the four-sided figure, so the area of the four-sided figure is

$$\frac{1}{2} \left(\frac{1}{2} ab \sin A + \frac{1}{2} cd \sin C + \frac{1}{2} ad \sin D + \frac{1}{2} bc \sin B \right)$$

$$\text{or } \frac{1}{4} (ab \sin A + ad \sin D + bc \sin B + cd \sin C).$$

(b) The difference between the Egyptian formula

$$\frac{1}{4} (ab + ad + bc + cd)$$
 and the correct formula

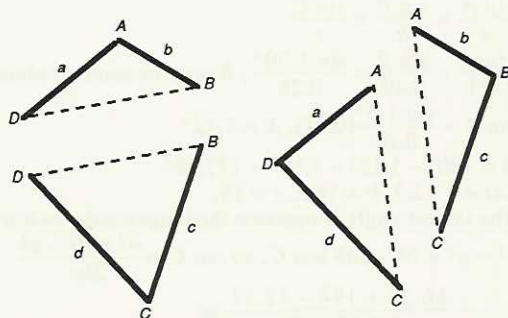
 $\frac{1}{4} (ab \sin A + ad \sin D + bc \sin B + cd \sin C)$ is the factors $\sin A$, $\sin B$, $\sin C$, and $\sin D$. Since we assumed each angle is between 0° and 180° the value of the sine of each angle is

between 0 and 1.

$$\begin{aligned}\text{Thus, } ab &\geq ab \sin A \\ ad &\geq ad \sin D \\ bc &\geq bc \sin B \\ cd &\geq cd \sin C\end{aligned}$$

$$\begin{aligned}ab + ad + bc + cd &\geq ab \sin A + ad \sin D + bc \sin B + cd \sin C, \\ \frac{1}{4}(ab + ad + bc + cd) &\geq \frac{1}{4}(ab \sin A + ad \sin D + bc \sin B + cd \sin C).\end{aligned}$$

If the figure is a rectangle $A = B = C = D = 90^\circ$, and $\sin A = \sin B = \sin C = \sin D = 1$, so both expressions give the same value.



Exercise 6-2

1. $(3, -4), (-2, 6): d = \sqrt{(-2 - 3)^2 + (6 - (-4))^2} = \sqrt{125} = 5\sqrt{5}$

5. $(0, -3), (-9, -12): d = \sqrt{(-9 - 0)^2 + (-12 - (-3))^2} = 9\sqrt{2}$

9. $b = 61.3, c = 23.9, A = 124.0^\circ$
 $a^2 = b^2 + c^2 - 2bc \cos A$
 $a^2 = 61.3^2 + 23.9^2 - 2(61.3)(23.9) \cos 124^\circ$
 $a = \sqrt{61.3^2 + 23.9^2 - 2(61.3)(23.9) \cos 124^\circ} \approx 77.249$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin 124^\circ}{77.249} = \frac{\sin B}{61.3} = \frac{\sin C}{23.9} \quad \text{Find angle } C \text{ first; it is the smallest and therefore acute.}$$

$$\sin C \approx \frac{23.9 \sin 124^\circ}{77.249}; C \approx 14.9^\circ$$

$$B \approx 180^\circ - 14.9^\circ - 124^\circ \approx 41.1^\circ$$

$$\text{Thus } a \approx 77.2, B \approx 41.1^\circ, C \approx 14.9^\circ.$$

13. $a = 23.5, b = 19.4, c = 35.0$

$$c^2 = a^2 + b^2 - 2ab \cos C, \text{ so } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{23.5^2 + 19.4^2 - 35^2}{2(23.5)(19.4)} \text{ so } C \approx 108.97^\circ \approx 109.0^\circ$$

$$23.5 \boxed{x^2} \boxed{+} 19.4 \boxed{x^2} \boxed{-} 35 \boxed{x^2} \boxed{=} \boxed{\div} \boxed{2} \boxed{\div} 23.5 \boxed{\div} 19.4 \boxed{=} \boxed{\text{SHIFT}} \boxed{\cos}$$

$$\boxed{\text{TI-81}} \boxed{2\text{nd}} \boxed{\cos} \boxed{(} 23.5 \boxed{x^2} \boxed{+} 19.4 \boxed{x^2} \boxed{-} 35 \boxed{x^2} \boxed{)} \boxed{\div} \boxed{2} \boxed{\div} 23.5 \boxed{\div} 19.4 \boxed{\text{ENTER}}$$

Since C is the largest angle in the triangle we know A and B are acute.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin A}{23.5} = \frac{\sin B}{19.4} = \frac{\sin 108.97^\circ}{35}$$

$$\sin A = \frac{\sin 108.97^\circ}{35} (23.5); A \approx 39.4^\circ; B \approx 180^\circ - 109.0^\circ - 39.4^\circ \approx 31.6^\circ$$

17. $a = 13.2, b = 5.9, C = 139.4^\circ$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$c^2 = 13.2^2 + 5.9^2 - 2(13.2)(5.9) \cos 139.4^\circ$$

$$c = \sqrt{13.2^2 + 5.9^2 - 2(13.2)(5.9) \cos 139.4^\circ} \approx 18.09 \approx 18.1$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin A}{13.2} = \frac{\sin B}{5.9} = \frac{\sin 139.4^\circ}{18.09}; \sin A \approx \frac{\sin 139.4^\circ}{18.09} (13.2), \text{ so } A \approx 28.3^\circ (A \text{ is acute.})$$

$$B \approx 180^\circ - 139.4^\circ - 28.3^\circ \approx 12.3^\circ.$$

21. $a = 30.0, c = 20.0, B = 112.0^\circ$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$b^2 = 30^2 + 20^2 - 2(30)(20) \cos 112^\circ$$

$$b \approx 41.83 \approx 41.8$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin A}{30} = \frac{\sin 112^\circ}{41.8} = \frac{\sin C}{20}; A \text{ and } C \text{ must both be acute.}$$

$$\sin C \approx \frac{\sin 112^\circ}{41.8} (20), \text{ so } C \approx 26.3^\circ$$

$$A \approx 180^\circ - 26.3^\circ - 112^\circ \approx 41.7^\circ$$

25. $a = 0.21, b = 0.49, C = 1.50^\circ$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$c^2 = 0.21^2 + 0.49^2 - 2(0.21)(0.49) \cos 1.50^\circ, \text{ so } c \approx 0.280$$

$$\approx 0.28$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin A}{0.21} = \frac{\sin B}{0.49} = \frac{\sin 1.50^\circ}{0.28}; B \text{ may be acute or obtuse, so find } A \text{ first.}$$

$$\sin A \approx \frac{\sin 1.50^\circ}{0.28}(0.21), A \approx 1.12^\circ$$

$$B \approx 180^\circ - 1.12^\circ - 1.50^\circ \approx 177.38^\circ.$$

29. Let $a = 12.3$, $b = 16.2$, $c = 19$.

The largest angle is opposite the longest side, so it is angle C .

$$c^2 = a^2 + b^2 - 2ab \cos C, \text{ so } \cos C = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{16.2^2 + 19^2 - 12.3^2}{2(16.2)(19)}, \text{ so}$$

$$C \approx 82.4^\circ, \text{ to the nearest tenth.}$$

33. $31.5^2 = 17.6^2 + 22.5^2 - 2(17.6)(22.5) \cos \theta^\circ$

$$\cos \theta = \frac{17.6^2 + 22.5^2 - 31.5^2}{2(17.6)(22.5)}; \theta \approx 102.9^\circ$$

37. Angle A is the largest. $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$$= \frac{53 + 34 - 65}{2\sqrt{53}(\sqrt{34})} \approx 0.2591, A \approx 75.0^\circ.$$

Exercise 6-3

We use $A = (|A|, \theta_A) = (|A| \cos \theta_A, |A| \sin \theta_A)$.

1. $(40, 30^\circ) = (40 \cos 30^\circ, 40 \sin 30^\circ) = (40 \cdot \frac{\sqrt{3}}{2}, 40 \cdot \frac{1}{2}) = (20\sqrt{3}, 20)$.

5. $(10.0, 200.0^\circ) = (10 \cos 200^\circ, 10 \sin 200^\circ) \approx (-9.4, -3.4)$

9. $(6, -45^\circ) = (6 \cos(-45^\circ), 6 \sin(-45^\circ)) \approx$
 $(6 \cos 45^\circ, -6 \sin 45^\circ)$ cosine is an even function,
sine is an odd function.

$$= (6 \cdot \frac{\sqrt{2}}{2}, -6 \cdot \frac{\sqrt{2}}{2}) = (3\sqrt{2}, -3\sqrt{2}).$$

29. $(30.0, 30^\circ) = (30 \cos 30^\circ, 30 \sin 30^\circ) = (25.980, 15.000)$
 $(15.2, 33.6^\circ) = (15.2 \cos 33.6^\circ, 15.2 \sin 33.6^\circ) = (12.660, 8.412)$

$$(38.641, 23.412) = (45.2, 31.2^\circ)$$

33. $(3.2, -45.0^\circ) = (3.2 \cos(-45^\circ), 3.2 \sin(-45^\circ)) = (2.263, -2.263)$

$$(5.9, -59.2^\circ) = (5.9 \cos(-59.2^\circ), 5.9 \sin(-59.2^\circ)) = (3.021, -5.068)$$

$$(5.284, -7.331) = (9.0, -54.2^\circ)$$

37. $(3.5, 19.2^\circ) = (3.5 \cos 19.2^\circ, 3.5 \sin 19.2^\circ) = (3.305, 1.151)$

$$(2.7, 83.1^\circ) = (2.7 \cos 83.1^\circ, 2.7 \sin 83.1^\circ) = (0.324, 2.680)$$

$$(4.3, 145.7^\circ) = (4.3 \cos 145.7^\circ, 4.3 \sin 145.7^\circ) = (-3.552, 2.423)$$

$$(0.077, 6.255) = (6.3, 89.3^\circ)$$

41. $(3.5, -25^\circ) = (3.5 \cos(-25^\circ), 3.5 \sin(-25^\circ)) = (3.172, -1.479)$

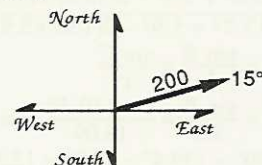
$$(6.8, 25^\circ) = (6.8 \cos 25^\circ, 6.8 \sin 25^\circ) = (6.163, 2.874)$$

$$(4.2, 50^\circ) = (4.2 \cos 50^\circ, 4.2 \sin 50^\circ) = (2.700, 3.217)$$

$$(12.035, 4.612) = (12.9, 21.0^\circ)$$

45. $V = (200, 15^\circ); V_x = 200 \cos 15^\circ \approx 193; V_y = 200 \sin 15^\circ \approx 52$.

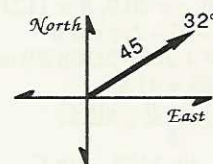
The aircraft is flying east at 193 knots and north at 52 knots.



49. 18 knots (18 nautical miles per hour) $\times$ 2.5 hours = 45 nm (nautical miles).

$$V = (45, 32^\circ); V_x = 45 \cos 32^\circ \approx 38 \text{ nm; distance east of the harbor (part b)}$$

$$V_y = 45 \sin 32^\circ \approx 24 \text{ nm; distance north of the harbor (part a).}$$



53. Force V Horizontal Component V_x Vertical Component V_y

$$(1000, 15^\circ) \quad 966$$

$$259$$

$$(2000, 15^\circ) \quad 1932$$

$$518$$

$$(1000, 30^\circ) \quad 866$$

$$500$$

Part (a).

Part (b); yes, the components double in value.

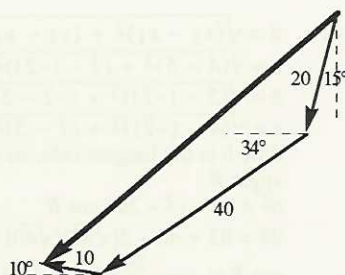
Part (c); no, the components do not double.

57. $(2.6, 18.3^\circ) = (2.6 \cos 18.3^\circ, 2.6 \sin 18.3^\circ) \approx (2.47, 0.82)$

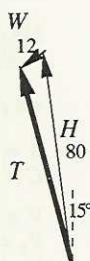
$$(15.8, -86.2^\circ) = (15.8 \cos(-86.2^\circ), 15.8 \sin(-86.2^\circ)) \approx (1.05, -15.77)$$

$$\approx (3.52, -14.95) \approx (15.4, -76.8^\circ)$$

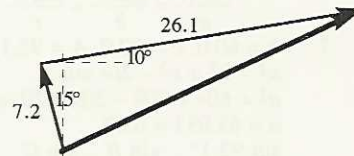
61. $(20, 255^\circ) + (40, 214^\circ) + (10, 170^\circ) \approx (62.6, -140.3^\circ)$.
Thus the ship is about 63 nautical miles from its starting position, at an angle of 40° south of west.



69. $H + W = T$, so $H = T - W$
 $= (80, 105^\circ) - (12, 225^\circ)$
 $= (80, 105^\circ) + (12, 225^\circ - 180^\circ)$
 $= (80, 105^\circ) + (12, 45^\circ)$
 $= (-20.71, 77.27) + (8.49, 8.49)$
 $= (-12.22, 85.76) \approx (87, 98^\circ)$.
 Thus the heading of the aircraft is 8° west of north, and its airspeed is 87 knots.



65. $(26.1, 10^\circ) + (7.2, 105^\circ) \approx (26.5, 25.7^\circ)$
 Thus the ship is traveling at 26.5 knots in a direction 25.7° north of east.



73. The sign is stationary, so the forces acting on it are balanced (they add to zero).

$$T_1 + T_2 + W = 0$$

$$T_1 = -T_2 - W$$

$$= -(456, 63^\circ) - (650, 270^\circ)$$

$$= (456, 63^\circ + 180^\circ) + (650, 270^\circ - 180^\circ)$$

To negate a vector, add or subtract 180° from its direction angle.

$$= (456, 243^\circ) + (650, 90^\circ)$$

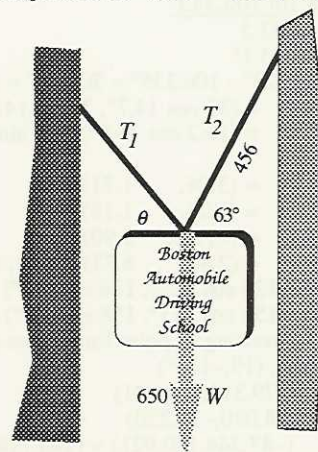
$$\approx (-207.02, -406.3) + (0, 650) \text{ Convert to rectangular form.}$$

$$\approx (-207.02, 243.7)$$

$$\approx (320, 130^\circ)$$

Convert back to polar form.

Thus the tension in the second cable is 320 pounds, and it makes an angle θ of 50° ($180^\circ - 130^\circ$) with the horizontal.



Chapter 6 Review

1. $a = 10.6, A = 47.9^\circ, B = 10.3^\circ$
 $C = 180^\circ - 47.9^\circ - 10.3^\circ = 121.8^\circ$
 $\frac{\sin 47.9^\circ}{10.6} = \frac{\sin 10.3^\circ}{b} = \frac{\sin 121.8^\circ}{c}$

$$b = \frac{10.6 \sin 10.3^\circ}{\sin 47.9^\circ} \approx 2.6$$

$$c = \frac{10.6 \sin 121.8^\circ}{\sin 47.9^\circ} \approx 12.1$$

3. $a = 10.0, b = 13.0, B = 79.0^\circ$

$$\frac{\sin A}{10} = \frac{\sin 79^\circ}{13} = \frac{\sin C}{c}$$

Observe that A must be acute, since side a is not the longest side of the triangle.

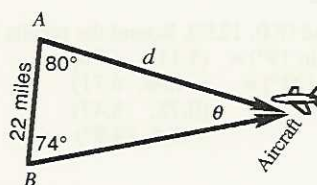
$$\sin A = \frac{10 \sin 79^\circ}{13}, \text{ so } A \approx 49.03^\circ \approx 49.0^\circ$$

$$C \approx 180^\circ - 79^\circ - 49.03^\circ \approx 51.97^\circ \approx 52.0^\circ$$

$$\frac{\sin 79^\circ}{13} \approx \frac{\sin 51.97^\circ}{c}, \text{ so } c \approx \frac{13 \sin 51.97^\circ}{\sin 79^\circ} \approx 10.4$$

5. $\theta = 180^\circ - 80^\circ - 74^\circ = 26^\circ; \frac{\sin 26^\circ}{22} = \frac{\sin 74^\circ}{d}$

$$d = \frac{22 \sin 74^\circ}{\sin 26^\circ}; d \approx 48 \text{ miles}$$



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\begin{aligned} 7. \quad b &= 60.0, c = 20.0, A = 92.1^\circ \\ a^2 &= b^2 + c^2 - 2bc \cos A \\ a^2 &= 60^2 + 20^2 - 2(60)(20) \cos 92.1^\circ \\ a &\approx 63.937 \approx 63.9 \\ \frac{\sin 92.1^\circ}{63.937} &= \frac{\sin B}{60} = \frac{\sin C}{20} \end{aligned}$$

Neither B nor C can be obtuse, since angle A must be the largest angle in the triangle.

$$\sin B = \frac{60 \sin 92.1^\circ}{63.937}, \text{ so } B \approx 69.7^\circ$$

$$C \approx 180^\circ - 69.7^\circ - 92.1^\circ \approx 18.2^\circ$$

9. $a = 43.5, b = 17.8, c = 35.0$
Find the largest angle first, since the two smallest angles must be acute, and the law of cosines does not have an ambiguous case. The largest angle is opposite the longest side, so it is A .

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \approx \frac{17.8^2 + 35^2 - 43.5^2}{2(17.8)(35)}$$

$$A \approx 106.335^\circ \approx 106.3^\circ$$

$$\frac{\sin 106.335^\circ}{43.5} = \frac{\sin B}{17.8} = \frac{\sin C}{35}$$

$$\sin B = \frac{17.8 \sin 106.335^\circ}{43.5}$$

$$B \approx 23.123^\circ \approx 23.1^\circ$$

$$C \approx 180^\circ - 23.123^\circ - 106.335^\circ \approx 50.542^\circ \approx 50.5^\circ$$

$$\begin{aligned} 17. \quad (33.0, 14.7^\circ) &= (33 \cos 14.7^\circ, 33 \sin 14.7^\circ) \approx (31.92, 8.37) \\ (15.2, 33.6^\circ) &= (15.2 \cos 33.6^\circ, 15.2 \sin 33.6^\circ) \approx (12.66, 8.41) \\ &\quad (44.58, 16.79) \approx (47.6, 20.6^\circ) \end{aligned}$$

$$\begin{aligned} 19. \quad (3.5, 29.2^\circ) &\approx (3.06, 1.71) \\ (1.7, 43.1^\circ) &\approx (1.24, 1.16) \\ (4.3, 115.0^\circ) &\approx (-1.82, 3.90) \\ &\approx (2.48, 6.77) \approx (7.2, 69.9^\circ) \end{aligned}$$

$$\begin{aligned} 21. \quad (126, 223^\circ) &= (126 \cos 223^\circ, 126 \sin 223^\circ) \approx (-92.15, -85.93) \\ (158, 311^\circ) &= (158 \cos 311^\circ, 158 \sin 311^\circ) \approx (103.66, -119.24) \end{aligned}$$

Adding and converting to polar form gives magnitude ≈ 205 pounds; direction $\approx -87^\circ$.

$$\begin{aligned} 23. \quad \text{Add } (195, 114^\circ), (19, -115^\circ) \\ (195, 114^\circ) &\approx (-79.314, 178.141) \\ (19, -115^\circ) &\approx (-8.030, -17.220) \\ &\quad (-87.344, 160.921) \approx (183, 118^\circ) \end{aligned}$$

Ground speed is 183 knots, true course is 62° north of west.

$$\begin{aligned} 11. \quad d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ a &= \sqrt{(4 - 5)^2 + (7 - (-2))^2} = \sqrt{82} \\ b &= \sqrt{(5 - (-2))^2 + (-2 - 5)^2} = \sqrt{98} = 7\sqrt{2} \\ c &= \sqrt{(4 - (-2))^2 + (7 - 5)^2} = \sqrt{40} = 2\sqrt{10} \end{aligned}$$

Side b is the longest side, so we use the law of cosines to find angle B .

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$98 = 82 + 40 - 2(\sqrt{82})(\sqrt{40}) \cos B$$

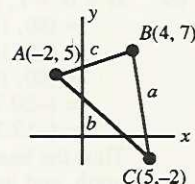
$$\cos B = \frac{24}{2(\sqrt{82})(\sqrt{40})} \approx 0.20953, \text{ so } B \approx 77.905^\circ \approx 77.9^\circ$$

$$\frac{\sin A}{\sqrt{82}} = \frac{\sin 77.905^\circ}{\sqrt{98}}$$

$$\sin A = \frac{\sqrt{82}}{\sqrt{98}} \sin 77.905^\circ$$

$$A \approx 63.4^\circ$$

$$C \approx 180^\circ - 77.9^\circ - 63.4^\circ \approx 38.7^\circ$$



$$\begin{aligned} 13. \quad V &= (27.2, 29^\circ) \\ V_x &= 27.2 \cos 29^\circ \approx 23.8 \\ V_y &= 27.2 \sin 29^\circ \approx 13.2 \end{aligned}$$

15. Convert $(450, 34.6^\circ)$ to rectangular coordinates.
 $(450, 34.6^\circ) = (450 \cos 34.6^\circ, 450 \sin 34.6^\circ) \approx (370, 256)$
Thus the horizontal component is 370 knots, and the vertical component is 256 knots.



Chapter 6 Test

$$\begin{aligned} 1. \quad B &= 180^\circ - 13.5^\circ - 82.1^\circ = 84.4^\circ \\ \frac{\sin 13.5^\circ}{a} &= \frac{\sin 84.4^\circ}{22.6} = \frac{\sin 82.1^\circ}{c} \end{aligned}$$

$$a = \frac{22.6 \sin 13.5^\circ}{\sin 84.4^\circ} \approx 5.3$$

$$c = \frac{22.6 \sin 82.1^\circ}{\sin 84.4^\circ} \approx 22.5$$

$$\begin{aligned} 3. \quad b^2 &= a^2 + c^2 - 2ac \cos B \\ b^2 &= 25.9^2 + 16.2^2 - 2(25.9)(16.2) \cos 100^\circ \approx 1078.97 \\ b &\approx 32.848 \approx 32.8 \end{aligned}$$

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

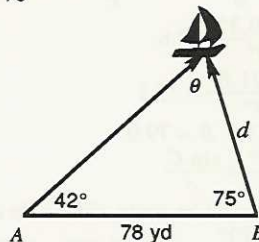
$$\frac{\sin A}{25.9} = \frac{\sin 100^\circ}{32.848}$$

$$\sin A = \frac{25.9 \sin 100^\circ}{32.848}; A \approx 50.9^\circ$$

$$C \approx 180^\circ - 50.9^\circ - 100^\circ \approx 29.1^\circ$$

9. Add the vectors $(5.4, 19.0^\circ)$ and $(8.0, 123^\circ)$. Round the results to the nearest tenth.
 $(5.4, 19^\circ) = (5.4 \cos 19^\circ, 5.4 \sin 19^\circ) \approx (5.11, 1.76)$
 $(8, 123^\circ) = (8 \cos 123^\circ, 8 \sin 123^\circ) \approx (-4.36, 6.71)$
 $\approx (0.75, 8.47)$
 $\approx (8.5, 84.9^\circ)$

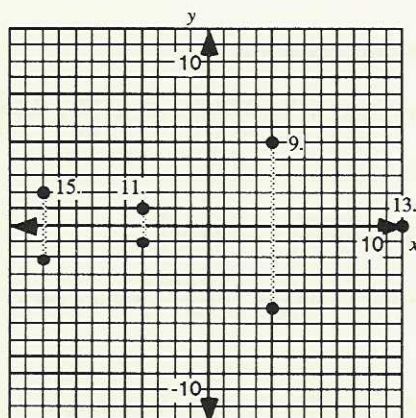
$$\begin{aligned} 5. \quad \theta &= 180^\circ - 42^\circ - 75^\circ = 63^\circ \\ \frac{\sin 42^\circ}{d} &= \frac{\sin 63^\circ}{78}; d \approx 59 \text{ yards} \end{aligned}$$



$$\begin{aligned} 7. \quad (2, 30^\circ) &= (2 \cos 30^\circ, 2 \sin 30^\circ) \\ &= \left(2 \cdot \frac{\sqrt{3}}{2}, 2 \cdot \frac{1}{2}\right) = (\sqrt{3}, 1) \end{aligned}$$

Exercise 7-1

1. real: 4; im: -5
5. real: 12; im: 0
9. $4 + 5i$
11. $-4 - i$
13. 12
15. $-10 - 2i$



17. $(5 + 4i) + (-3 + 2i) = 2 + 6i$
21. $13 - 7i + 3i - 4 = 9 - 4i$
25. $-5i(6 - 4i) = -30i + 20i^2 = -20 - 30i$
29. $\frac{5 + 4i}{5 + 2i} \cdot \frac{5 - 2i}{5 - 2i} = \frac{33 + 10i}{29} = \frac{33}{29} + \frac{10}{29}i$

Remember: Given vector $z = a + bi = r \operatorname{cis} \theta$. Then

$$r = \sqrt{a^2 + b^2}, \quad \tan \theta = \tan^{-1} \frac{b}{a}, \text{ and}$$

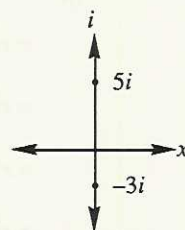
$$\theta = \begin{cases} \theta' & \text{if } a > 0 \\ \theta' - 180^\circ & \text{if } \theta' > 0 \\ \theta' + 180^\circ & \text{if } \theta' < 0 \end{cases}$$

33. $5 - 2i$
 $a = 5, b = -2; r = \sqrt{a^2 + b^2} = \sqrt{5^2 + (-2)^2} = \sqrt{29} \approx 5.4$
 $\theta' = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{-2}{5} \approx -21.8^\circ$
 Since $a > 0$, $\theta = \theta'$. Thus the point is $5.4 \operatorname{cis}(-21.8^\circ)$.
37. $-3 + 4i$ $r = \sqrt{(-3)^2 + 4^2} = 5$
 $\theta' = \tan^{-1} \frac{4}{-3} \approx -53.1^\circ$. $a < 0, \theta' < 0$ so $\theta \approx -53.1^\circ + 180^\circ \approx 126.9^\circ$. The point is $5 \operatorname{cis} 126.9^\circ$.
41. $3 + 3i$ $r = \sqrt{3^2 + 3^2} = 3\sqrt{2}$
 $\theta' = \tan^{-1} 1 = 45^\circ$; the point is $3\sqrt{2} \operatorname{cis} 45^\circ$.

81. Find the 3 cube roots of $75 - 100i$ to the nearest tenth.
 $75 - 100i \approx 125 \operatorname{cis}(306.8699^\circ); (75 - 100i)^{1/3} \approx (125 \operatorname{cis}(306.8699^\circ))^{1/3}$
 Evaluate $\sqrt[3]{125} \operatorname{cis}\left(\frac{306.8699^\circ}{3} + \frac{k \cdot 360^\circ}{3}\right) = 5 \operatorname{cis}(102.29^\circ + k \cdot 120^\circ)$ for $k = 0, 1, 2$.
 $k = 0: 5 \operatorname{cis}(102.29^\circ) = 5 \cos 102.29^\circ + 5 \sin 102.29^\circ i \approx -1.1 + 4.9i$
 $k = 1: 5 \operatorname{cis}(102.29^\circ + 120^\circ) = 5 \cos 222.29^\circ + 5 \sin 222.29^\circ i \approx -3.7 - 3.4i$
 $k = 2: 5 \operatorname{cis}(102.29^\circ + 240^\circ) = 5 \cos 342.29^\circ + 5 \sin 342.29^\circ i \approx 4.8 - 1.5i$

85. $I = \frac{V}{Z}$, so $V = IZ = (10 \operatorname{cis} 15^\circ)(5 \operatorname{cis} 30^\circ) = 50 \operatorname{cis} 45^\circ$.

45. $5i$ Plotting the point shows that $r = 5$ and $\theta = 90^\circ$. Thus the point is $5 \operatorname{cis} 90^\circ$.



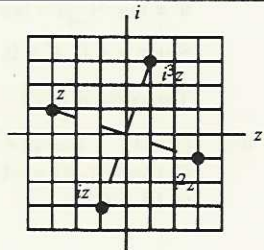
49. $4.5 \operatorname{cis} 35^\circ$ $4.5 \cos 35^\circ + (4.5 \sin 35^\circ)i = 3.7 + 2.6i$
53. $13.6 \operatorname{cis} (-25^\circ)$ $13.6 \cos (-25^\circ) + (13.6 \sin(-25^\circ))i = 12.3 - 5.7i$
57. $10 \operatorname{cis} 300^\circ$ $10 \cos 300^\circ + (10 \sin 300^\circ)i = 10 \cdot \frac{1}{2} + 10(-\frac{\sqrt{3}}{2})i = 5 - 5\sqrt{3}i$
61. $\sqrt{8} \operatorname{cis} 315^\circ$ $\sqrt{8} \cos 315^\circ + (\sqrt{8} \sin 315^\circ)i = \sqrt{8} \cdot \frac{\sqrt{2}}{2} + \sqrt{8} \cdot (-\frac{\sqrt{2}}{2})i = 2 - 2i$
65. $(5.4 \operatorname{cis} 300^\circ)(2 \operatorname{cis} 300^\circ) = 10.8 \operatorname{cis}(600^\circ) = 10.8 \operatorname{cis}(600^\circ - 2 \cdot 360^\circ) = 10.8 \operatorname{cis}(-120^\circ)$
69. $\frac{40 \operatorname{cis} 80^\circ}{18 \operatorname{cis} 160^\circ} = \frac{40}{18} \operatorname{cis}(80^\circ - 160^\circ) = \frac{20}{9} \operatorname{cis}(-80^\circ)$
73. $(3 \operatorname{cis} 200^\circ)^3 = 3^3 \operatorname{cis}(3 \cdot 200^\circ) = 27 \operatorname{cis} 600^\circ = 27 \operatorname{cis}(600^\circ - 2 \cdot 360^\circ) = 27 \operatorname{cis}(-120^\circ)$

$$\text{Use: } (r \operatorname{cis} \theta)^{\frac{1}{n}} = r^{\frac{1}{n}} \operatorname{cis} \left(\frac{\theta}{n} + \frac{k \cdot 360^\circ}{n} \right), 0 \leq k < n.$$

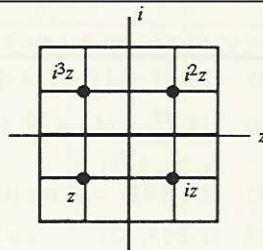
77. Find the 3 cube roots of 8 in exact form.
 $8 = 8 \operatorname{cis} 0^\circ$
 Evaluate $\sqrt[3]{8} \operatorname{cis} \left(\frac{0^\circ}{3} + \frac{k \cdot 360^\circ}{3} \right) = 2 \operatorname{cis} (k \cdot 120^\circ)$ for $k = 0, 1, 2$.
 $k = 0: 2 \operatorname{cis} 0^\circ = 2 \cos 0^\circ + 2 \sin 0^\circ i = 2 + 0i = 2$
 $k = 1: 2 \operatorname{cis} 120^\circ = 2 \cos 120^\circ + 2 \sin 120^\circ i = 2\left(-\frac{1}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right)i = -1 + \sqrt{3}i$
 $k = 2: 2 \operatorname{cis} 240^\circ = 2 \cos 240^\circ + 2 \sin 240^\circ i = 2\left(-\frac{1}{2}\right) + 2\left(-\frac{\sqrt{3}}{2}\right)i = -1 - \sqrt{3}i$
 Thus the three cube roots of 8 are $2, -1 + \sqrt{3}i, -1 - \sqrt{3}i$.

89. $P = 5 + 2i \approx 5.385 \operatorname{cis} 21.801^\circ$
 $Z = 1 - 4i \approx 4.123 \operatorname{cis} 284.036^\circ$
 $\sqrt{\frac{P}{Z}} \approx \sqrt{\frac{5.385 \operatorname{cis} 21.801^\circ}{4.123 \operatorname{cis} 284.036^\circ}} = (1.3061 \operatorname{cis}(-262.235^\circ))^{1/2} \approx \sqrt{1.3061} \operatorname{cis}\left(\frac{-262.235^\circ}{2}\right) \approx 1.143 \operatorname{cis}(-131.118^\circ) \approx 0.75 + 0.86i$

93. (a) $z: -3 + i$
 (b) $iz: i(-3 + i) = -3i + i^2 = -1 - 3i$
 (c) $i^2 z: (-1)(-3 + i) = 3 - i$
 (d) $i^3 z: (-i)(-3 + i) = 1 + 3i$



97. (a) $z: -1 - i$
 (b) $iz: i(-1 - i) = -i - i^2 = 1 - i$
 (c) $i^2 z: (-1)(-1 - i) = 1 + i$
 (d) $i^3 z: (-i)(-1 - i) = -1 + i$



$$\begin{aligned}
 101. \quad & \frac{1}{r^n} \operatorname{cis} \left(\frac{\theta}{n} + \frac{k \cdot 360^\circ}{n} \right) \\
 &= \frac{1}{r^n} \operatorname{cis} \left[\frac{\theta}{n} + \frac{(an + b) \cdot 360^\circ}{n} \right] \quad \text{Replace } k \text{ by } an + b. \\
 &= \frac{1}{r^n} \operatorname{cis} \left[\frac{\theta}{n} + \frac{an \cdot 360^\circ}{n} + \frac{b \cdot 360^\circ}{n} \right] = \frac{1}{r^n} \operatorname{cis} \left[\frac{\theta}{n} + a \cdot 360^\circ + \frac{b \cdot 360^\circ}{n} \right] = \frac{1}{r^n} \operatorname{cis} \left[\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n} \right) + a \cdot 360^\circ \right] \\
 &= \frac{1}{r^n} \operatorname{cis} \left[\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n} \right) + a \cdot 360^\circ \right] \\
 &= \frac{1}{r^n} \cos \left[\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n} \right) + a \cdot 360^\circ \right] + i \frac{1}{r^n} \sin \left[\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n} \right) + a \cdot 360^\circ \right] \\
 \cos \left[\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n} \right) + a \cdot 360^\circ \right] &= \cos \left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n} \right) \cos(a \cdot 360^\circ) - \sin \left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n} \right) \sin(a \cdot 360^\circ) \\
 &= \cos \left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n} \right), \\
 &\quad \text{because } \cos(a \cdot 360^\circ) = 1 \text{ and } \sin(a \cdot 360^\circ) = 0, \text{ when } a \text{ is an integer.} \\
 \sin \left[\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n} \right) + a \cdot 360^\circ \right] &= \sin \left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n} \right) \cos(a \cdot 360^\circ) + \cos \left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n} \right) \sin(a \cdot 360^\circ) \\
 &= \sin \left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n} \right), \\
 &\quad \text{because } \cos(a \cdot 360^\circ) = 1 \text{ and } \sin(a \cdot 360^\circ) = 0, \text{ when } a \text{ is an integer.}
 \end{aligned}$$

Thus,

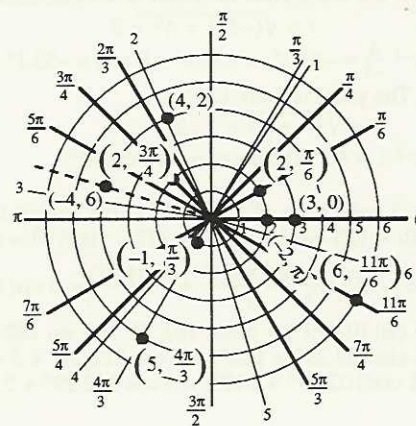
$$\begin{aligned}
 & \frac{1}{r^n} \cos \left[\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n} \right) + a \cdot 360^\circ \right] + i \frac{1}{r^n} \sin \left[\left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n} \right) + a \cdot 360^\circ \right] \\
 &= \frac{1}{r^n} \cos \left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n} \right) + i \frac{1}{r^n} \sin \left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n} \right) \\
 &= \frac{1}{r^n} \operatorname{cis} \left(\frac{\theta}{n} + \frac{b \cdot 360^\circ}{n} \right), \text{ where } b < n.
 \end{aligned}$$

This last expression is one of the previous roots.

Exercise 7-2

The points for problems 1 through 18 are plotted in the figures. Some special considerations are also listed.

9. $(-2, \pi) = (2, \pi - \pi) = (2, 0)$
11. $\left(-1, \frac{\pi}{3}\right) = \left(1, \frac{\pi}{3} + \pi\right) = \left(1, \frac{4\pi}{3}\right)$
13. $(4, 2)$; Note: $2 = 2 \cdot \frac{180^\circ}{\pi} \approx 115^\circ$ (for plotting)
15. $(-5, 6) = (5, 6 - \pi)$; $6 - \pi = (6 - \pi) \left(\frac{180^\circ}{\pi}\right) \approx 164^\circ$. To plot $(-5, 6)$, plot a point 5 units from the center, at an angle about 164° .



Solutions to 1,3,5,7,9,11,13,15,17.

Many answers possible. To change the sign of r add an odd multiple of π to θ ; for the rest add or subtract an even multiple of π .

21. $\left(6, \frac{11\pi}{6}\right)$ $\left(-6, \frac{11\pi}{6} - \pi\right)$ $\left(6, \frac{11\pi}{6} + 2\pi\right)$ $\left(6, \frac{11\pi}{6} - 2\pi\right)$
 $\left(-6, \frac{5\pi}{6}\right)$ $\left(6, \frac{23\pi}{6}\right)$ $\left(6, -\frac{\pi}{6}\right)$

Use $(r, \theta) = (r \cos \theta, r \sin \theta) = (x, y)$.

25. $\left(4, \frac{\pi}{2}\right) = \left(4 \cos \frac{\pi}{2}, 4 \sin \frac{\pi}{2}\right) = (4 \cdot 0, 4 \cdot 1) = (0, 4)$
29. $\left(4, \frac{4\pi}{3}\right) = \left(4 \cos \frac{4\pi}{3}, 4 \sin \frac{4\pi}{3}\right) = \left(4 \cdot \left(-\frac{1}{2}\right), 4 \cdot \left(-\frac{\sqrt{3}}{2}\right)\right) = (-2, -2\sqrt{3})$
33. $(3, 0.82) = (3 \cos 0.82, 3 \sin 0.82) \approx (2.05, 2.19)$
37. $(-2\sqrt{3}, -2)$ $r = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{4 \cdot 3 + 4} = 4$.
 $\theta' = \tan^{-1} \left(\frac{-2}{-2\sqrt{3}} \right) = \tan^{-1} \frac{\sqrt{3}}{3} = \frac{\pi}{6}$.

Since $x = -2\sqrt{3} < 0$, and $\theta' > 0$, $\theta = \theta' - \pi = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}$.

Thus the polar coordinates are $\left(4, -\frac{5\pi}{6}\right)$.

41. $(-4, -4)$ $r = \sqrt{4^2 + 4^2} = 4\sqrt{2}$
 $\theta' = \tan^{-1} \left(\frac{-4}{-4} \right) = \tan^{-1} 1 = \frac{\pi}{4}$.

Since $x < 0$, $\theta' > 0$, $\theta = \theta' - \pi = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$.

The point is $\left(4\sqrt{2}, -\frac{3\pi}{4}\right)$ in polar coordinates.

45. $(1, -4)$ $r = \sqrt{1^2 + 4^2} = \sqrt{17} \approx 4.12$
 $\theta' = \tan^{-1}(-4) \approx -1.326$; $x > 0$ so $\theta = \theta'$.
 $(4.12, -1.33)$

Use $x = r \cos \theta$, $y = r \sin \theta$.

49. $y = 4x$
 $r \sin \theta = 4r \cos \theta$
 $\sin \theta = 4 \cos \theta$
 $\frac{\sin \theta}{\cos \theta} = 4$
 $\tan \theta = 4$
 53. $y = mx + b$, $b \neq 0$
 $r \sin \theta = mr \cos \theta + b$

Use $\sin \theta = \frac{y}{r}$; $\cos \theta = \frac{x}{r}$;
 $r^2 = x^2 + y^2$.

61. $r = 2 \sec \theta$
 $r = \frac{2}{\cos \theta}$
 $r = \frac{2}{\frac{x}{r}}$

65. $r = \frac{2r}{x}$
 $rx = 2r$
 $x = 2$
 Assumes $r \neq 0$. It is not since 2
 $\sec \theta$ is never 0.
 $r^2 = \sin 2\theta$
 $r^2 = 2 \sin \theta \cos \theta$

$r \sin \theta - mr \cos \theta = b$
 $r(\sin \theta - m \cos \theta) = b$
 $r = \frac{b}{\sin \theta - m \cos \theta}$

57. $3x^2 + 2y^2 = 1$
 $3(r \cos \theta)^2 + 2(r \sin \theta)^2 = 1$
 $3r^2 \cos^2 \theta + 2r^2 \sin^2 \theta = 1$
 $r^2(3 \cos^2 \theta + 2 \sin^2 \theta) = 1$
 $r^2 = \frac{1}{3 \cos^2 \theta + 2 \sin^2 \theta}$

$r^2 = \frac{1}{3 \cos^2 \theta + 2(1 - \cos^2 \theta)}$
 $r^2 = \frac{1}{\cos^2 \theta + 2}$

$r^2 = 2 \frac{y}{r} \cdot \frac{x}{r}$
 $r^4 = 2xy$
 $(x^2 + y^2)^2 = 2xy$
 $x^4 + 2x^2y^2 + y^4 = 2xy$
 69. $r = \frac{3}{1 - 2 \sin \theta}$
 $r(1 - 2 \sin \theta) = 3$

$r(1 - \frac{2y}{r}) = 3$
 $r - 2y = 3$
 $r = 2y + 3$
 $r^2 = (2y + 3)^2$
 $x^2 + y^2 = 4y^2 + 12y + 9$

73. Consider a point $P = (r, \theta)$, where $r < 0$. Then $P = (-r, \theta + \pi)$, where $-r > 0$. Therefore, since $-r > 0$, $y = -r \sin(\theta + \pi)$ is true.

$y = -r \sin(\theta + \pi)$ A true statement.
 $= -r(\sin \theta \cos \pi + \cos \theta \sin \pi)$ $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$
 $= -r(\sin \theta (-1) + \cos \theta (0)) = -r(-\sin \theta) = r \sin \theta$

Thus $y = r \sin \theta$, even if $r < 0$.

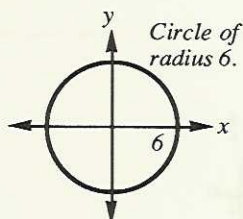
77. $r = 1 + 2 \sin 2\theta$
 $r = 1 + 2(2 \sin \theta \cos \theta)$

$r = 1 + 4 \frac{y}{r} \cdot \frac{x}{r}$
 $r^3 = r^2 + 4xy$

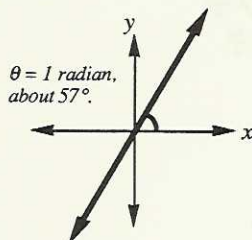
$r^6 = (r^2 + 4xy)^2$
 $(r^2)^3 = (x^2 + y^2 + 4xy)^2$
 $(x^2 + y^2)^3 = (x^2 + y^2 + 4xy)^2$

Exercise 7-3

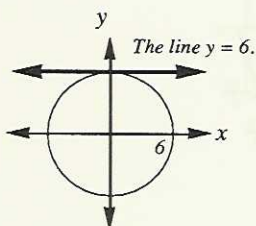
1. $r = 6$



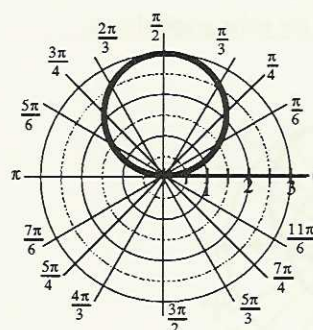
5. $\theta = 1$



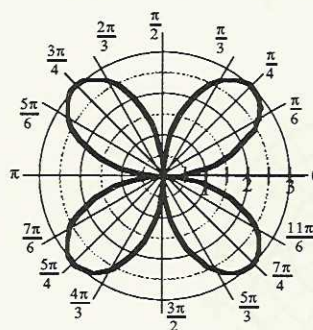
9. $r \sin \theta = 6$
 $y = 6$



13. $r = 3 \sin \theta$



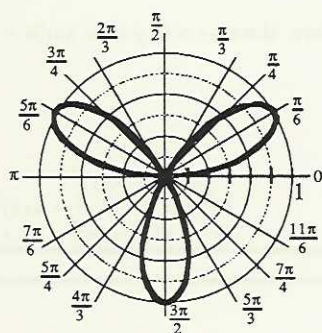
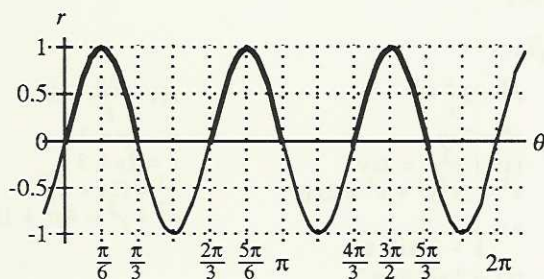
17. $r = 3 \sin 2\theta$



20. $r = \sin 3\theta$

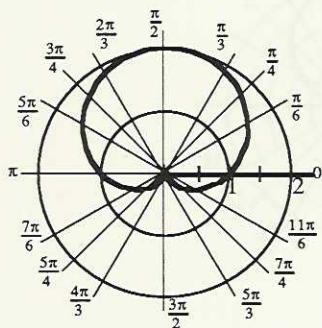
The rectangular coordinate graph of $r = \sin 3\theta$ is shown below. The negative lobes can be ignored because shifting them by π units and flipping them about the θ axis causes them to overlap the positive lobes.

We see there are lobes between 0 and $\frac{\pi}{3}$, with maximum at $\frac{\pi}{6}$, $\frac{2\pi}{3}$ and π , with maximum at $\frac{5\pi}{6}$, and $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$, with maximum at $\frac{3\pi}{2}$.

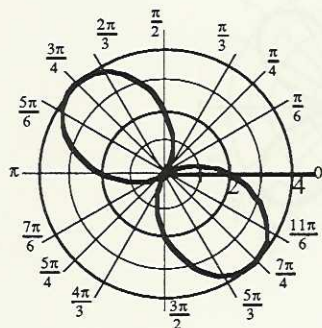


These lobes produce the polar graph shown.

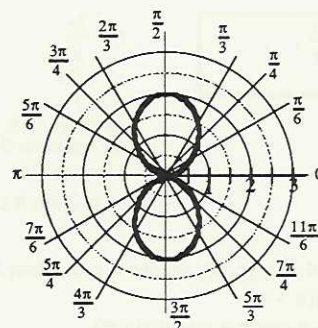
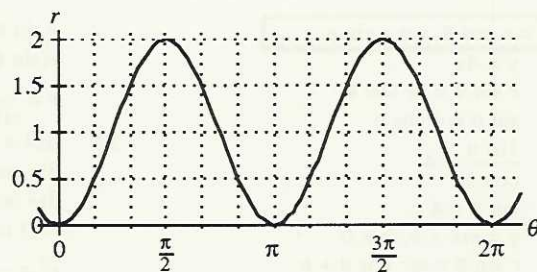
21. $r = 1 + \sin \theta$



25. $r = 2 - \sin 2\theta$



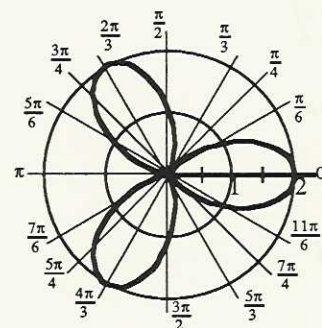
26. $r = 1 - \cos 2\theta$



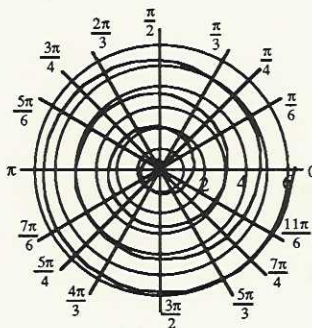
The rectangular graph shows two lobes, one between 0 and π , with maximum at $\frac{\pi}{2}$, and one between π and 2π , with

maximum at $\frac{3\pi}{2}$. The two positive lobes produce the polar graph shown. The two lobes are not circles, which would require algebra beyond the scope of this text to show. Of the problems in this text, only those equations categorized in the text as producing circles will in fact produce circles.

27. $r = 1 + \cos 3\theta$



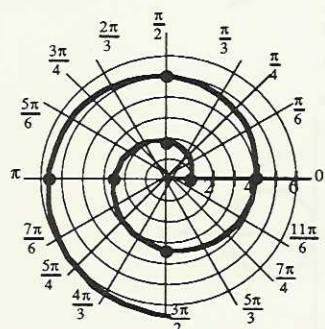
31. $r = \frac{\theta}{4}, \theta > 0$



32. $r = 1 + \frac{\theta}{2}$

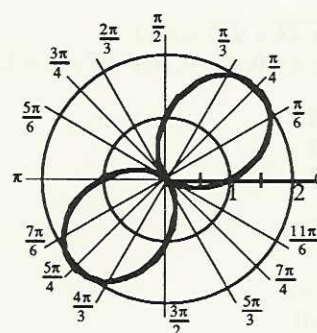
This graph is not periodic, and thus an analysis of its rectangular graph is less helpful than in the case where the trigonometric functions are involved. We simply plot points to obtain the graph. It is clear that as the value of θ increases, r increases. This property produces a spiral, which starts at $r = 1$.

A table of values and the graph are shown.



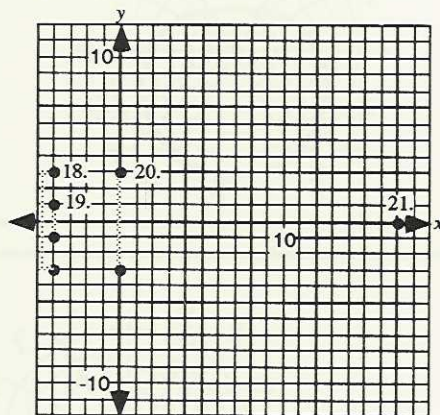
θ	r
0	1
$\frac{\pi}{2}$	1.8
π	2.6
$\frac{3\pi}{2}$	3.4
2π	4.1
$\frac{5\pi}{2}$	4.9
3π	5.7

35. $r = 1 + \sin 2\theta$



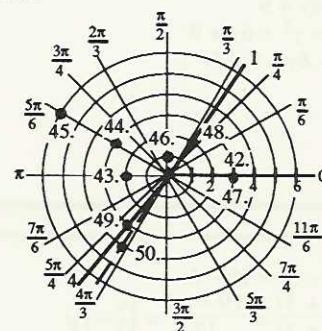
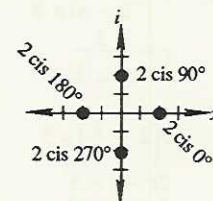
Chapter 7 Review

1. real: 3; im: -1
3. real: 12; im: 0
5. $2 - 4i - 3 + 2i = -1 - 2i$
7. $15 + 60i - 12i - 48i^2$
 $15 + 48i + 48$
 $63 + 48i$
9. $8i - 12i^2$
 $8i + 12$
 $12 + 8i$
11. $\frac{10 - 4i}{5 + 2i} \cdot \frac{5 - 2i}{5 - 2i} = \frac{42 - 40i}{29} = \frac{42}{29} - \frac{40}{29}i$
13. $\frac{8 + 6i}{14i} \cdot \frac{-i}{-i} = \frac{-8i - 6i^2}{-14i^2} = \frac{6 - 8i}{14} = \frac{3}{7} - \frac{4}{7}i$
15. $(2 + i)(3 - (3 - 2i)) = (2 + i)(2i) = 4i + 2i^2 = -2 + 4i$
17. $\frac{(2 + i)((2 + i) - (3 - 2i))}{2(3 - 2i)} = \frac{(2 + i)(-1 + 3i)}{(6 - 4i)}$
 $= \frac{-50 + 10i}{52} = -\frac{25}{26} + \frac{5}{26}i$
18. $-4 - 3i$
19. $-4 - i$
20. $-3i$
21. 17



25. $\sqrt{3} - i$ $r = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$
 $\theta' = \tan^{-1}(\frac{y}{x}) = \tan^{-1}(-\frac{1}{\sqrt{3}}) = -30^\circ$; $a > 0$ so $\theta = \theta'$
 $2 \text{ cis}(-30^\circ)$
27. $-4i$ $r = \sqrt{0^2 + (-4)^2} = 4$
 $\theta = -90^\circ$ (plot the point $0 - 4i$ to see this)
 $4 \text{ cis}(-90^\circ)$

29. $3 \text{ cis } 243^\circ = 3 \cos 243^\circ + 3 \sin 243^\circ \approx -1.4 - 2.7i$
31. $10 \text{ cis } 330^\circ = 10 \cos 330^\circ + 10i \sin 330^\circ = 10 \cdot \frac{\sqrt{3}}{2} + 10i(-\frac{1}{2})$
 $= 5\sqrt{3} - 5i$
33. $(2 \text{ cis } 18^\circ)(6.5 \text{ cis } 122^\circ) = 2 \cdot 6.5 \text{ cis}(18^\circ + 122^\circ) = 13 \text{ cis } 140^\circ$
35. $\frac{50 \text{ cis } 45^\circ}{100 \text{ cis } 9^\circ} = \frac{50}{100} \text{ cis}(45^\circ - 9^\circ) = \frac{1}{2} \text{ cis } 36^\circ$
37. $(2 \text{ cis } 150^\circ)^4 = 2^4 \text{ cis}(4 \cdot 150^\circ) = 16 \text{ cis } 600^\circ = 16 \text{ cis}(600^\circ - 720^\circ) = 16 \text{ cis}(-120^\circ)$
39. $16 = 16 \text{ cis } 0^\circ$; evaluate
 $16^{1/4} \text{ cis}(\frac{0^\circ}{4} + \frac{k \cdot 360^\circ}{4}) = 2 \text{ cis}(k \cdot 90^\circ)$ for $k = 0, 1, 2, 3$.
 $k = 0$: $2 \text{ cis } 0^\circ = 2$
 $k = 1$: $2 \text{ cis } 90^\circ = 2i$
 $k = 2$: $2 \text{ cis } 180^\circ = -2$
 $k = 3$: $2 \text{ cis } 270^\circ = -2i$
41. $I = \frac{130 \text{ cis } 25^\circ}{30 \text{ cis } 75^\circ} = \frac{130}{30} \text{ cis}(25^\circ - 75^\circ) = 4\frac{1}{3} \text{ cis}(-50^\circ)$



Answers to 42 through 50.

53. $(-2, \frac{5\pi}{3}) = (-2 \cos \frac{5\pi}{3}, -2 \sin \frac{5\pi}{3}) = (-2 \cdot \frac{1}{2}, -2(-\frac{\sqrt{3}}{2})) = (-1, \sqrt{3})$
55. $(5, 2) = (5 \cos 2, 5 \sin 2) \approx (-2.08, 4.55)$
57. $(2, 1)$ $r = \sqrt{2^2 + 1^2} = \sqrt{5} \approx 2.24$
 $\theta = \theta' = \tan^{-1} \frac{1}{2} \approx 0.46$; $(2.24, 0.46)$

59. $(-1, -4) \quad r = \sqrt{1^2 + 4^2} = \sqrt{17} \approx 4.12$
 $\theta' = \tan^{-1} 4 \approx 1.326$; $x < 0, \theta' > 0$, so $\theta = \theta' - \pi \approx 1.326 - \pi \approx -1.82$; $(4.12, -1.82)$

61. $y = 4x + 2$
 $r \sin \theta = 4r \cos \theta + 2$
 $r \sin \theta - 4r \cos \theta = 2$
 $r(\sin \theta - 4 \cos \theta) = 2$
 $r = \frac{2}{\sin \theta - 4 \cos \theta}$

63. $y^2 - 3x = 0$
 $(r \sin \theta)^2 - 3r \cos \theta = 0$
 $r^2 \sin^2 \theta - 3r \cos \theta = 0$
 $r \sin^2 \theta - 3 \cos \theta = 0$
 $r \sin^2 \theta = 3 \cos \theta$
 $r = \frac{3 \cos \theta}{\sin^2 \theta}$
 $r = 3 \frac{\cos \theta}{\sin \theta} \cdot \frac{1}{\sin \theta}$

Alternate form
of answer.

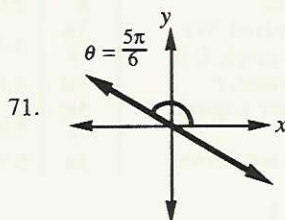
65. $r = 2 \sec \theta$
 $r = 2 \cdot \frac{1}{\cos \theta}$

$r = 2 \cdot \frac{r}{x}$
 $1 = \frac{2}{x}$ Divide each member by r .
 $x = 2$

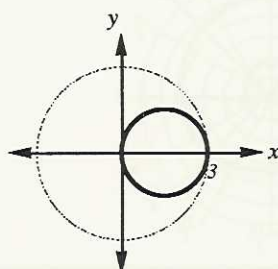
67. $r^2 = \tan \theta$
 $x^2 + y^2 = \frac{y}{x}$
 $x^3 + xy^2 = y$

69. $r = \frac{3}{2 - \sin \theta}$
 $r = \frac{3}{2 - \frac{y}{r}}$

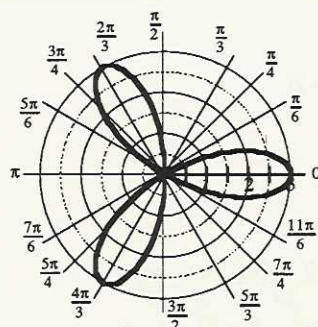
$r\left(2 - \frac{y}{r}\right) = 3$
 $2r - y = 3$
 $2r = y + 3$
 $(2r)^2 = (y + 3)^2$
 $4r^2 = y^2 + 6y + 9$
 $4(x^2 + y^2) = y^2 + 6y + 9$
 $4x^2 + 3y^2 - 6y - 9 = 0$



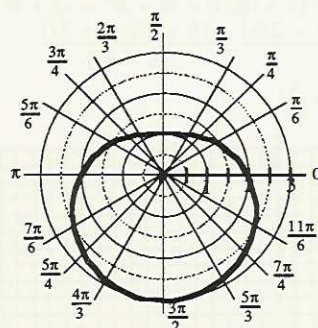
73. $r = 3 \cos \theta$



75. $r = 3 \cos 3\theta$



77. $r = 2 - \sin \theta$



Chapter 7 Test

1. $12 - 4i + 3 - 2i = 15 - 6i$

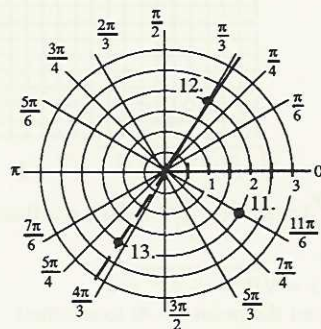
3. $\frac{2 - 5i}{2 - 3i} \cdot \frac{2 + 3i}{2 + 3i} = \frac{19 - 4i}{13} = \frac{19}{13} - \frac{4}{13}i$

5. $4 - 5i \quad r = \sqrt{4^2 + 5^2} = \sqrt{41} \approx 6.40$
 $\theta = \theta' - \tan^{-1}\left(-\frac{5}{4}\right) \approx -51.3$ $\theta = \theta'$ because $a > 0$
 $6.40 \operatorname{cis}(-51.3^\circ)$

7. $(2 \operatorname{cis} 325^\circ)(7 \operatorname{cis} 145^\circ) = 2 \cdot 7 \operatorname{cis} (325^\circ + 145^\circ) = 14 \operatorname{cis} 470^\circ$
 $= 14 \operatorname{cis}(470^\circ - 360^\circ) = 14 \operatorname{cis} 110^\circ$

9. $(3 \operatorname{cis} 150^\circ)^3 = 3^3 \operatorname{cis}(3 \cdot 150^\circ) = 27 \operatorname{cis} 450^\circ$
 $= 27 \operatorname{cis}(450^\circ - 360^\circ) = 27 \operatorname{cis} 90^\circ$

11. $(2, \frac{11\pi}{6})$ 12. $(2, \frac{\pi}{3})$ 13. $(-2, 1)$



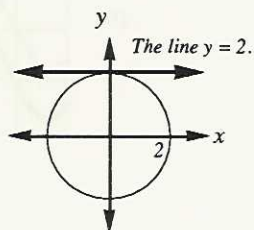
15. $(-5, 2) = (-5 \cos 2, -5 \sin 2) \approx (2.08, -4.55)$

17. $(-5, 5)$
 $r = \sqrt{25 + 25} = 5\sqrt{2}$
 $\theta' = \tan^{-1} \frac{5}{-5} = \tan^{-1}(-1) = -\frac{\pi}{4}$
 Since $x < 0$, $\theta' < 0$ then θ
 $= \theta' + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

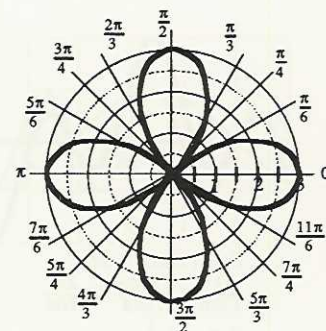
19. $2y^2 - x = 5$
 $2(r \sin \theta)^2 - r \cos \theta = 5$
 $2r^2 \sin^2 \theta - r \cos \theta = 5$

21. $r^2 = \cos 2\theta$
 $x^2 + y^2 = \cos^2 2\theta - \sin^2 2\theta$
 $x^2 + y^2 = \frac{x^2 - y^2}{r^2} - \frac{y^2}{r^2}$
 $x^2 + y^2 = \frac{x^2 + y^2}{r^2}$
 $x^2 + y^2 = \frac{x^2 - y^2}{x^2 + y^2}$
 $(x^2 + y^2)^2 = x^2 - y^2$
 $x^4 + 2x^2y^2 + y^4 = x^2 - y^2$

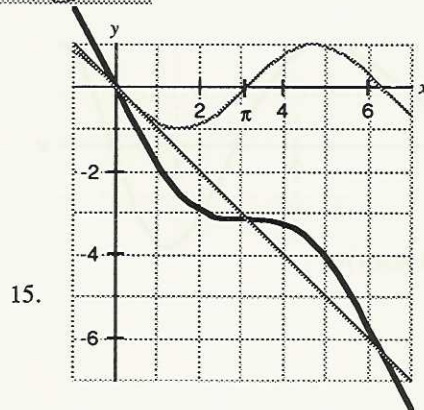
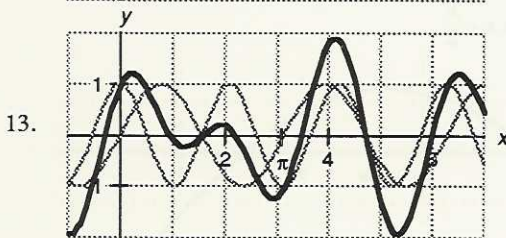
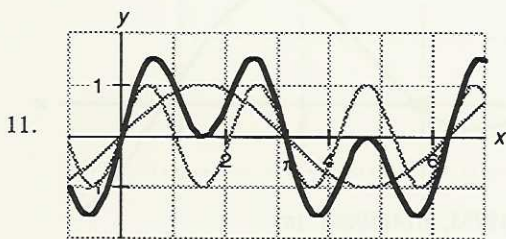
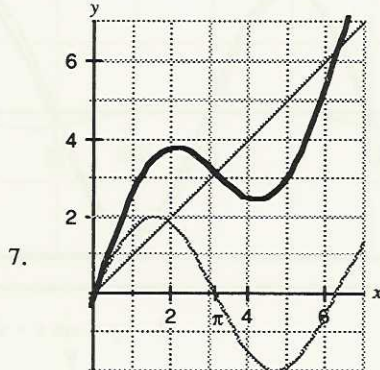
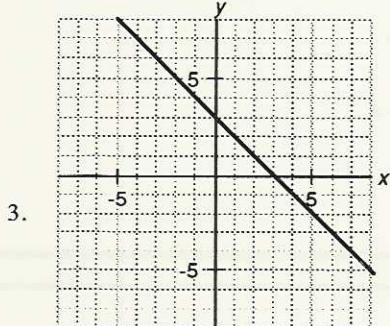
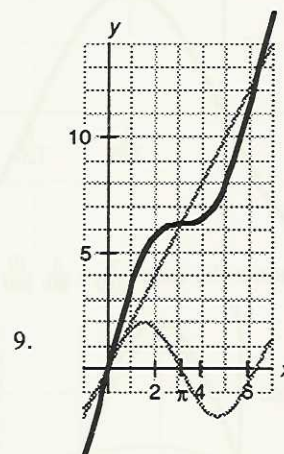
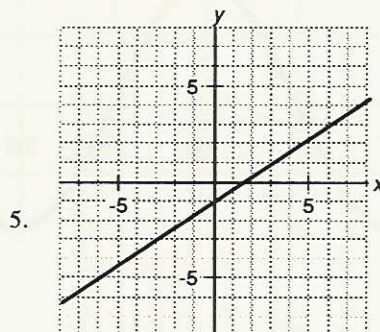
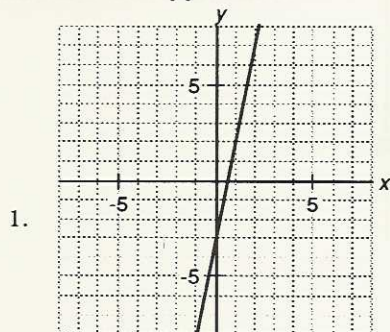
23. $r = 2 \csc \theta$
 $r = \frac{2}{\sin \theta}$
 $r \sin \theta = 2$
 $y = 2$

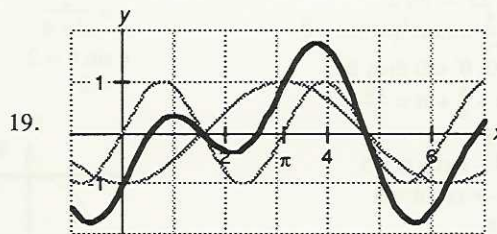
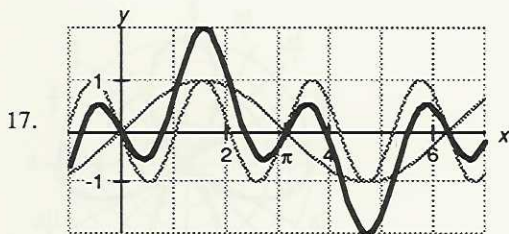


24. $r = 3 \cos 2\theta$



Exercises for Appendix A



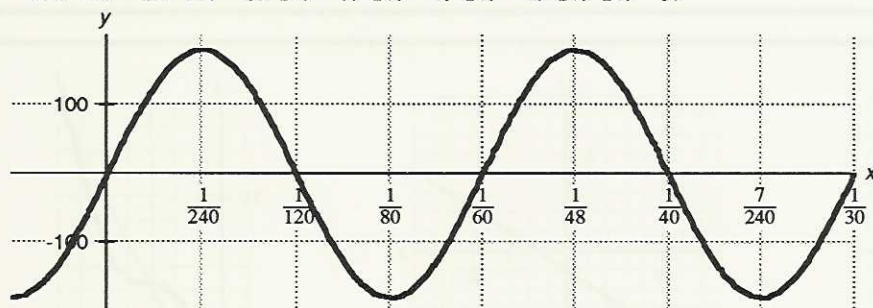


21. a. $0 \leq 120\pi x \leq 2\pi$
 $\frac{0}{120\pi} \leq \frac{120\pi x}{120\pi} \leq \frac{2\pi}{120\pi}$
 $0 \leq x \leq \frac{1}{60}$

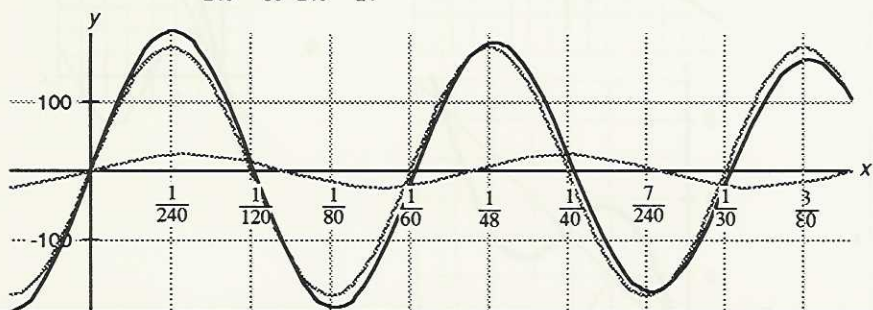
Two cycles are $2 \cdot \frac{1}{60} = \frac{1}{30}$ (seconds)

$\frac{1}{30} \div \frac{1}{240} = \frac{1}{30} \cdot \frac{240}{1} = 8$, so there are 8 tick marks for the two cycles. These are multiples of $\frac{1}{240}$.

$0, \frac{1}{240}, \frac{2}{240} = \frac{1}{120}, \frac{3}{240} = \frac{1}{80}, \frac{4}{240} = \frac{1}{60}, \frac{5}{240} = \frac{1}{48}, \frac{6}{240} = \frac{1}{40}, \frac{7}{240}, \frac{8}{240} = \frac{1}{30}$

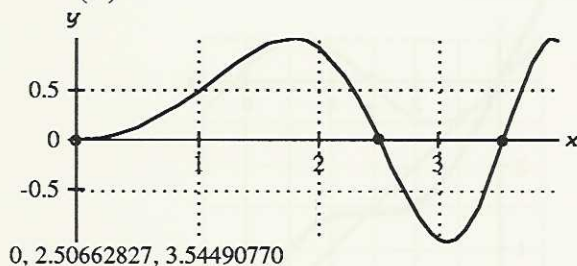


b. Continue the units: $\frac{9}{240} = \frac{3}{80}, \frac{10}{240} = \frac{1}{24}$

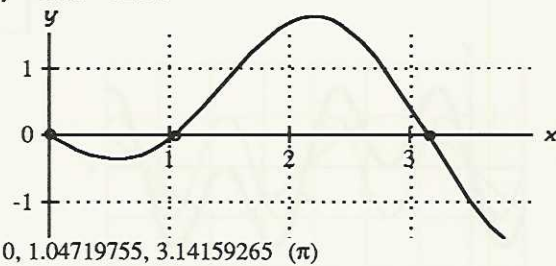


Exercises for Appendix B

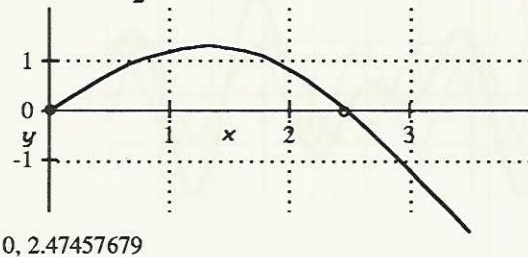
1. $y = \sin\left(\frac{x^2}{2}\right)$



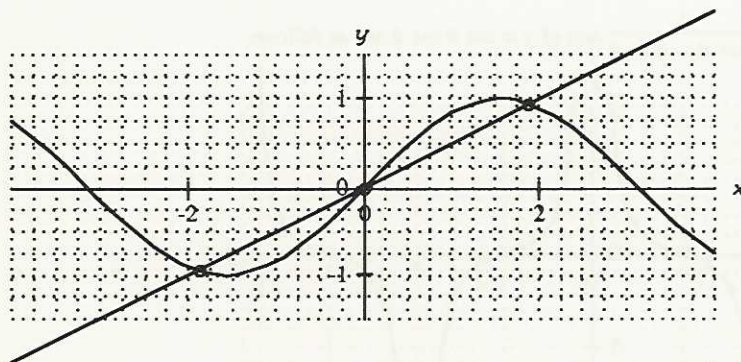
3. $y = \sin x - \sin 2x$



5. $y = 2 \sin x - \frac{x}{2}$

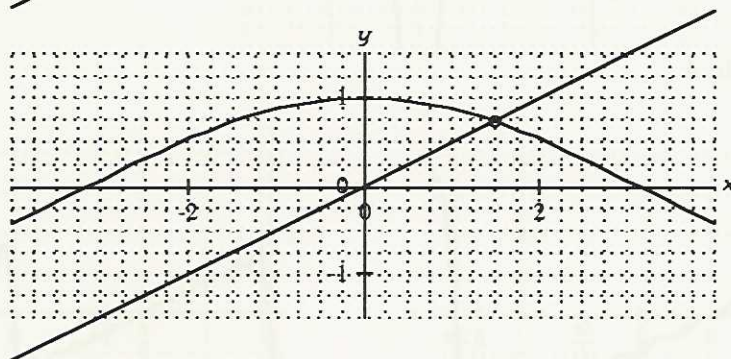


7. $y = \sin x$; $y = \frac{x}{2}$



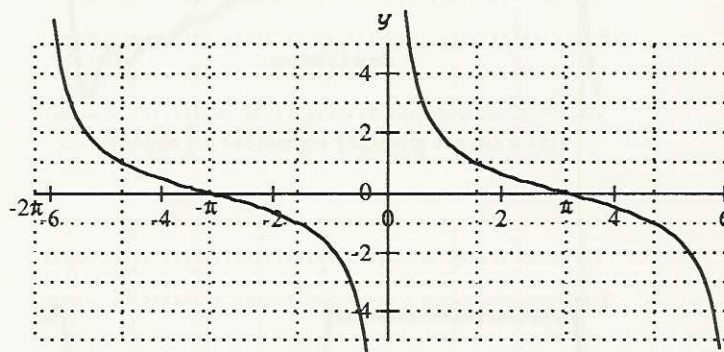
$0, \pm 1.89549427$

9. $y = \cos \frac{x}{2}$; $y = \frac{x}{2}$

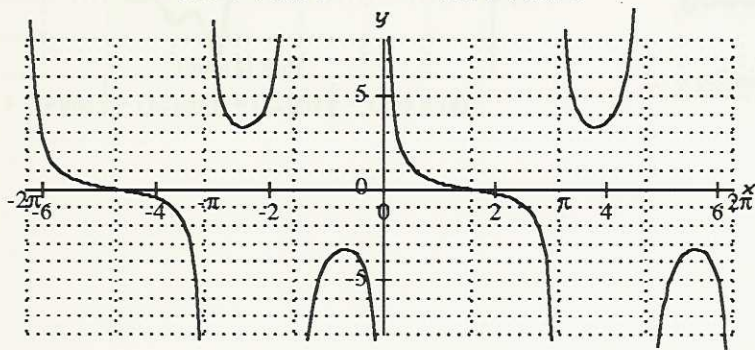


1.47817027

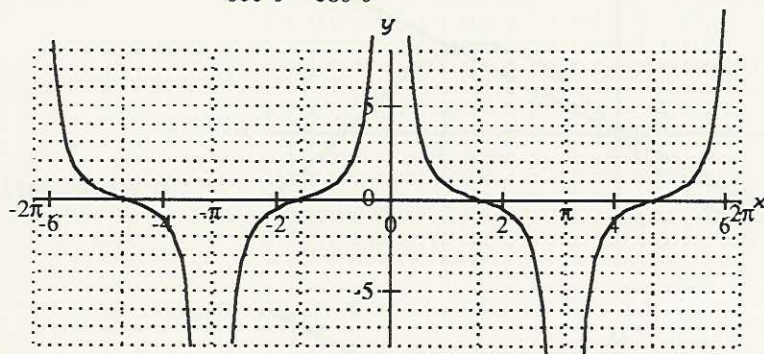
11. The graph of $y = \csc \theta + \cot \theta$, and $y = \frac{1 + \cos \theta}{\sin \theta}$ both look like:



13. The graphs of $y = \frac{\csc \theta}{\sec \theta + \tan \theta}$ and of $y = \frac{\cos \theta}{\sin \theta + \sin^2 \theta}$ both look like the following.



15. The graph of both $y = \frac{1}{\sec \theta - \cos \theta}$ and of $y = \cot \theta \csc \theta$ are as follows.



- 17, 19 See the answers in appendix A.
21.

